11.6 Exercises - Part 6

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Exercise 21. Let Λ be a finite-dimensional algebra over an algebraically closed field. Let M and N be Λ-modules and $f\colon M \to N$ be a minimal right almost split map.

(a) Show that N is indecomposable.

(b) Show that f is not a split monomorphism.

(c) Suppose there exists a module L and morphisms $f_1\colon M \to L$ and $f_2\colon L \to N$ such that $f = f_2 f_1$ and f_2 is not a split epimorphism. Show that f_1 is a split monomorphism.

(d) Conclude that f is an irreducible morphism.

Exercise 22. Using the knitting algorithm on dimension vectors, compute the following.

(a) The AR quiver of the path algebra of the quiver
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \\ & & 6 & & \end{tikzcd}.$$

(b) The AR quiver of the first 4 inverse AR translates of the indecomposable projective modules for the path algebra of the quiver
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}.$$

Exercise 23. Let $K_0(\Lambda)$ be the Grothendieck group of Λ, i.e. the free abelian group on isomorphism classes $[M]$ of modules $M \in \Lambda$-mod subject to the relations $[L] = [M] + [N]$ for each short exact sequence $0 \to M \to L \to N \to 0$ in Λ-mod.

(a) Show the set $\{[S_1], \dots, [S_n]\}$, where the S_i are the simple left Λ-modules, generates $K_0(\Lambda)$. (Hint: For any $M \in \Lambda$-mod consider a composition series and use the additivity of the dimension vector on short exact sequences.)

(b) Show that the set $\{[S_1], \dots, [S_n]\}$ is $\mathbb{Z}$-linearly independent in $K_0(\Lambda)$ and deduce that $\underline{\dim}\colon K_0(A) \to \mathbb{Z}^n$ defines an isomorphism of abelian groups.

Exercise 24. Let Q be a quiver without oriented cycles and $\chi\colon \mathbb{Z}^n \to \mathbb{Z}$ be the quadratic form given by $\chi(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$. Let $M \in \Lambda$-mod have dimension vector $\underline{\dim}\, M = \mathbf{d}$. Show that

$$\chi(\mathbf{d}) = \dim \operatorname{Hom}_\Lambda(M, M) - \dim \operatorname{Ext}^1_\Lambda(M, M).$$

(You may assume that any such M has a projective resolution of the form

$$0 \to \bigoplus_{\alpha \in Q_1} P(t(\alpha))^{(d_{s(\alpha)})} \to \bigoplus_{i \in Q_0} P(i)^{(d_i)} \to M \to 0.$$

Hints: Use the long exact Hom-Ext sequence; recall $\dim \operatorname{Hom}_\Lambda(P(i), M) = d_i$.)