

11.6 Exercises - Part 6

(Published on January 3, **solutions to be submitted January 19, 2017.**)

Exercise 21. Let Λ be a finite-dimensional algebra over an algebraically closed field. Let M and N be Λ -modules and $f: M \rightarrow N$ be a minimal right almost split map.

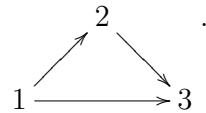
- Show that N is indecomposable.
- Show that f is not a split monomorphism.
- Suppose there exists a module L and morphisms $f_1: M \rightarrow L$ and $f_2: L \rightarrow N$ such that $f = f_2 f_1$ and f_2 is not a split epimorphism. Show that f_1 is a split monomorphism.
- Conclude that f is an irreducible morphism.

Exercise 22. Using the knitting algorithm on dimension vectors, compute the following.

- The AR quiver of the path algebra of the quiver $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 \\ & & & & & & & & \downarrow \\ & & & & & & & & 6 \end{array}$$

- The AR quiver of the first 4 inverse AR translates of the indecomposable projective modules for the path algebra of the quiver



Exercise 23. Let $K_0(\Lambda)$ be the Grothendieck group of Λ , i.e. the free abelian group on isomorphism classes $[M]$ of modules $M \in \Lambda\text{-mod}$ subject to the relations $[L] = [M] + [N]$ for each short exact sequence $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$ in $\Lambda\text{-mod}$.

- Show the set $\{[S_1], \dots, [S_n]\}$, where the S_i are the simple left Λ -modules, generates $K_0(\Lambda)$. (Hint: For any $M \in \Lambda\text{-mod}$ consider a composition series and use the additivity of the dimension vector on short exact sequences.)
- Show that the set $\{[S_1], \dots, [S_n]\}$ is \mathbb{Z} -linearly independent in $K_0(\Lambda)$ and deduce that $\underline{\dim}: K_0(\Lambda) \rightarrow \mathbb{Z}^n$ defines an isomorphism of abelian groups.

Exercise 24. Let Q be a quiver without oriented cycles and $\chi: \mathbb{Z}^n \rightarrow \mathbb{Z}$ be the quadratic form given by $\chi(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$. Let $M \in \Lambda\text{-mod}$ have dimension vector $\underline{\dim} M = \mathbf{d}$. Show that

$$\chi(\mathbf{d}) = \dim \text{Hom}_\Lambda(M, M) - \dim \text{Ext}_\Lambda^1(M, M).$$

(You may assume that any such M has a projective resolution of the form

$$0 \rightarrow \bigoplus_{\alpha \in Q_1} P(t(\alpha))^{d_{s(\alpha)}} \rightarrow \bigoplus_{i \in Q_0} P(i)^{d_i} \rightarrow M \rightarrow 0.$$

Hints: Use the long exact Hom-Ext sequence; recall $\dim \text{Hom}_\Lambda(P(i), M) = d_i$.)