Discrete Time Signals and Systems
Time-frequency Analysis

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Time-frequency Analysis

- Fourier transform (1D and 2D)

- Reference textbook:

  *Discrete time signal processing*, A.W. Oppenheim and R.W. Shafer
  
  - Chapter 1: Introduction
  
  - Chapter 2: Discrete time Signals and Systems
  
  - Chapter 3: Sampling of Continuous Time Signals
  
  - (Chapter 4: The z-Transform)
Signal Classification

- **Continuous time signals** $x(t)$ are functions of a continuous independent variable $t$
  \[ x = x(t), t \in \mathbb{R} \]

- **Discrete time signals** are functions of a discrete variable
  \[ x = \{x[n]\}, n \in \mathbb{Z}, -\infty < n < +\infty \]
  - Are defined at discrete time intervals
  - Are represented as *sequences of numbers*

- **Digital signals** both the independent variable **and** the amplitude are discrete

Digital **systems**: both the input and the output are digital signals
⇒ Digital Signal Processing: processing of signals that are discrete in both time and amplitude
Periodic Sampling

The sequences are obtained by sampling the analog signal at equi-spaced points:

\[ x[n] = x(nT), \quad n \in I \]

\( T \): sampling period, interval between samples

\( f = \frac{1}{T} \): sampling frequency
Basic Sequences

- Products and sums among sequences: element by element operations

- Delayed or shifted sequence: \( y[n] = x[n - k] \)

- Unit sample sequence \( \delta[n] \) (Dirac function, impulse):
  \[
  \delta[n] = \begin{cases} 
  0, & n \neq 0 \\
  1, & n = 0 
  \end{cases}
  \] (1)

- Unit step sequence:
  \[
  u[n] = \begin{cases} 
  0, & n < 0 \\
  1, & n \geq 0 
  \end{cases}
  \] (2)
Basic Sequences

→ relation between the two functions:

\[ u[n] = \sum_{k=0}^{\infty} \delta[n - k] \]  \hspace{1cm} (3)
\[ \delta[n] = u[n] - u[n - 1] \]  \hspace{1cm} (4)

- Sinusoidal sequence:

\[ x[n] = A\cos(\omega_0 n + \Phi), \forall n \]
Basic Sequences

- Complex exponential

\[ x[n] = A e^{(\alpha + j(\omega_0 n + \phi))} \]

- \(|\alpha| < 1\): exponentially decaying envelop
- \(|\alpha| > 1\): exponentially growing envelop
- \(|\alpha| = 1\): complex exponential sequence:
  \[ x[n] = |A| \cos(\omega_0 n + \phi) + j|A| \sin(\omega_0 n + \phi) \]

The real and imaginary parts oscillate sinusoidally with \(n\). The fact that \(n\) is an integer leads to important differences with respect to the corresponding functions in the continuous domain. By analogy, \(\omega_0\) and \(\Phi\) are called frequency and phase.
Graphically

\[
\begin{align*}
\text{delta} & \quad n = 0 \\
\text{unit step} & \quad n = 0, 1, 2, 3 \\
\text{real exponential} & \quad n = -2, -1, 0, 1, 2, 3
\end{align*}
\]
1. Complex exponentials and sinusoids are $2\pi$-periodic

$$x[n] = A \exp^{j(\omega_0 + 2\pi)n} = A \exp^{j\omega_0 n} \exp^{j2\pi n} = A \exp^{j\omega_0 n} \quad (5)$$

$\Rightarrow$ Complex exponentials with frequencies $(\omega_0 + 2\pi r)$ are indistinguishable from one another $\Rightarrow$ only frequencies in an interval of length $2\pi$ need to be considered

2. Complex exponentials in general are not periodic

Periodicity condition: $x[n] = x[n + N], \forall n, N \in I$. For this condition to be true, the following relation must hold true:

$$\omega_0 N = 2\pi k, \; k \in I \quad (6)$$

$\Rightarrow$ Complex exponentials and sinusoidal sequences are not necessarily periodic with period $T_0 = 2\pi / \omega_0$ as in the continuous domain and, depending on the value of $\omega_0$, might not be periodic at all.
Discrete vs. continuous

Hint: relation (??) can be written as:

\[ T_0 = \frac{2\pi}{\omega_0} = \frac{N}{k} \]  

(7)

where \( T_0 \) is the period. If one thinks of the discrete time signal as to the sampled version of a continuous time signal of period \( T_0 \) with unitary sampling interval \( (T_s = 1) \), then relation (??) can be naturally interpreted as follows: in order to obtain a discrete time periodic signal of period \( N \) by sampling a continuous time signal of period \( T_0 \), the period \( T_0 \) must be an integer factor of \( N \), or, viceversa, \( N \) must be a multiple of \( T_0 \). This can be easily generalized to the case \( T_s \neq 1 \) (not necessarily integer) as follows:

\[ \omega_0 N T_s = 2\pi k \quad T_0 = \frac{N}{k} T_s \]  

(8)

namely, \( N = kT_0/T_s \) must be an integer multiple of the ratio between the period and the sampling step.
Examples

1. Example 1

\[ x(t) = A \sin(\omega_0 t) \]
\[ x[n] = A \sin(\omega_0 n) \]
\[ \omega_0 = \frac{3}{4}\pi \]
\[ \rightarrow \frac{3}{4}\pi N = 2\pi k \rightarrow 3N = 8k \rightarrow \]

the first two integers satisfying the condition are \( N = 8, k = 3 \), thus the signal \( x[n] \) obtained by sampling a continuous sinusoid \( x(t) \) with frequency \( \omega_0 = \frac{3}{4}\pi \) is periodic with period \( N = 8 \); each period of \( x[n] \) corresponds to three \( (k = 3) \) periods of the original signal.

2. Example 2

\[ \omega_0 = 1 \rightarrow \text{no solutions!} \]
Summary

\[ \omega_0 \leftrightarrow \omega_0 + 2\pi r, \quad r \in I \] \hspace{1cm} (9)
\[ \omega_0 = \frac{2k\pi}{N}, \quad k, N \in I \text{ condition for periodicity} \] \hspace{1cm} (10)

Let's assume that, given \( N \), \( \omega_0 \) satisfies relation \((9)\) for a certain \( k \). Relation \((9)\) implies that only \( N \) distinguishable frequencies satisfy relation \((9)\).

Proof: let \( \omega_k = \frac{2k\pi}{N} \), \( k = 0, 1, \ldots, N - 1 \)
\[ \omega_d = 0 \quad \text{degenerate case} \]
\[ \omega_1 = \frac{2\pi}{N} \]
\[ \omega_2 = \frac{2 \times 2\pi}{N} \]
\[ \ldots \]
\[ \omega_{N-1} = \frac{(N - 1) \times 2\pi}{N} \]
\[ \omega_N = \frac{N \times 2\pi}{N} = 2\pi \]
\[ \ldots \]
\[ \omega_{N+i} = \frac{(N + i) \times 2\pi}{N} = 2\pi + \frac{2\pi i}{N} \leftrightarrow \omega_i \]
Figure 4.16: DT sinusoids of various frequencies: (a) lowest rate $\Omega = 0$ to (d) highest rate $\Omega = \pi$. 

Examples
Conclusion: for a sinusoidal signal $x[n] = A \sin(\omega_0 n)$, as $\omega_0$ increases from $\omega_0 = 0$ to $\omega_0 = \pi$, $x[n]$ oscillates more and more rapidly. Conversely, from $\omega_0 = \pi$ to $\omega_0 = 2\pi$ the oscillations become slower. In fact, because of the periodicity in $\omega_0$ of complex exponentials and sinusoidal sequences, $\omega_0 = 0$ is indistinguishable from $\omega_0 = 2\pi$ and, more in general, frequencies around $\omega_0 = 2\pi$ are indistinguishable from frequencies around $\omega_0 = 0$. As a consequence, values of $\omega_0$ in the vicinity of $\omega_0 = 2k\pi$ are referred to as low frequencies, whereas frequencies in the vicinity of $\omega_0 = (2k + 1)\pi$ are referred to as high frequencies. This is a fundamental difference from the continuous case, where the speed of the oscillations increases monotonically with the frequency $\omega_0$. 
Operations and Properties

• Periodicity: a sequence is periodic with period $N$ iif:

$$x[n + N] = x[n], \forall n$$

• Energy: $E = \sum_{n=\infty}^{+\infty} |x[n]|^2$

• Sample-wise operations:
  - Product: $x \cdot y = \{x[n]y[n]\}$
  - Sum: $x + y = \{x[n] + y[n]\}$
  - Scaling: $\alpha x = \{\alpha x[n]\}$
  - Delay: $y[n] = x[n - n_0], n, n_0 \in I$
Operations and Properties

Any sequence can be represented as a sum of *scaled* and *delayed* unit samples:

\[
x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]
\]  \hspace{1cm} (11)

\[
\delta[n - k] = \begin{cases} 
1 & \text{for } n = k \\
0 & \text{for } n \neq k 
\end{cases}
\]  \hspace{1cm} (12)

\[\delta[n-3] \]
Discrete Time Fourier Transform

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \]  
\[ x[n] = \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \]  

- \( x[n] \) is a discrete time signal
- The DTFT \( X(e^{j\omega}) \) is a complex continuous function of the independent variable \( \omega \). As such, can be put in the equivalent forms:

\[ X(e^{j\omega}) = X_R(e^{j\omega}) + X_I(e^{j\omega}) \]  
\[ X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})} \]  

\( |X(e^{j\omega})| \) is the magnitude or Fourier spectrum of the signal and \( e^{j\angle X(e^{j\omega})} \) is the phase spectrum of the transformed signal.
- \( x[n] \) and \( X(e^{j\omega}) \) form the Fourier representation for the sequence.
- Eq. (13): analysis formula. It projects the signal to the frequency domain.
- Eq. (14): synthesis or reconstruction formula. It is used to recover the signal from its frequency domain representation.
The DTFT represents the sequence $x[n]$ as a linear superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi}X(e^{j\omega n})d\omega$$

with $-\pi \leq \omega \leq \pi$ and $X(e^{j\omega n})$ representing the relative amount of each complex sinusoidal component.

By comparing (17) with relation (??) it is easy to realize that the frequency response of a LTIS is the Fourier transform of its impulse response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \leftrightarrow h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}$$
Symmetry properties of the DTFT: see Table 2.1, page 53 of the reference textbook.

The DTFT is a periodic function of $\omega$ with period $2\pi$.

\[
X(e^{j(\omega + 2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega + 2\pi)n}
\]

\[
= e^{-j2\pi n} \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j\omega})
\]  

(18)

More in general:

\[
X(e^{j(\omega + 2r\pi)}) = X(e^{j\omega}), \forall r \in I
\]

(20)

The Fourier spectrum of a discrete (sampled) signal is periodic.
Euler's formula

\[ e^{jx} = \cos x + j \sin x \rightarrow e^{-j2\pi n} = e^{-j\omega n} e^{-j2\pi n} = e^{-j\omega n}, \forall n \in \mathbb{Z} \]

Reminder:

\[
\sin x = \frac{e^{jx} - e^{-jx}}{2j} \quad \cos x = \frac{e^{jx} + e^{-jx}}{2}
\]
The `rect()` function is important because it represents the impulse response of the ideal band-pass filter in the frequency domain. Its Fourier transform is a `sinc()` function which spreads over \((\infty, +\infty)\).
Ideal low pass filter

\[ H(e^{j\omega n}) = H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases} \]  \hspace{1cm} (22)

\[ h[n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(e^{i\omega})e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \]

\[ = \frac{1}{2\pi} \frac{1}{jn} \left[ e^{j\omega_c n} - e^{-j\omega_c n} \right] = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]

\[ h[n] = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \]  \hspace{1cm} (23)

\( \omega_c \): cutting frequency of the filter \( \leftrightarrow \) bandwidth

Let \( \omega_s \) be the sampling frequency. If we let \( \omega_c = \frac{\omega_s}{2} \) then

\[ h[n] = \frac{1}{T_s} \text{sinc} \left( \frac{n\pi}{T_s} \right) \]  \hspace{1cm} (24)
## To keep in mind

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta[n] )</td>
<td>1</td>
</tr>
<tr>
<td>( \delta[n - n_0] )</td>
<td>( e^{-j\omega n_0} )</td>
</tr>
<tr>
<td>1</td>
<td>( \sum_k 2\pi \delta(\omega + 2k\pi) )</td>
</tr>
<tr>
<td>( \frac{\sin(\omega_c n)}{n} )</td>
<td>( X(\omega) = \begin{cases} 1 &amp;</td>
</tr>
<tr>
<td>( x[n] = \text{rect}_T[n] )</td>
<td>( 2T \text{sinc}(\omega T) )</td>
</tr>
<tr>
<td>( e^{j\omega_0 n} )</td>
<td>( \sum_k 2\pi \delta(\omega - \omega_0 + 2k\pi) )</td>
</tr>
<tr>
<td>( \cos(\omega_0 n + \Phi) )</td>
<td>( \pi \sum_k [e^{j\Phi} \delta(\omega - \omega_0 + 2k\pi) + e^{-j\Phi} \delta(\omega + \omega_0 + 2k\pi)] )</td>
</tr>
</tbody>
</table>

Table 2.3, page 61, reference textbook
Discrete Time Systems

A DTS is an operator that maps a discrete sequence $x[n]$ at its input to a discrete sequence $y[n]$ at its output:

$$y[n] = T\{x[n]\}$$

- Linearity:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}, \text{ additivity} \quad (25)$$

$$T\{ax[n]\} = aT\{x[n]\}, \text{ scaling or homogeneity} \quad (26)$$

⇒ Principle of superposition:

$$T\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 T\{x_1[n]\} + a_2 T\{x_2[n]\} \quad (27)$$
Discrete Time Systems

- Time-invariance: a delayed input sequence maps to a delayed output sequence:

\[ x[n] \rightarrow y[n] \quad \text{(28)} \]
\[ x[n - n_0] \rightarrow y[n - n_0] \quad \text{(29)} \]

- Causality: the output value \( y[n] \) for \( n = n_0 \) only depends on previous input samples \( x[n] : n < n_0 \)
  
  In images causality doesn’t matter!

- Stability: a bounded input generates a bounded output
Linear Time Invariant Systems

Linear systems:

\[ y[n] = T \left\{ \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right\} \] 

(30)

Principle of superposition:

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] T \{ \delta[n-k] \} = \] 

(31)

\[ = \sum_{k=-\infty}^{\infty} x[k] h_k[n] \] 

(32)

Time invariance:

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \leftrightarrow \text{Convolution sum} \] 

(33)

→ The system is completely characterized by the impulse response \( h[k] \)

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Discrete Time Signals and Systems

Time-frequency Analysis
Convolution operator

\[ x[n] \ast y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot y[n-k] \] (34)

\[ = \sum_{k=-\infty}^{+\infty} x[n-k] \cdot y[k] \] (35)

Recipe for the convolution (refer to (??)):

1. Reflect \( y[k] \) about the origin to get \( y[-k] \)
2. Shift the reflected sequence of \( n \) steps
3. Multiply (point wise) the resulting sequence by \( x[n] \)
4. Sum over the samples of the resulting signal to get the value of the convolution at position \( n \)
5. Increment \( n \) and go back to point 2.
Properties of LTIS

- **Commutativity** (see ??)
  
  A LTIS with input \( x[n] \) and impulse response \( h[n] \) has the same output of a system with input \( h[n] \) and impulse response \( x[n] \)
  
  → The *cascade* of LTIS systems has an impulse response that is the convolution of the IRs of the individual systems

- **Distributivity:**

\[
x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]
\]

→ The *parallel connection* of LTIS systems has an impulse response that is the sum of the IRs of the single systems
Properties of LTIS

Cascade connection

Parallel connection
The transfer function is a $N$-th order linear constant coefficient difference equation:

$$
\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \rightarrow (36)
$$

$$
y[n] = \frac{1}{a_0} \left\{ \sum_{k=1}^{N-1} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \right\} \rightarrow (37)
$$

$\rightarrow$ The output value $y[n]$ is the linear combination of the $N-1$ last values of the output and the $M$ last values of the input $x[n]$.

System classification:

- Finite Impulse Response (FIR): the impulse response involves a *finite* number of samples

- Infinite Impulse Response (IIR): the impulse response involves a *infinite* number of samples
Frequency domain

- Complex exponential sequences are eigenfunctions of LTIS

- The response to a complex sinusoid $x[n] = e^{j\omega n}, -\infty < n < +\infty$ is a sinusoid with same frequency and amplitude and phase determined by the system (i.e. by the impulse response)

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n} \left( \sum_{k=-\infty}^{+\infty} h[k] e^{-j\omega k} \right)$$  \hspace{1cm} (38)

Let:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} h[k] e^{-j\omega k}$$  \hspace{1cm} (39)

Then:

$$y[n] = H(e^{j\omega}) e^{j\omega n}$$  \hspace{1cm} (40)
Frequency domain

$H(e^{j\omega})$ frequency response of the system

$$H(e^{j\omega}) = H_R(e^{j\omega}) + H_I(e^{j\omega})$$  \hspace{1cm} (41)

$$= |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$  \hspace{1cm} (42)

→ If we manage to put the input signal in the form of a sum of complex exponentials then the output can be obtained as the sum of the responses to such signal components:

$$x[n] = \sum_{k=-\infty}^{+\infty} \alpha_k e^{j\omega_k n} \hspace{1cm} \text{Fourier representation} \rightarrow$$  \hspace{1cm} (43)

$$y[n] = \sum_{k=-\infty}^{+\infty} \alpha_k H(e^{j\omega_k}) e^{j\omega_k} \hspace{1cm} \text{output of the LTIS}$$  \hspace{1cm} (44)
Frequency response of a LTIS

\(H(e^{j\omega})\) is always a periodic function of \(\omega\) with period \(2\pi\)

\[
H(e^{j(\omega+2r\pi)}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j(\omega+2r\pi)n} = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} = H(e^{j\omega})
\]

Due to this property as well as to the fact that frequencies differing for multiples of \(2\pi\) are indistinguishable, \(H(e^{j\omega})\) only needs to be specified over an interval of length \(2\pi\). The inherent periodicity defines the frequency response on the entire frequency axis. It is common use to specify \(H\) over the interval \(-\pi \leq \omega \leq \pi\). Then, the frequencies around even multiples of \(\pi\) are referred to as low frequencies, while those around odd multiples of \(\pi\) are high frequencies.