Discrete Time Signals and Systems
Time-frequency Analysis

Gloria Menegaz

Time-frequency Analysis

- Fourier transform (1D and 2D)

- Reference textbook:

  Discrete time signal processing, A.W. Oppenheim and R.W. Shafer

  – Chapter 1: Introduction

  – Chapter 2: Discrete time Signals and Systems

  – Chapter 3: Sampling of Continuos Time Signals

  – (Chapter 4: The z-Transform)

Signal Classification

- *Continuous time signals* $x(t)$ are functions of a continuous independent variable t
$$x = x(t), t \in \mathcal{R}$$

- *Discrete time signals* are functions of a discrete variable
$$x = \{x[n]\}, n \in \mathcal{Z}, -\infty < n < +\infty$$
  - Are defined at discrete time intervals
  - Are represented as *sequences of numbers*

- *Digital signals* both the independent variable **and** the amplitude are discrete

Digital *systems*: both the input and the output are digital signals
$\Rightarrow$ Digital Signal Processing: processing of signals that are discrete in both time and amplitude

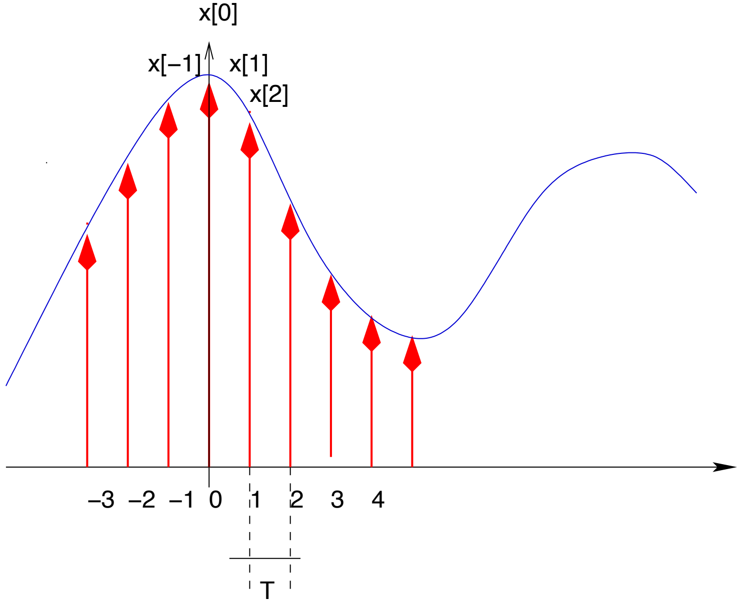
Periodic Sampling

The sequences are obtained by sampling the analog signal at equi-spaced points:

$$x[n] = x(nT), n \in I$$

T: *sampling period*, interval between samples

$f = \frac{1}{T}$: *sampling frequency*



Basic Sequences

- Products and sums among sequences: element by element operations
- Delayed or shifted sequence: $y[n] = x[n-k]$
- Unit sample sequence $\delta[n]$ (Dirac function, impulse):

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{1}$$

- Unit step sequence:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{2}$$

Basic Sequences

$\rightarrow$ relation between the two functions:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \quad (3)$$

$$\delta[n] = u[n] - u[n-1] \quad (4)$$

- Sinusoidal sequence:

$$x[n] = Acos(\omega_0 n + \Phi), \forall n$$

Basic Sequences

- Complex exponential

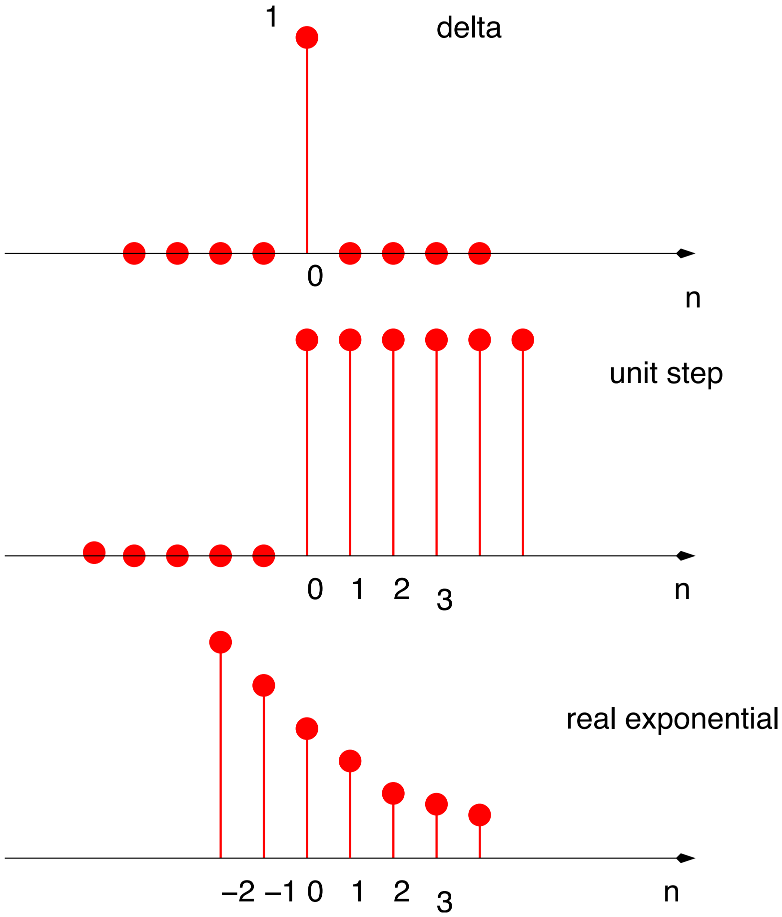
$$x[n] = Ae^{(\alpha + j(\omega_0 n + \phi))}$$

  - $|\alpha| < 1$: exponentially decaying envelop
  - $|\alpha| > 1$: exponentially growing envelop
  - $|\alpha| = 1$: *complex exponential sequence*:

$$x[n] = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi)$$

The real and imaginary parts oscillate sinusoidally with n. The fact that n is an integer leads to important differences with respect to the corresponding functions in the continuous domain. By analogy, ω_0 and Φ are called *frequency* and *phase*.

Graphically



Discrete vs. continuous

1. Complex exponentials and sinusoids are 2π-periodic

$$x[n] = A \exp^{j(\omega_0+2\pi)n} = A \exp^{j\omega_0 n} \exp^{j2\pi n} = A \exp^{j\omega_0 n} \tag{5}$$

$\Rightarrow$ Complex exponentials with frequencies $(\omega_0 + 2\pi r)$ are *indistinguishable* from one another $\Rightarrow$ only frequencies in an interval of length 2π need to be considered

2. Complex exponentials in general are not periodic
Periodicity condition: $x[n] = x[n+N], \; \forall n, N \in I$. For this condition to be true, the following relation must hold true:

$$\omega_0 N = 2\pi k, \; k \in I \tag{6}$$

$\Rightarrow$ Complex exponentials and sinusoidal sequences are *not* necessarily periodic with period $T_0 = 2\pi/\omega_0$ as in the continuous domain and, depending on the value of ω_0, might not be periodic at all.

Discrete vs. continuous

Hint: relation (**??**) can be written as:

$$T_0 = \frac{2\pi}{\omega_0} = \frac{N}{k} \tag{7}$$

where T_0 is the period. If one thinks of the discrete time signal as to the sampled version of a continuous time signal of period T_0 with unitary sampling interval ($T_s = 1$), then relation (**??**) can be naturally interpreted as follows: in order to obtain a discrete time *periodic* signal of period N by sampling a continuous time signal of period T_0, the period T_0 must be an integer factor of N, or, viceversa, N must be a multiple of T_0. This can be easily generalized to the case $T_s \neq 1$ (not necessarily integer) as follows:

$$\omega_0 N T_s = 2\pi k \quad T_0 = \tfrac{N}{k} T_s \tag{8}$$

namely, $N = kT_0/T_s$ must be an integer multiple of the ratio between the period and the sampling step.

Examples

1. Example 1

$$
\begin{aligned}
x(t) &= A\sin(\omega_0 t) \\
x[n] &= A\sin(\omega_0 n) \\
\omega_0 &= \frac{3}{4}\pi \\
\rightarrow \frac{3}{4}\pi N &= 2\pi k \rightarrow 3N = 8k \rightarrow
\end{aligned}
$$

   the first two integers satisfying the condition are $N = 8, k = 3$, thus the signal $x[n]$ obtained by sampling a continuous sinusoid $x(t)$ with frequency $\omega_0 = \frac{3}{4}\pi$ is periodic with period $N = 8$; each period of $x[n]$ corresponds to three $(k = 3)$ periods of the original signal.

2. Example 2

   $\omega_0 = 1 \rightarrow$ no solutions!

Summary

$$
\begin{aligned}
\omega_0 &\leftrightarrow \omega_0 + 2\pi r, \qquad r \in I && (9)\\
\omega_0 &= \frac{2k\pi}{N}, \qquad k, N \in I \text{ condition for periodicity} && (10)
\end{aligned}
$$

Let's assume that, given N, ω_0 satisfies relation (**??**) for a certain k. Relation (**??**) implies that only N distinguishable frequencies satisfy relation (**??**).
Proof: let $\omega_k = \frac{2k\pi}{N}, \qquad k = 0, 1, ..., N-1$

$$
\begin{aligned}
\omega_d &= 0 \qquad \text{degenerate case}\\
\omega_1 &= \frac{2\pi}{N}\\
\omega_2 &= \frac{2 \times 2\pi}{N}\\
\ldots\\
\omega_{N-1} &= \frac{(N-1) \times 2\pi}{N}\\
\omega_N &= \frac{N \times 2\pi}{N} = 2\pi\\
\ldots\\
\omega_{N+i} &= \frac{(N+i) \times 2\pi}{N} = 2\pi + \frac{2\pi i}{N} \leftrightarrow \omega_i
\end{aligned}
$$

Examples

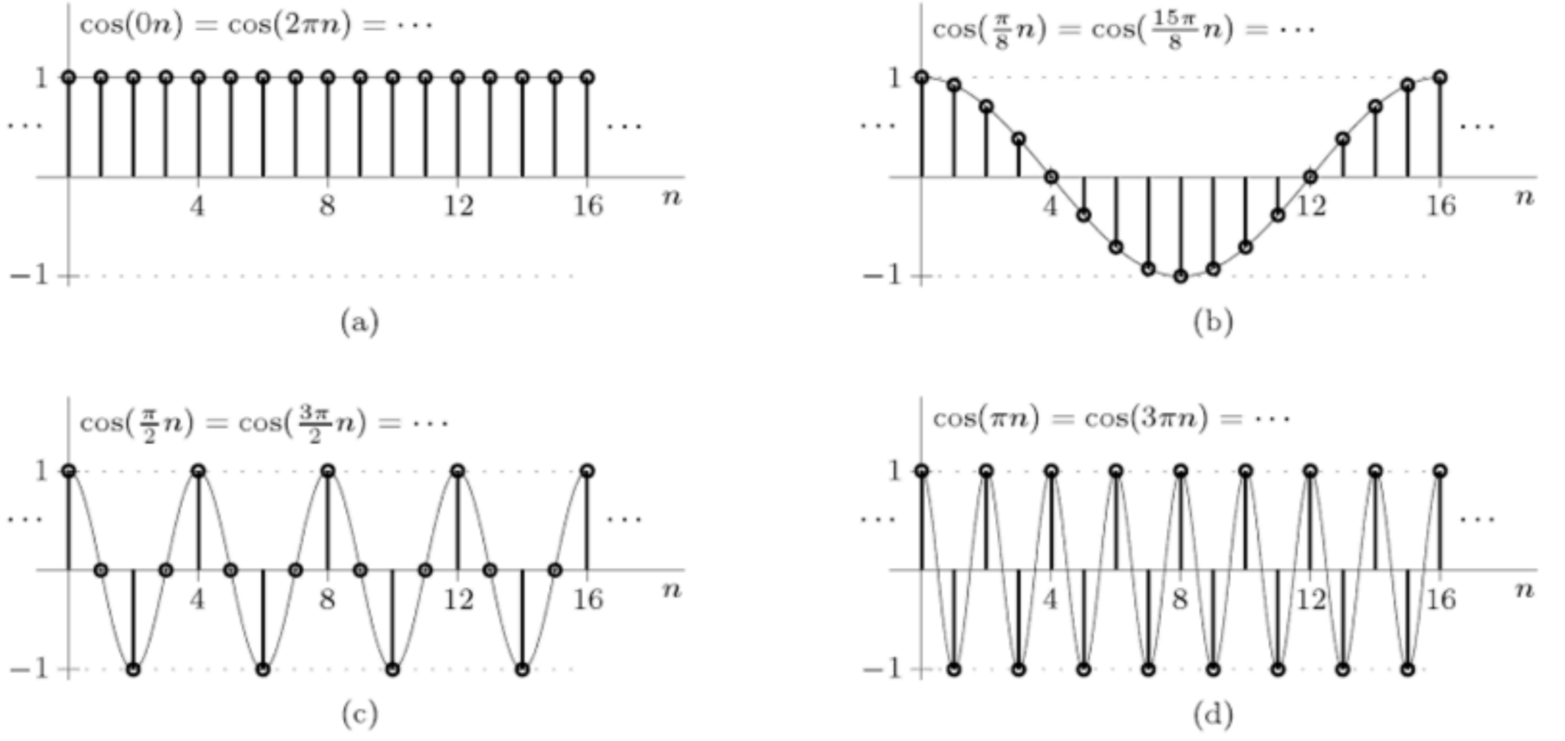


Figure 4.16: DT sinusoids of various frequencies: (a) lowest rate $\Omega = 0$ to (d) highest rate $\Omega = \pi$.

Discrete vs Continuous

Conclusion: for a sinusoidal signal $x[n] = A\sin(\omega_0 n)$, as ω_0 increases from $\omega_0 = 0$ to $\omega_0 = \pi$, $x[n]$ oscillates more and more rapidly. Conversely, from $\omega_0 = \pi$ to $\omega_0 = 2\pi$ the oscillations become slower. In fact, because of the periodicity in ω_0 of complex exponentials and sinusoidal sequences, $\omega_0 = 0$ is indistinguishable from $\omega_0 = 2\pi$ and, more in general, frequencies around $\omega_0 = 2\pi$ are indistinguishable from frequencies around $\omega_0 = 0$. As a consequence, values of ω_0 in the vicinity of $\omega_0 = 2k\pi$ are referred to as *low frequencies*, whereas frequencies in the vicinity of $\omega_0 = (2k+1)\pi$ are referred to as *high frequencies*. This is a fundamental difference from the continuous case, where the speed of the oscillations increases monotonically with the frequency ω_0.

Operations and Properties

- Periodicity: a sequence is periodic with *period* N *iif*:

$$x[n+N] = x[n], \forall n$$

- Energy: $E = \sum_{n=-\infty}^{+\infty} |x[n]|^2$

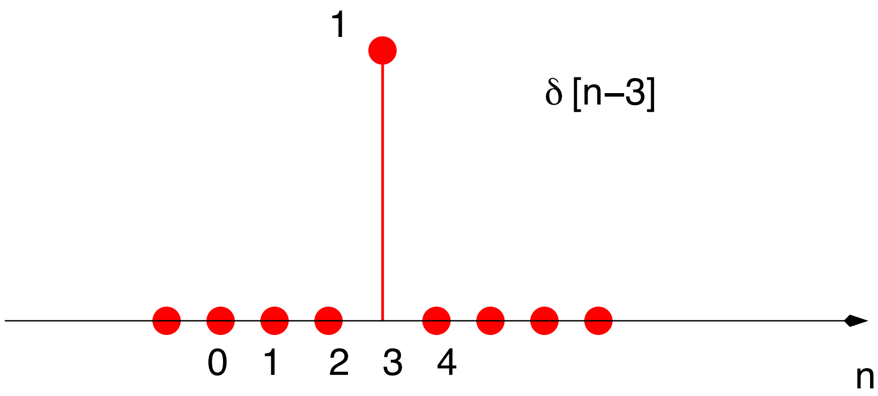
- Sample-wise operations:
  - Product: $x \cdot y = \{x[n]y[n]\}$
  - Sum: $x + y = \{x[n] + y[n]\}$
  - *Scaling*: $\alpha x = \{\alpha x[n]\}$
  - Delay: $y[n] = x[n - n_0], n, n_0 \in I$

Operations and Properties

Any sequence can be represented as a sum of *scaled* and *delayed* unit samples:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \tag{11}$$

$$\delta[n-k] = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases} \tag{12}$$



Discrete Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \qquad (13)$$

$$x[n] = \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad (14)$$

- $x[n]$ is a *discrete time* signal
- The DTFT $X(e^{j\omega})$ is a complex *continuos function* of the independent variable ω. As such, can be put in the equivalent forms:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + X_I(e^{j\omega}) \qquad (15)$$

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{\angle X(e^{j\omega})} \qquad (16)$$

  $|X(e^{j\omega})|$ is the *magnitude* or Fourier spectrum of the signal and $e^{\angle X(e^{j\omega})}$ is the phase spectrum of the transformed signal.
- $x[n]$ and $X(e^{j\omega})$ form the *Fourier representation* for the sequence.
- Eq. (**??**): analysis formula. It *projects* the signal to the frequency domain.
- Eq. (**??**): synthesis or reconstruction formula. It is used to recover the signal from its frequency domain representation.

DTFT: Interpretation

The DTFT represents the sequence $x[n]$ as a linear superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi}X(e^{j\omega n})d\omega \tag{17}$$

with $-\pi \leq \omega \leq \pi$ and $X(e^{j\omega n})$ representing the relative amount of each complex sinusoidal component.

By comparing (**??**) with relation (**??**) it is easy to realize that the frequency response of a LTIS is the Fourier transform of its impulse response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \quad \leftrightarrow \quad h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}$$

DTFT

Symmetry properties of the DTFT: see Table 2.1, page 53 of the reference textbook

The DTFT is a *periodic* function of ω with period 2π.

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n} \quad (18)$$

$$= e^{-j2\pi n} \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = X(e^{j\omega}) \quad (19)$$

More in general:

$$X(e^{j(\omega+2r\pi)}) = X(e^{j\omega}), \forall r \in I \quad (20)$$

The Fourier spectrum of a discrete (sampled) signal is periodic

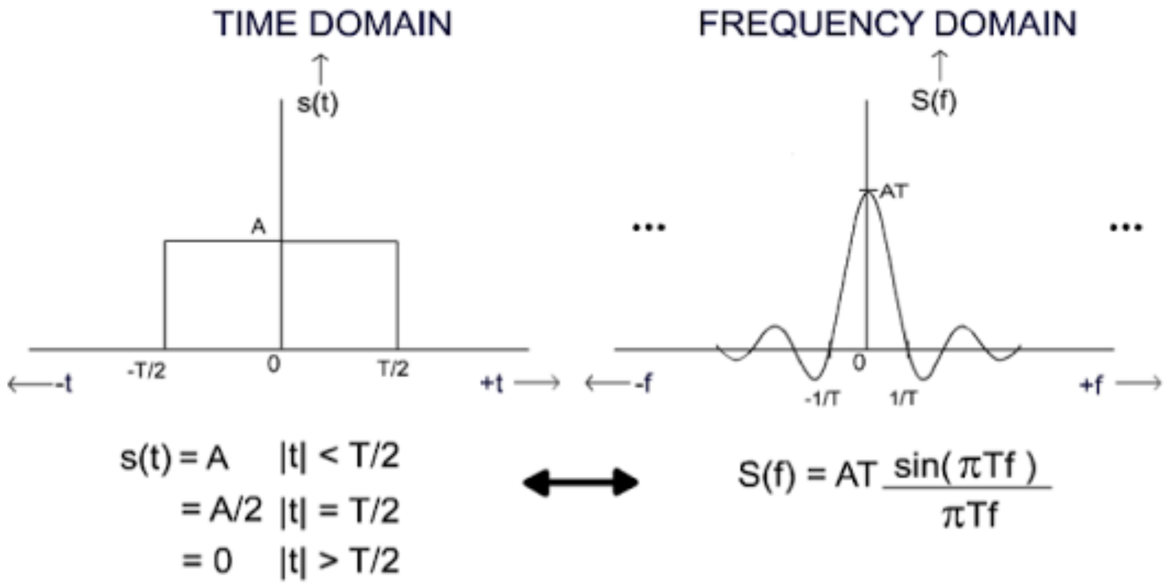
Eulero's formula

$$e^{jx} = \cos x + j \sin x \rightarrow \qquad e^{-j\omega 2n\pi} = e^{-j\omega n} e^{-j2\pi n} = e^{-j\omega n}, \forall n \in \mathbb{Z} \tag{21}$$

Reminder:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

Example: rect() function



The `rect()` function is important because it represents the impulse response of the ideal band-pass filter in the frequency domain. Its Fourier transform is a `sinc()` function which spreads over $(\infty, +\infty)$.

Ideal low pass filter

$$H(e^{j\omega n}) = H(\omega) \quad = \quad \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases} \tag{22}$$

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(e^{i\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \\ &= \frac{1}{2\pi}\frac{1}{jn}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n) \end{aligned}$$

$$h[n] = \frac{\omega_c}{\pi}\text{sinc}(\omega_c n) \tag{23}$$

ω_c : cutting frequency of the filter $\leftrightarrow$ bandwidth

Let ω_s be the sampling frequency. If we let $\omega_c = \frac{\omega_s}{2}$ then

$$h[n] = \frac{1}{T_s}\text{sinc}\left(\frac{n\pi}{T_s}\right) \tag{24}$$

To keep in mind

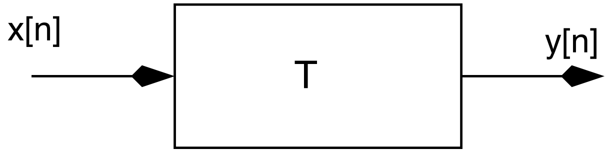
Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
1	$\sum_k 2\pi\delta(\omega+2k\pi)$
$\frac{\sin(\omega_c n)}{n}$	$X(\omega) = \begin{cases} 1 & \lvert\omega\rvert \leq \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \leq \pi \end{cases}$
$x[n] = \text{rect}_T[n]$	$2T\,\text{sinc}(\omega T)$
$e^{j\omega_0 n}$	$\sum_k 2\pi\delta(\omega-\omega_0+2k\pi)$
$\cos(\omega_0 n+\Phi)$	$\pi\sum_k \left[e^{j\Phi}\delta(\omega-\omega_0+2k\pi+e^{-j\Phi}\delta(\omega+\omega_0+2k\pi)\right]$

Table 2.3, page 61, reference textbook

Discrete Time Systems

A DTS is an operator that maps a discrete sequence $x[n]$ at its input to a discrete sequence $y[n]$ at its output:

$$y[n] = T\{x[n]\}$$



- Linearity:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}, \textit{ additivity} \quad (25)$$
$$T\{ax[n]\} = aT\{x[n]\}, \textit{ scaling or homogeneity} \quad (26)$$

$\Rightarrow$ Principle of superposition:

$$T\{a_1x_1[n] + a_2x_2[n]\} = a_1T\{x_1[n]\} + a_2T\{x_2[n]\} \quad (27)$$

Discrete Time Systems

- Time-invariance: a delayed input sequence maps to a delayed output sequence:

$$x[n] \rightarrow y[n] \tag{28}$$
$$x[n-n_0] \rightarrow y[n-n_0] \tag{29}$$

- Causality: the output value $y[n]$ for $n = n_0$ only depends on *previous* input samples $x[n] : n < n_0$

In images causality doesn't matter!

- Stability: a bounded input generates a bounded output

Linear Time Invariant Systems

Linear systems:

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \tag{30}$$

Principle of superposition:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \tag{31}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \tag{32}$$

Time invariance:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \leftrightarrow \text{Convolution sum} \tag{33}$$

$\rightarrow$ The system is *completely* characterized by the *impulse response* $h[k]$

Convolution operator

$$x[n] \star y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot y[n-k] \quad (34)$$

$$= \sum_{k=-\infty}^{+\infty} x[n-k] \cdot y[k] \quad (35)$$

Recipe for the convolution (refer to(**??**)):

1. Reflect $y[k]$ about the origin to get $y[-k]$
2. Shift the reflected sequence of n steps
3. Multiply (point wise) the resulting sequence by $x[n]$
4. Sum over the samples of the resulting signal to get the value of the convolution at position n
5. Increment n and ge back to point 2.

Properties of LTIS

- Commutativity (see(**??**))

  A LTIS with input $x[n]$ and impulse response $h[n]$ has the same output of a system with input $h[n]$ and impulse response $x[n]$

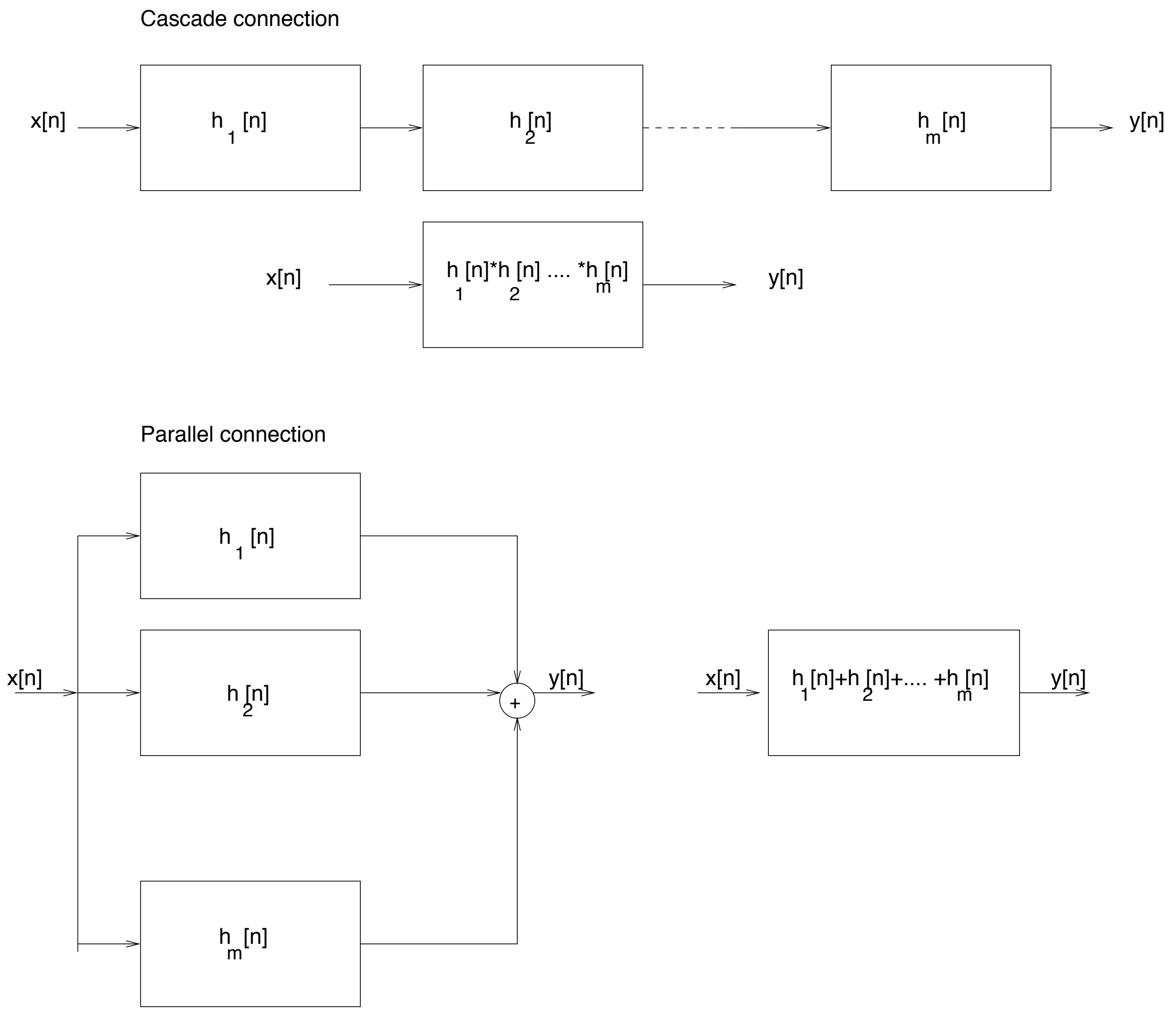
  $\rightarrow$ The *cascade* of LTIS systems has an impulse response that is the convolution of the IRs of the individual systems

- Distributivity:

$$x[n] \star (h_1[n] + h_2[n]) \quad = \quad x[n] \star h_1[n] + x[n] \star h_2[n]$$

  $\rightarrow$ The *parallel connection* of LTIS systems has an impulse response that is the sum of the IRs of the single systems

Properties of LTIS



Difference Equation

The transfer function is a N-th order linear constant coefficient difference equation:

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \rightarrow \tag{36}$$

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right\} \tag{37}$$

$\rightarrow$ The output value $y[n]$ is the linear combination of the $N-1$ last values of the output and the M last values of the input $x[n]$.
System classification:

- Finite Impulse Response (FIR): the impulse response involves a *finite* number of samples
- Infinite Impulse Response (IIR): the impulse response involves a *infinite* number of samples

Frequency domain

- Complex exponential sequences are eigenfunctions of LTIS

- The response to a complex sinusoid $x[n] = e^{j\omega n}, -\infty < n < +\infty$ is a sinusoid with *same* frequency and *amplitude* and *phase* determined by the system (i.e. by the impulse response)

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n}\left(\sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}\right) \tag{38}$$

Let:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k} \tag{39}$$

Then:

$$y[n] = H(e^{j\omega})e^{j\omega n} \tag{40}$$

Frequency domain

$H(e^{j\omega})$ *frequency response* of the system

$$
\begin{aligned}
H(e^{j\omega}) &= H_R(e^{j\omega}) + H_I(e^{j\omega}) && (41)\\
&= |H(e^{j\omega})| e^{\angle H(e^{j\omega})} && (42)
\end{aligned}
$$

$\rightarrow$ If we manage to put the input signal in the form of a sum of complex exponentials then the output can be obtained as the sum of the responses to such *signal components*:

$$
\begin{aligned}
x[n] &= \sum_{k=-\infty}^{+\infty} \alpha_k e^{j\omega_k n} \quad \text{Fourier representation} \rightarrow && (43)\\
y[n] &= \sum_{k=-\infty}^{+\infty} \alpha_k H(e^{j\omega_k}) e^{j\omega_k} \quad \text{output of the LTIS} && (44)
\end{aligned}
$$

Frequency response of a LTIS

$H(e^{j\omega})$ is *always* a periodic function of ω with period 2π

$$H(e^{j(\omega+2r\pi)}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j(\omega+2r\pi)n} = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} = H(e^{j\omega}) \tag{45}$$

Due to this property as well as to the fact that frequencies differing for multiples of 2π are indistinguishable, $H(e^{j\omega})$ only needs to be specified over an interval of length 2π. The inherent periodicity defines the frequency response on the entire frequency axis. It is common use to specify H over the interval $-\pi \leq \omega \leq \pi$. Then, the frequencies around even multiples of π are referred to as *low frequencies*, while those around odd multiples of π are *high frequencies*.