2D Discrete Fourier Transform (DFT)

2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size $N_x \mathrm{x} N_y$

or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.
- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a sampled version of the 2D continuous Fourier transform

2D Discrete Fourier Transform

- 2D Fourier Transform

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D DFT is a sampled version of 2D FT.

2D Discrete Fourier Transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- It is also possible to define DFT as follows

$$F[k,l] = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- Or, as follows

$$F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \mathrm{Real}(F)$ and $I = \mathrm{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

2D Discrete Fourier Transform

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v), f(ax, by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

2D Discrete Fourier Transform

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN}f^*(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting the function $F^*(u,v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x,y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution[†]	$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$
Correlation[†]	$f(x,y) \circ h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)$
Convolution theorem[†]	$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$; $f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$
Correlation theorem[†]	$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v)H(u,v)$; $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \circ H(u,v)$

2D Discrete Fourier Transform

Some useful FT pairs:

Impulse $\delta(x, y) \Leftrightarrow 1$

Gaussian $A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$

Rectangle $\text{rect}[a, b] \Leftrightarrow ab \dfrac{\sin(\pi ua)}{(\pi ua)} \dfrac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$

Cosine $\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$

$$\frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$$

Sine $\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$

$$j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$$

† Assumes that functions have been extended by zero padding.

Periodicity

- [M,N] point DFT is periodic with period [M,N]

$$F[k,l] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$F[k+M,l+N] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k+M}{M}m+\frac{l+N}{N}n\right)}$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}e^{-j2\pi\left(\frac{M}{M}m+\frac{N}{N}n\right)}$$

1

$$= F[k,l]$$

Periodicity

- [M,N] point DFT is periodic with period [M,N]

$$f[m,n]=\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$f[m+M,n+N]=\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}F[k,l]e^{j2\pi\left(\frac{k}{M}(m+M)+\frac{l}{N}(n+N)\right)}$$

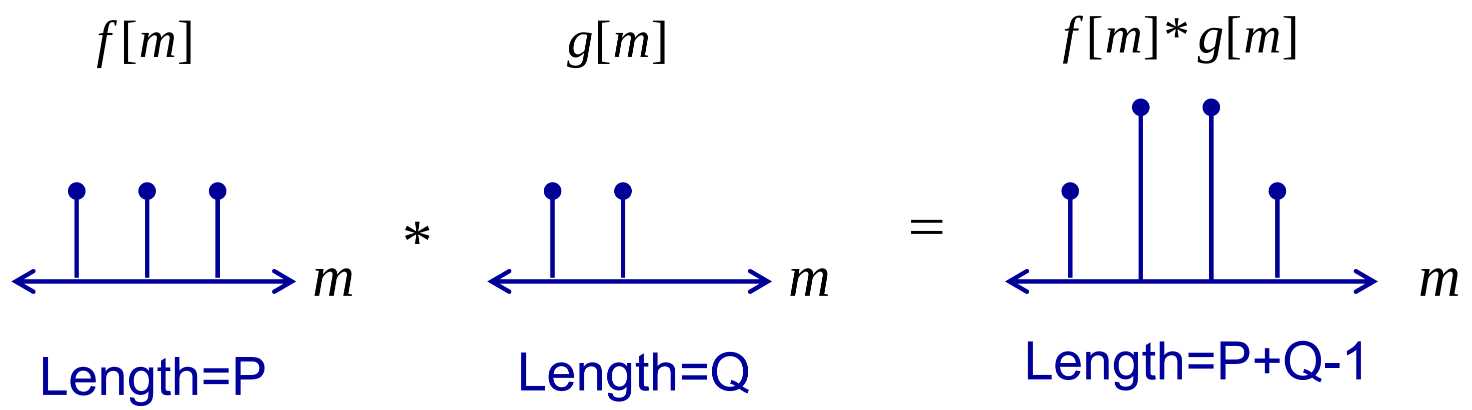
$$=\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\underbrace{e^{j2\pi\left(\frac{k}{M}M+\frac{l}{N}N\right)}}_{1}$$

$$=f[m,n]$$

Convolution

- Be careful about the convolution property!

$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$

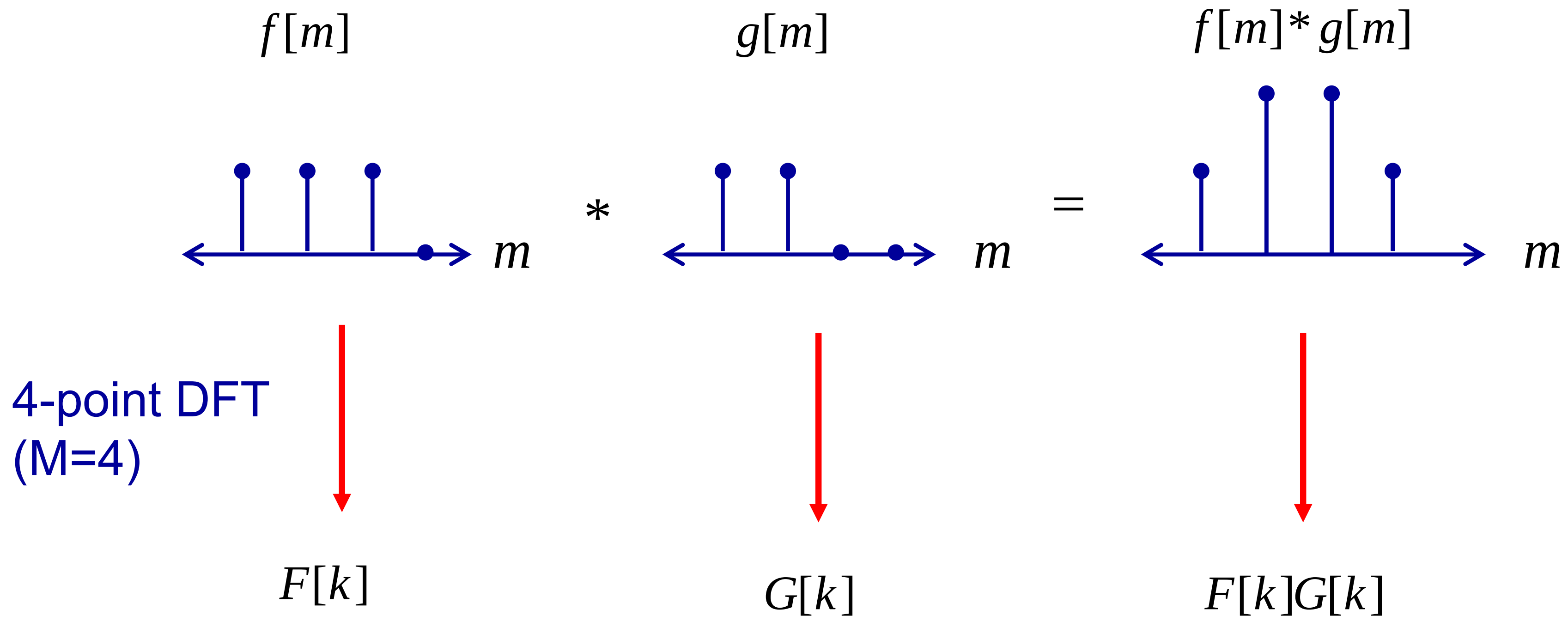


For the convolution property to hold, M must be *greater than or equal* to P+Q-1.

Convolution

$$f[m] * g[m] \Leftrightarrow F[k]G[l]$$

- Zero padding



2D DCT

- Separable product (equivalently, a composition) of DCTs along each dimension

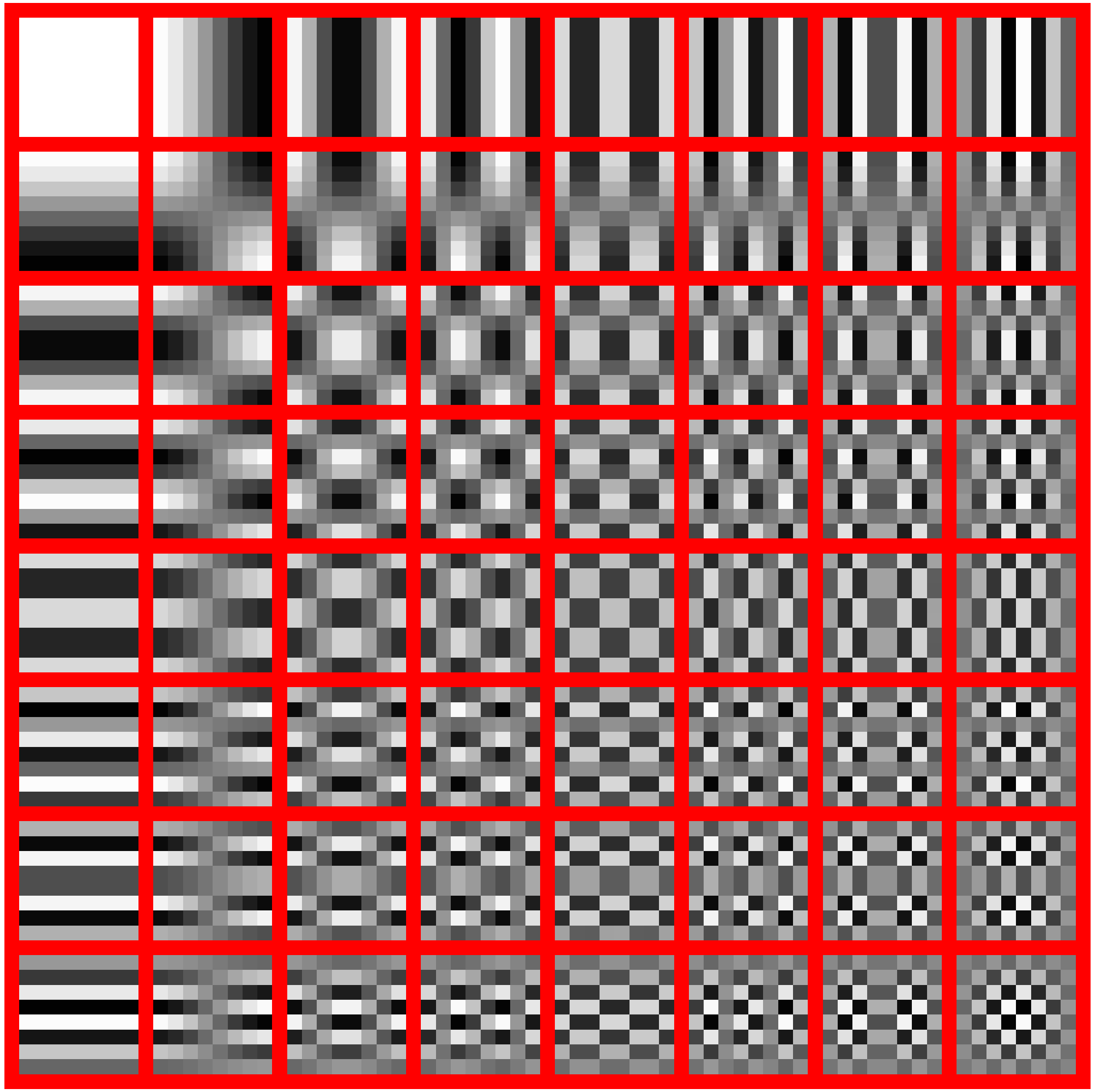
$$X_{k_1,k_2} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_1}\left(n_1+\frac{1}{2}\right)k_1\right] \cos\left[\frac{\pi}{N_2}\left(n_2+\frac{1}{2}\right)k_2\right]$$

- *Row-column* algorithm
- The inverse of a multi-dimensional DCT is just a separable product of the inverse(s) of the corresponding one-dimensional DCT
  - e.g. the one-dimensional inverses applied along one dimension at a time

DCT: basis functions

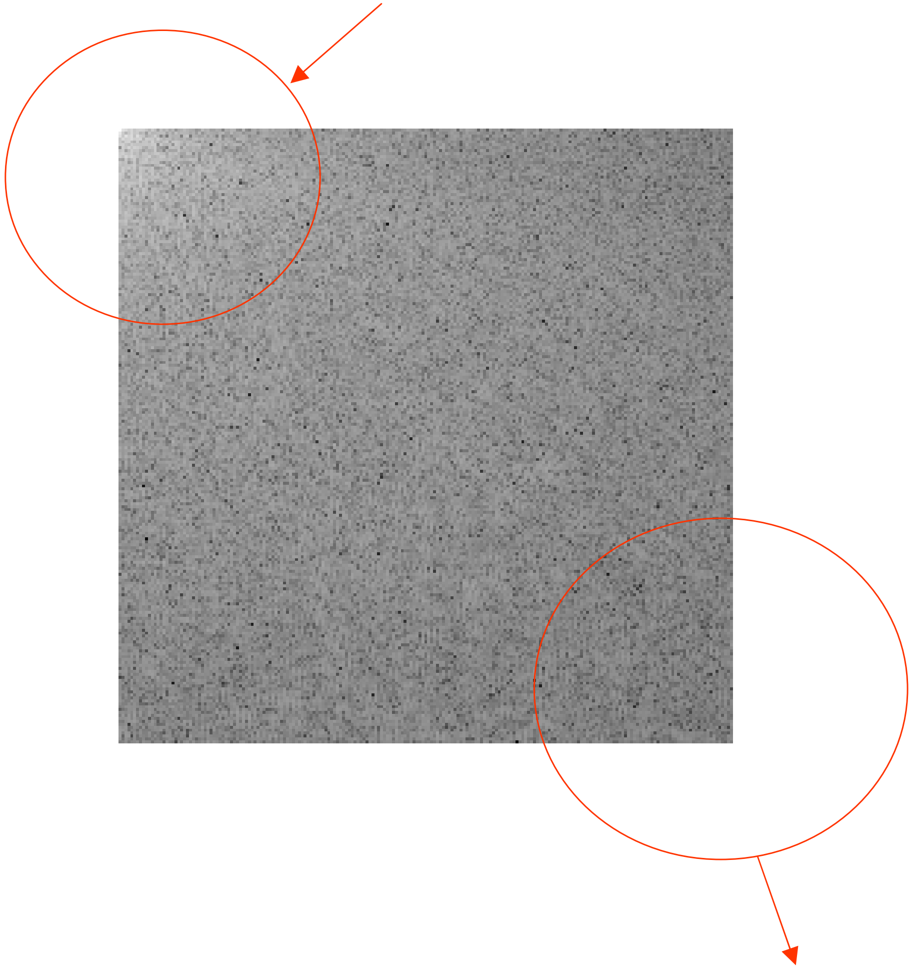
Block-based transform

Block size
$N_1=N_2=8$



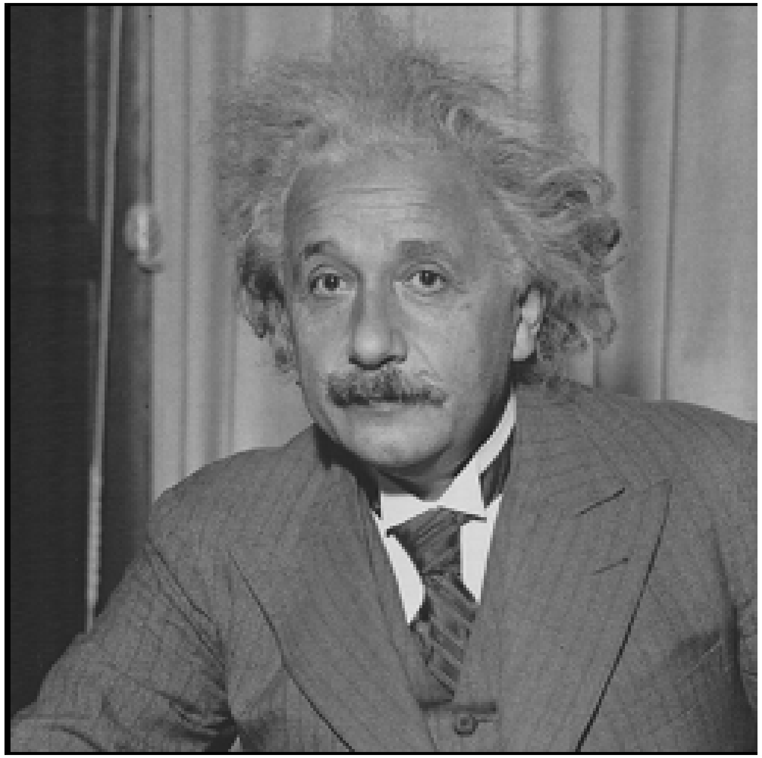
DCT: example

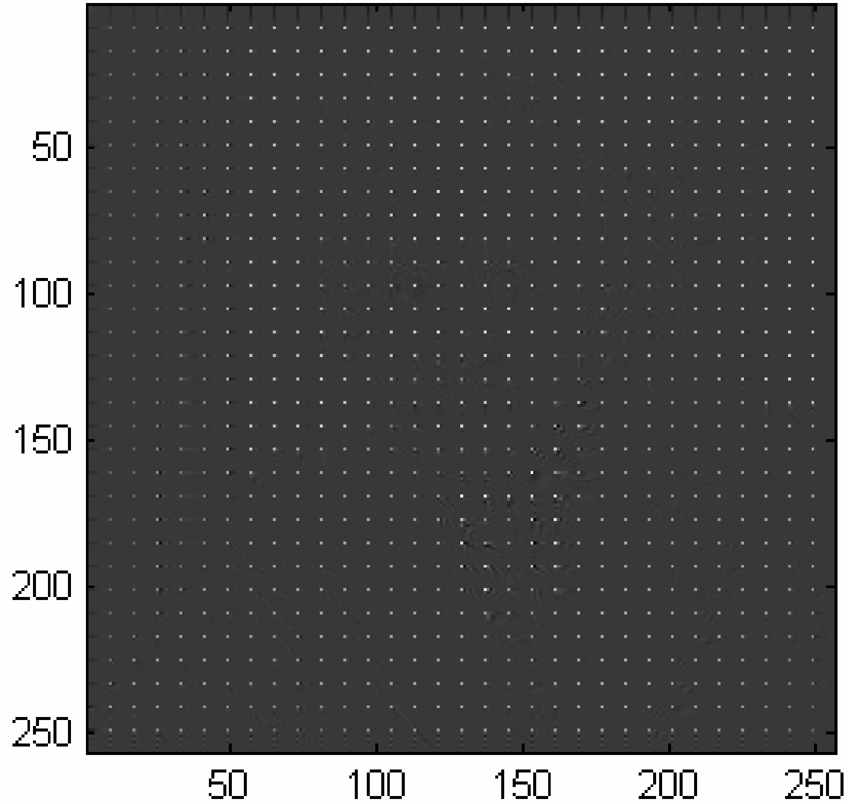
Low-frequency components



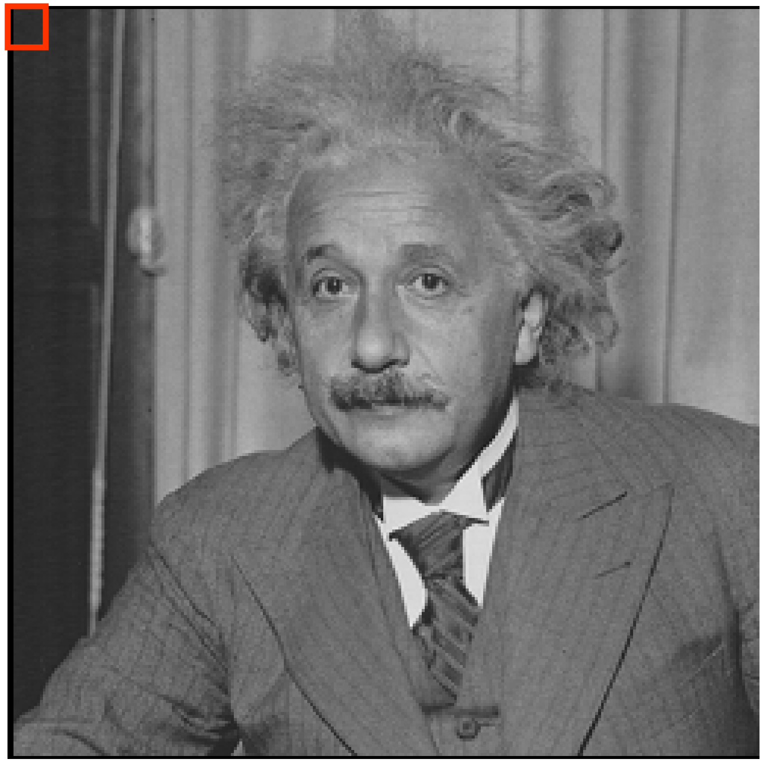
High-frequency components

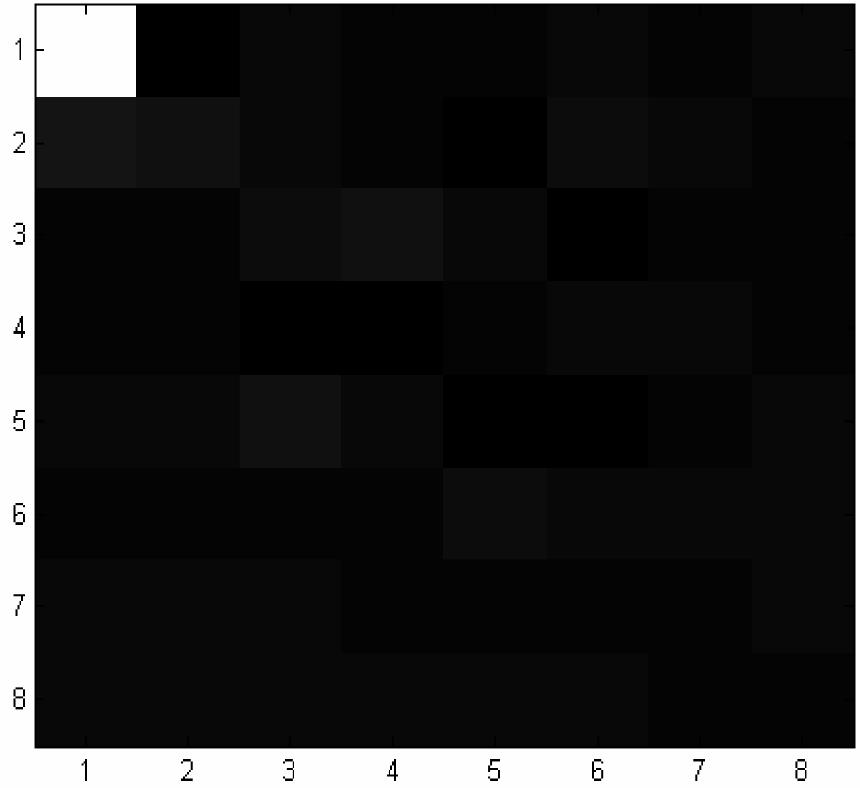
Block-based DCT



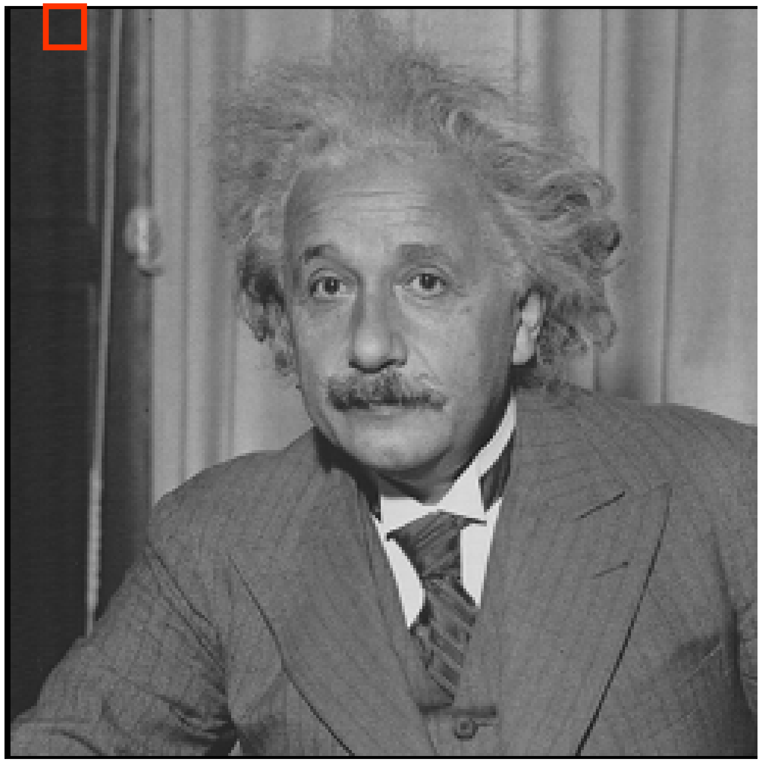


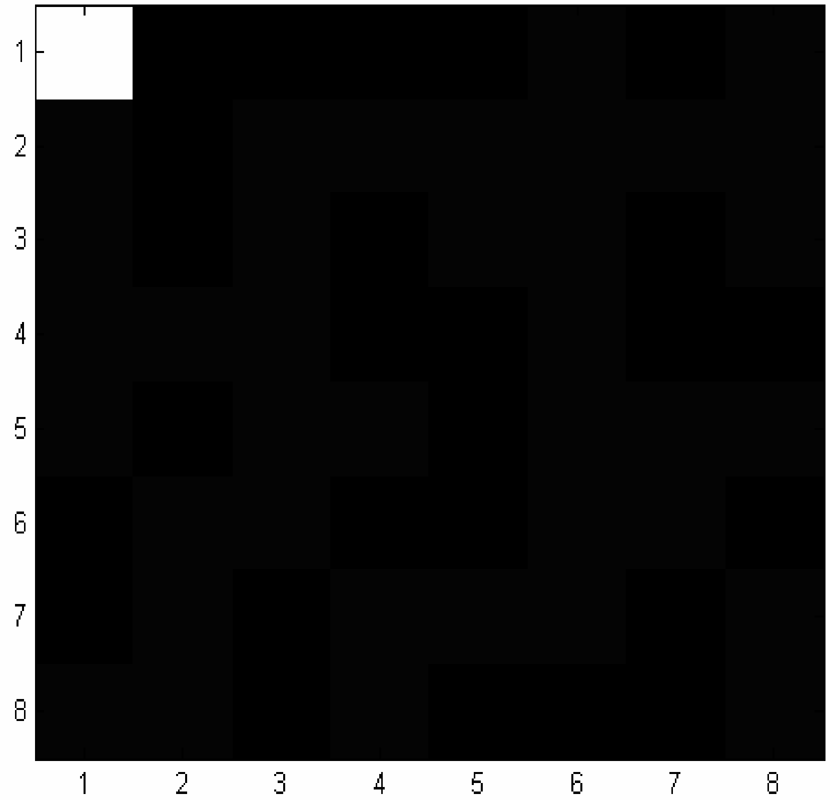
Block-based DCT



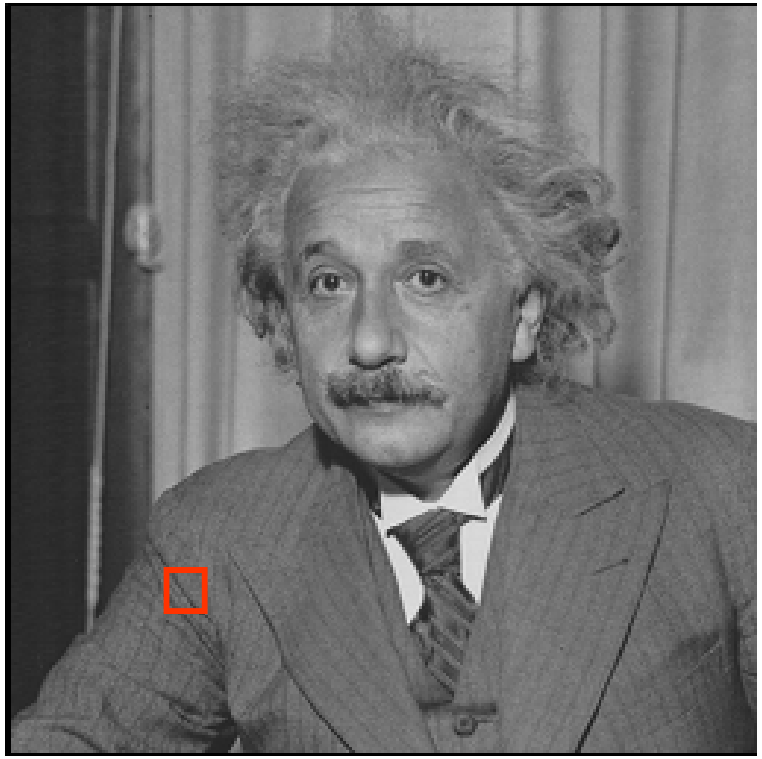


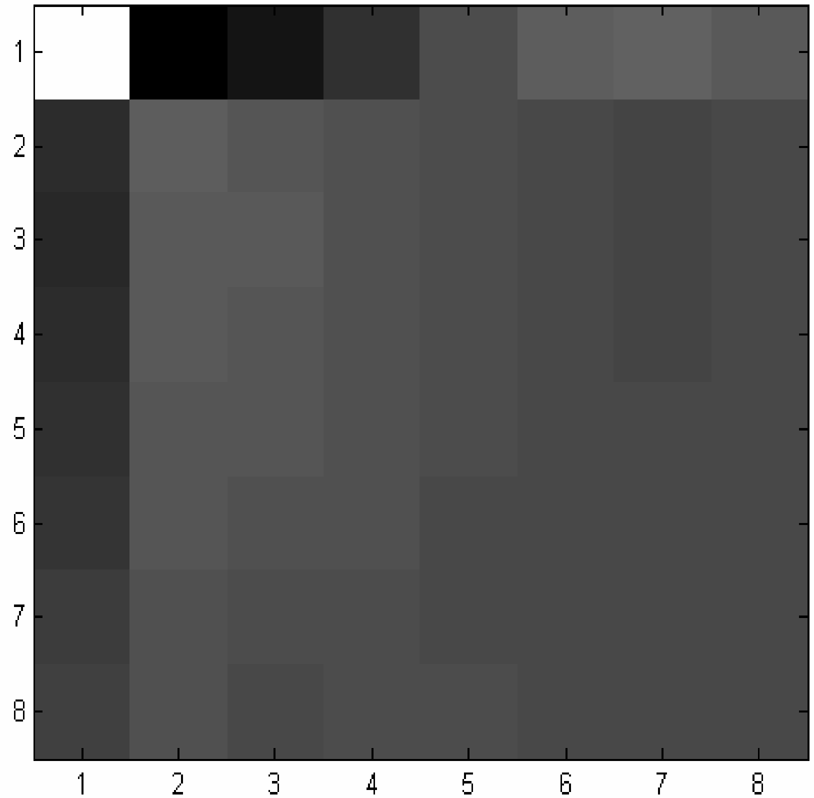
Block-based DCT



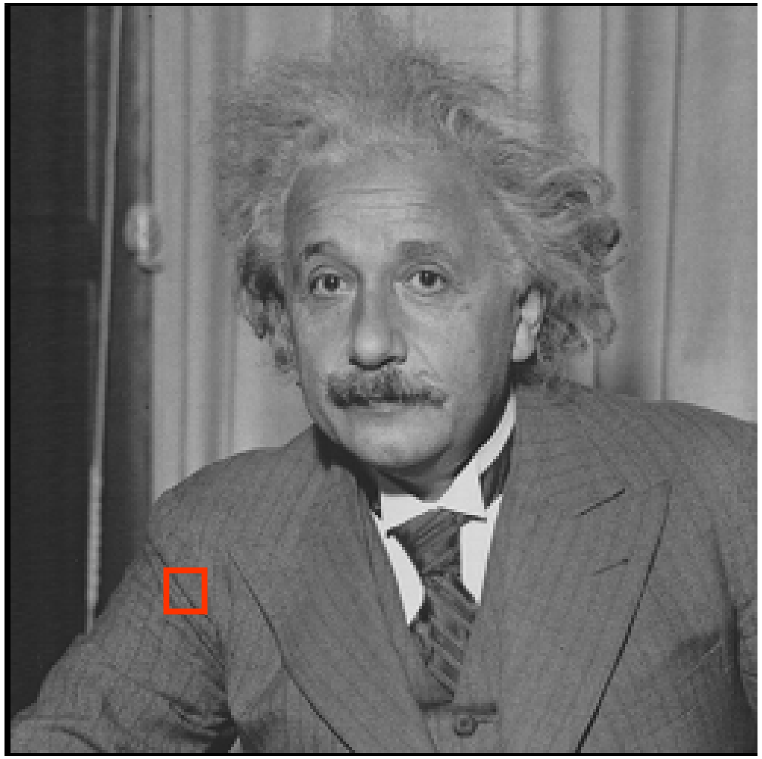


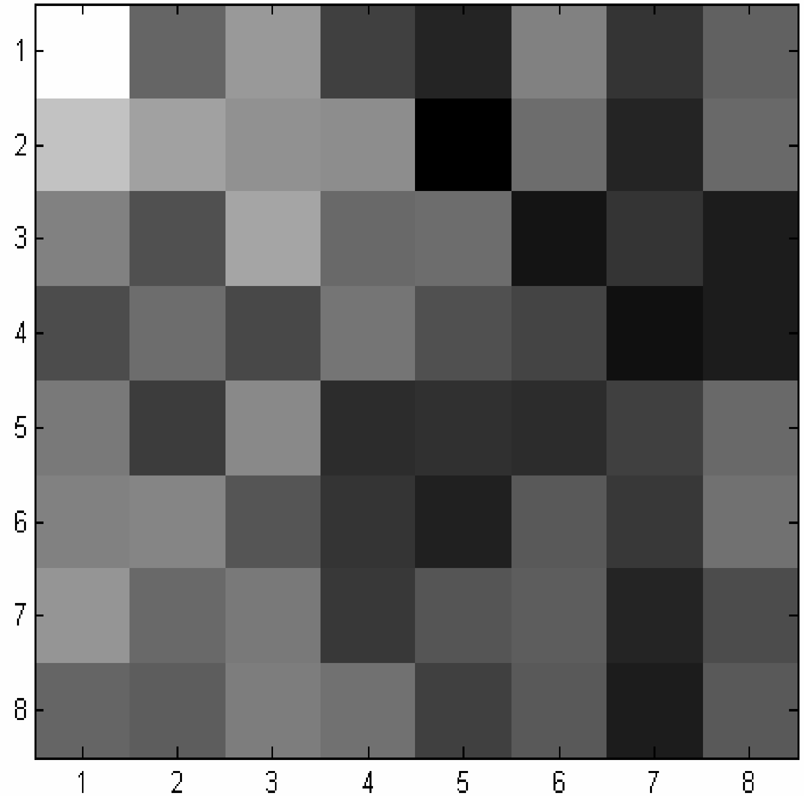
Block-based DCT





Block-based DCT



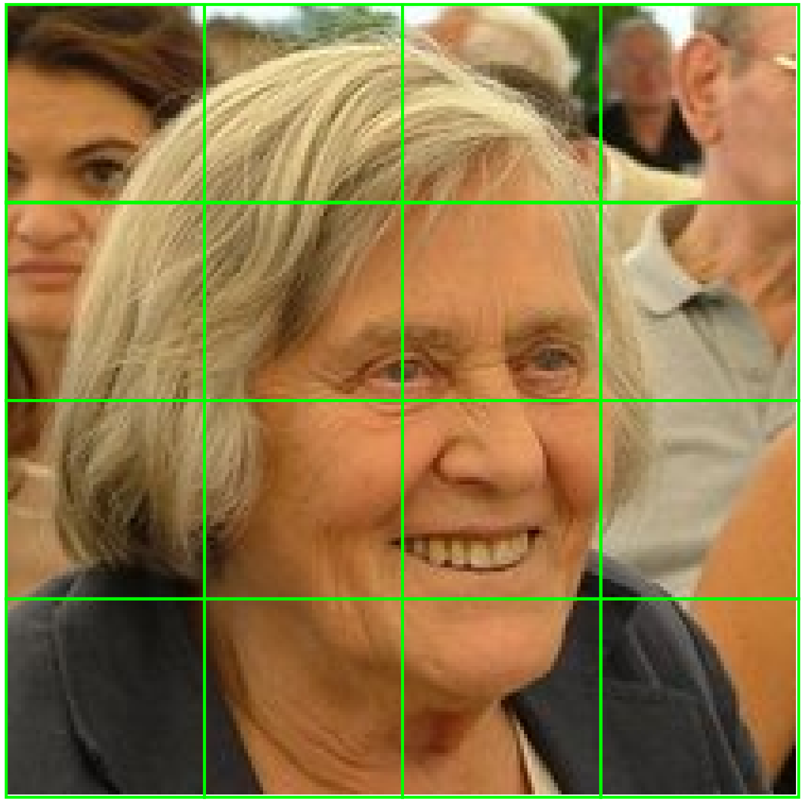


The mean value has been subtracted before calculating the DCT. This is usual in coding applications and allows to outline the value of the coefficients corresponding to non-zero spatial frequencies

JPEG preview

Block-based DCT

Block size
$N_1=N_2=8$



The source data (8x8) is transformed to a linear combination of these 64 frequency squares.

Appendix

Appendix: Impulse Train

- The Fourier Transform of a comb function is

$$F\left(comb_{M,N}[m,n]\right) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} comb_{M,N}[m,n]e^{-j2\pi(um+vn)}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta[m-kM,n-lN]\right]e^{-j2\pi(um+vn)}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta[m-kM,n-lN]e^{-j2\pi(um+vn)}\right]$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[e^{-j2\pi(ukM+vlN)}\right]$$

Impulse Train (cont'd)

- The Fourier Transform of a comb function is

$$F\left(comb_{M,N}[m,n]\right) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[e^{-j2\pi\left(ukM+vlN\right)}\right]$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[1e^{-j2\pi\left((uM)k+(vN)l\right)}\right]$$

(Fourier Trans. of 1)

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(uM-k, vN-l)$$

?

$$= \frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M}, v-\frac{l}{N}\right)$$

Impulse Train (cont'd)

- Proof

$$\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(uM-k)F(u)du = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(uM-k)F(u)du$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{M} \int_{-\infty}^{\infty} \delta(v-k)F\left(\frac{v}{M}\right)dv$$

$$= \frac{1}{M} \sum_{k=-\infty}^{\infty} F\left(\frac{k}{M}\right) = \frac{1}{M} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(v-\frac{k}{M})F(v)dv$$

$$= \int_{-\infty}^{\infty} \frac{1}{M} \sum_{k=-\infty}^{\infty} \delta(u-\frac{k}{M})F(u)du$$