IMAGE ENHANCEMENT: FILTERING IN THE FREQUENCY DOMAIN

Francesca Pizzorni Ferrarese
In this lecture we will look at image enhancement in the frequency domain

- Jean Baptiste Joseph Fourier
- The Fourier series & the Fourier transform
- Image Processing in the frequency domain
  - Image smoothing
  - Image sharpening
- Fast Fourier Transform
Jean Baptiste Joseph Fourier

Fourier was born in Auxerre, France in 1768

- Most famous for his work “La Théorie Analitique de la Chaleur” published in 1822

Nobody paid much attention when the work was first published

One of the most important mathematical theories in modern engineering
Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*. 

Notice how we get closer and closer to the original function as we add more and more frequencies.
The Big Idea (cont...)

Frequency domain signal processing example in Excel
The Discrete Fourier Transform (DFT)

The \textit{Discrete Fourier Transform} of \( f(x, y) \), for \( x = 0, 1, 2 \ldots M-1 \) and \( y = 0,1,2 \ldots N-1 \), denoted by \( F(u, v) \), is given by the equation:

\[
F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}
\]

for \( u = 0, 1, 2 \ldots M-1 \) and \( v = 0, 1, 2 \ldots N-1 \).
The DFT of a two dimensional image can be visualised by showing the spectrum of the images component frequencies.

DFT & Images
DFT & Images

DFT & Images (cont...)

Scanning electron microscope image of an integrated circuit magnified ~2500 times

DFT

Fourier spectrum of the image
DFT & Images (cont...)

DFT & Images (cont...)

The Inverse DFT

It is really important to note that the Fourier transform is completely reversible.

The inverse DFT is given by:

\[
 f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi (ux/M + vy/N)}
\]

for \( x = 0, 1, 2 \ldots M-1 \) and \( y = 0, 1, 2 \ldots N-1 \).
The DFT and Image Processing

To filter an image in the frequency domain:

1. Compute $F(u,v)$ the DFT of the image
2. Multiply $F(u,v)$ by a filter function $H(u,v)$
3. Compute the inverse DFT of the result
Some Basic Frequency Domain Filters

Low Pass Filter

High Pass Filter
Some Basic Frequency Domain Filters
Some Basic Frequency Domain Filters

Smoothing Frequency Domain Filters

Smoothing is achieved in the frequency domain by dropping out the high frequency components. The basic model for filtering is:

\[ G(u,v) = H(u,v)F(u,v) \]

where \( F(u,v) \) is the Fourier transform of the image being filtered and \( H(u,v) \) is the filter transform function.

*Low pass filters* – only pass the low frequencies, drop the high ones
Ideal Low Pass Filter

Simply cut off all high frequency components that are a specified distance $D_0$ from the origin of the transform.
The transfer function for the ideal low pass filter can be given as:

$$H(u, v) = \begin{cases} 
1 & \text{if } D(u, v) \leq D_0 \\
0 & \text{if } D(u, v) > D_0 
\end{cases}$$

where $D(u, v)$ is given as:

$$D(u, v) = \left[ (u - M / 2)^2 + (v - N / 2)^2 \right]^{1/2}$$
Above we show an image, its Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it.
Ideal Low Pass Filter (cont…)
Ideal Low Pass Filter (cont...)
Ideal Low Pass Filter (cont...)
Ideal Low Pass Filter (cont...)

Result of filtering with ideal low pass filter of radius 5
Ideal Low Pass Filter (cont…)

Result of filtering with ideal low pass filter of radius 15
The transfer function of a Butterworth lowpass filter of order $n$ with cutoff frequency at distance $D_0$ from the origin is defined as:

$$H(u, v) = \frac{1}{1 + \left[ D(u, v) / D_0 \right]^{2n}}$$
Butterworth Lowpass Filter (cont...)
Butterworth Lowpass Filter (cont...)

Result of filtering with Butterworth filter of order 2 and cutoff radius 5
Butterworth Lowpass Filter (cont...)  


Result of filtering with Butterworth filter of order 2 and cutoff radius 15
Gaussian Lowpass Filters

The transfer function of a Gaussian lowpass filter is defined as:

\[ H(u, v) = e^{-\frac{D^2(u,v)}{2D_0^2}} \]
Gaussian Lowpass Filters (cont...)


Original image

Result of filtering with Gaussian filter with cutoff radius 5

Result of filtering with Gaussian filter with cutoff radius 15

Result of filtering with Gaussian filter with cutoff radius 30

Result of filtering with Gaussian filter with cutoff radius 85

Result of filtering with Gaussian filter with cutoff radius 230
Lowpass Filters Compared

Result of filtering with ideal low pass filter of radius 15

Result of filtering with Butterworth filter of order 2 and cutoff radius 15

Result of filtering with Gaussian filter with cutoff radius 15

A low pass Gaussian filter is used to connect broken text

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
Lowpass Filtering Examples

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.
Lowpass Filtering Examples (cont...)

Different lowpass Gaussian filters used to remove blemishes in a photograph

Lowpass Filtering Examples (cont…)
Lowpass Filtering Examples (cont...)
Sharpening in the Frequency Domain

Edges and fine detail in images are associated with high frequency components

*High pass filters* – only pass the high frequencies, drop the low ones

High pass frequencies are precisely the reverse of low pass filters, so:

\[ H_{hp}(u, v) = 1 - H_{lp}(u, v) \]
The ideal high pass filter is given as:

\[
H(u, v) = \begin{cases} 
0 & \text{if } D(u, v) \leq D_0 \\
1 & \text{if } D(u, v) > D_0 
\end{cases}
\]

where \( D_0 \) is the cut off distance as before.
Ideal High Pass Filters (cont...)

Results of ideal high pass filtering with $D_0 = 15$

Results of ideal high pass filtering with $D_0 = 30$

Results of ideal high pass filtering with $D_0 = 80$

Butterworth High Pass Filters

The Butterworth high pass filter is given as:

\[ H(u, v) = \frac{1}{1 + \left[ \frac{D_0}{D(u, v)} \right]^{2n}} \]

where \( n \) is the order and \( D_0 \) is the cut off distance as before.
Butterworth High Pass Filters (cont…)

Results of Butterworth high pass filtering of order 2 with $D_0 = 15$

Results of Butterworth high pass filtering of order 2 with $D_0 = 30$

Results of Butterworth high pass filtering of order 2 with $D_0 = 80$
The Gaussian high pass filter is given as:

$$H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

where $D_0$ is the cut off distance as before.
Gaussian High Pass Filters (cont...)


Results of Gaussian high pass filtering with $D_0 = 15$

Results of Gaussian high pass filtering with $D_0 = 80$

Results of Gaussian high pass filtering with $D_0 = 30$
Highpass Filter Comparison

Results of ideal high pass filtering with $D_0 = 15$

Results of Butterworth high pass filtering of order 2 with $D_0 = 15$

Results of Gaussian high pass filtering with $D_0 = 15$

Highpass Filter Comparison

Results of ideal high pass filtering with $D_0 = 15$

Highpass Filter Comparison

Results of Butterworth high pass filtering of order 2 with $D_0 = 15$
Highpass Filter Comparison

Results of Gaussian high pass filtering with $D_0 = 15$
Highpass Filtering Example

Fast Fourier Transform

The reason that Fourier based techniques have become so popular is the development of the Fast Fourier Transform (FFT) algorithm. Allows the Fourier transform to be carried out in a reasonable amount of time. Reduces the amount of time required to perform a Fourier transform by a factor of 100 – 600 times!
Similar jobs can be done in the spatial and frequency domains.

Filtering in the spatial domain can be easier to understand.

Filtering in the frequency domain can be much faster – especially for large images.
Summary

In this lecture we examined image enhancement in the frequency domain

- The Fourier series & the Fourier transform
- Image Processing in the frequency domain
  - Image smoothing
  - Image sharpening
- Fast Fourier Transform