Introduction to Wavelets

The Fourier kingdom

- CTFT
  - Continuous time signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} dt$$

  - The amplitude ˆ*f ()* of each sinusoidal wave e*it* is equal to its correlation with *f* , also called Fourier transform
  - If *f (t)* is uniformly regular, then its Fourier transform coefficients also have a fast decay when the frequency increases, so it can be easily approximated with few low-frequency Fourier coefficients.

The Fourier kingdom

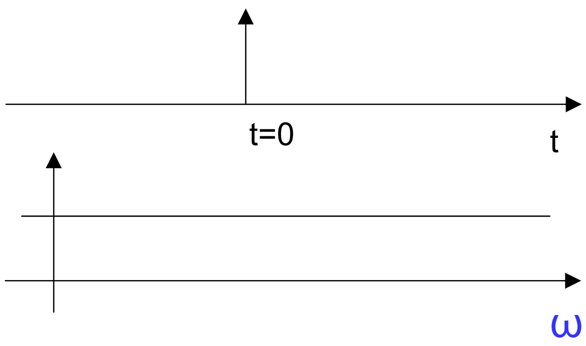
- DTFT
  - Over discrete signals, the Fourier transform is a decomposition in a discrete orthogonal Fourier basis $\{e^{i2kn/N}\}_{0\leq k<N}$ of C^N , which has properties similar to a Fourier transform on functions.
  - Its embedded structure leads to fast Fourier transform(FFT) algorithms,which compute discrete Fourier coefficients with$O(N \log N)$ instead of N^2. This FFT algorithm is a cornerstone of discrete signal processing.
- The Fourier transform is unsuitable for representing transient phenomena
  - the support of $e^{\omega it}$ covers the whole real line, so $\hat{f}(\omega)$ depends on the values $f(t)$ for all times $t \in R$. This global "mix" of information makes it difficult to analyze or represent any local property of $f(t)$ from $\hat{f}()$.
    - As long as we are satisfied with linear time-invariant operators or uniformly regular signals, the Fourier transform provides simple answers to most questions. Its richness makes it suitable for a wide range of applications such as signal transmissions or stationary signal processing. However, to represent a transient phenomenon—a word pronounced at a particular time, an apple located in the left corner of an image—the Fourier transform becomes a cumbersome tool that requires many coefficients to represent a localized event.

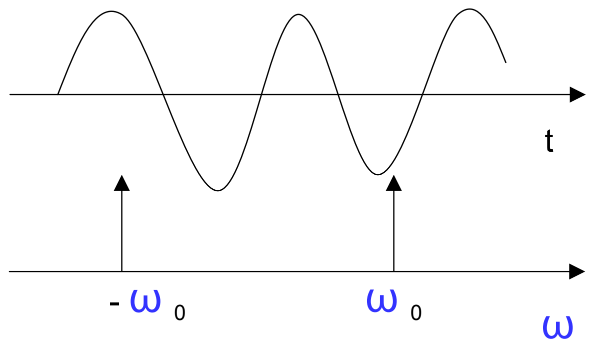
The Fourier kingdom

- The F-transform is not suitable for representing transient phenomena
  - Intuition

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

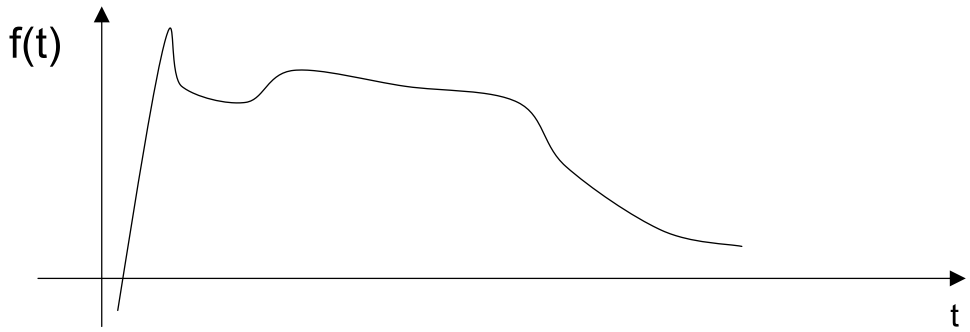
  - F(ω) depends on the values taken by f(t) on the entire temporal axis, which is not suitable for analyzing local properties
  - Need of a transformation which is well localized in *time and frequency*





The Fourier kingdom

- Transient phenomena

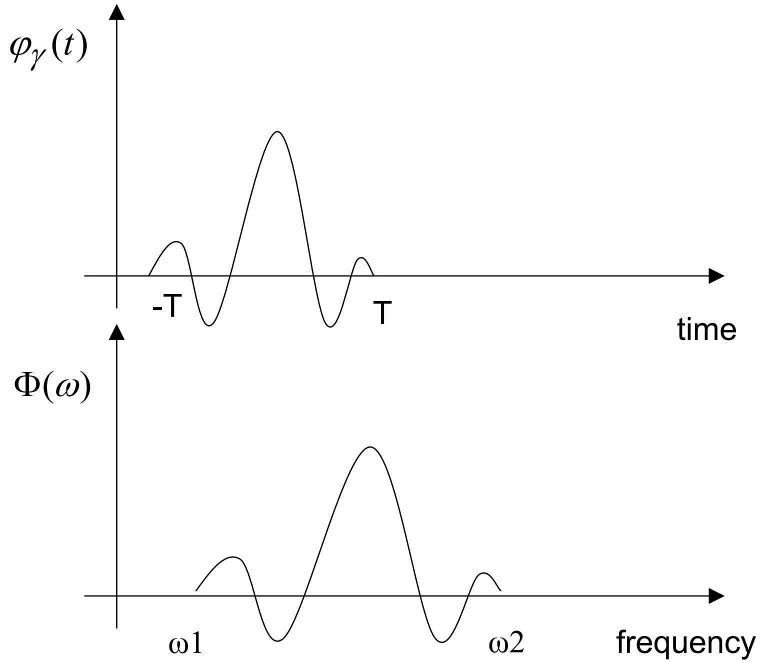


The two transients present in the signal contribute **differently** to the spectrum. The F-transform does not allow to characterize them **separately** to get a local description of the frequency content of the signal.

The basis functions of the FT are complex sinusoids, thus **F(ω) is a measure of the correlation of the signal f(t) with the complex exponential at frequency ω**, which spreads over the whole frequency axis.

Time-frequency localization

- Time-frequency atoms: basis functions that are well localized in *both* time and frequency



$\langle f(t), \varphi_\gamma(t) \rangle$ only depends on the values of f in the neighborhood of T

$F(\omega)\Phi(\omega)$ only depends on the values of F in the neighborhood of ω

Discrete Wavelet Transform

- A wavelet is a function of zero average centered in the neighborhood of t=0 and is normalized

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0$$

$$\|\psi\| = 1$$

- The translations and dilations of the wavelet generate a family of functions over which the signal is projected

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

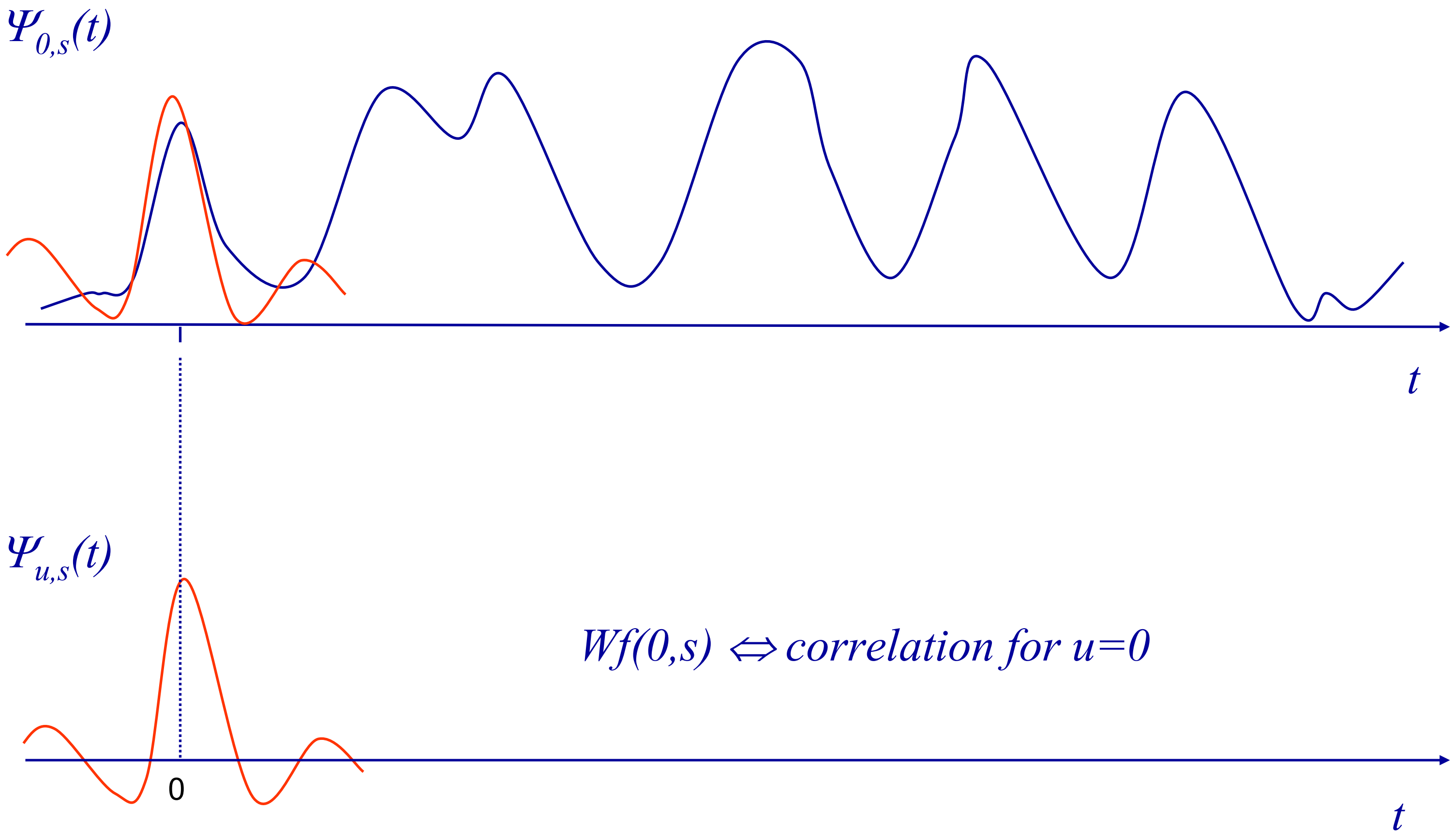
- Wavelet transform of f in $L^2(R)$ at position u and scale s is

$$Wf(u,s) = \left\langle f, \psi_{u,s} \right\rangle = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt$$

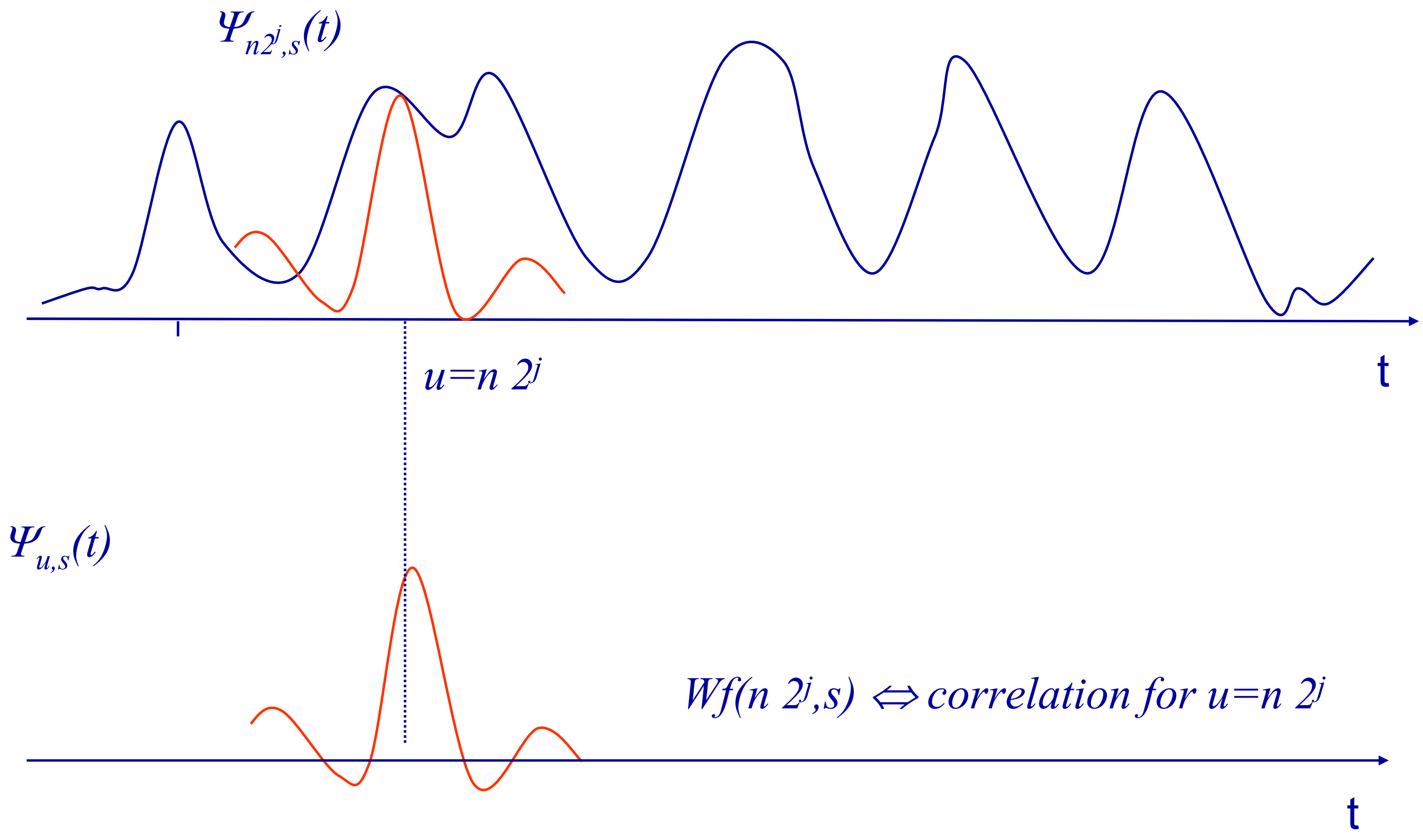
$$s = 2^j$$

$$u = k \cdot 2^j$$

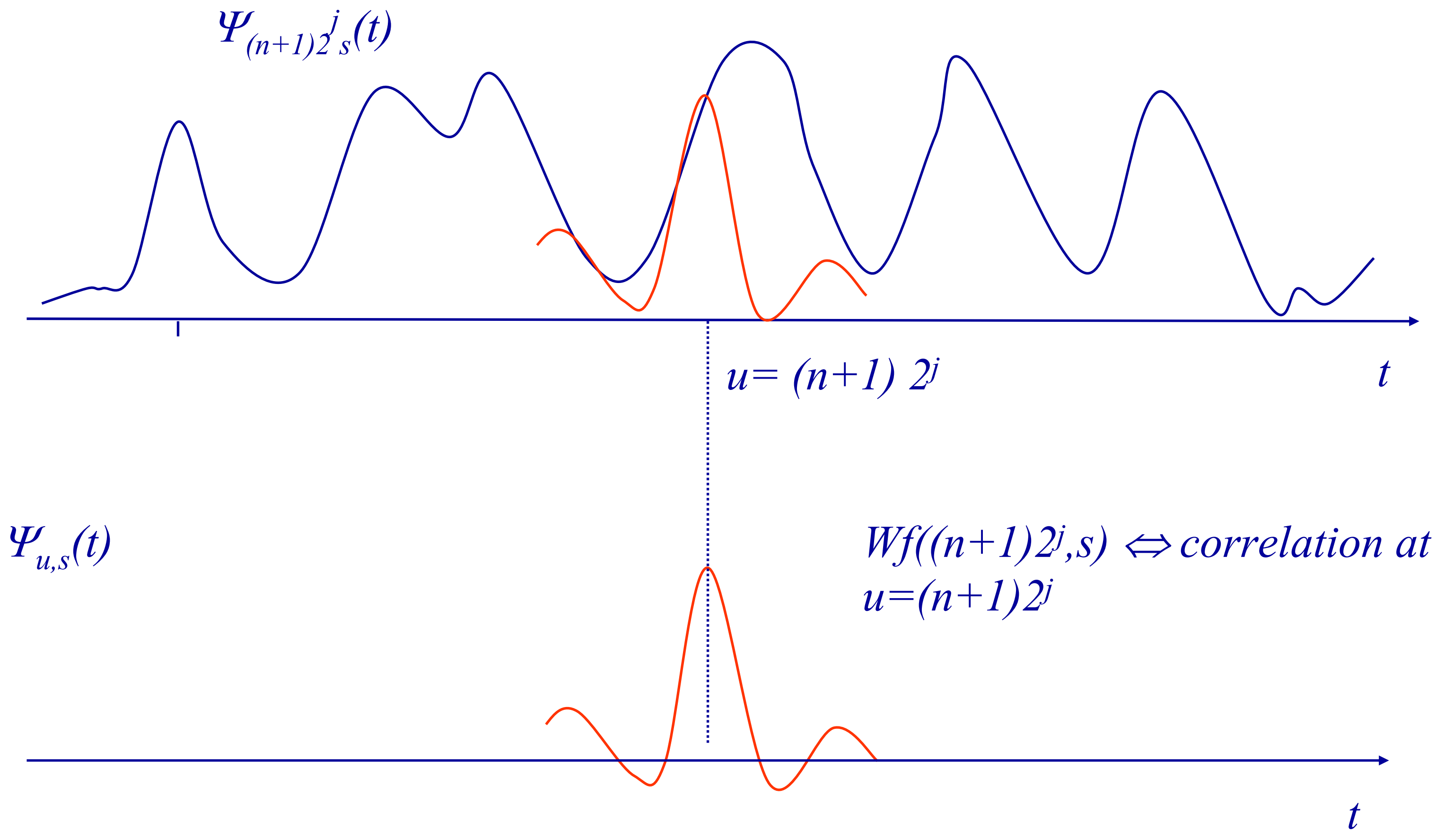
Wavelet transform



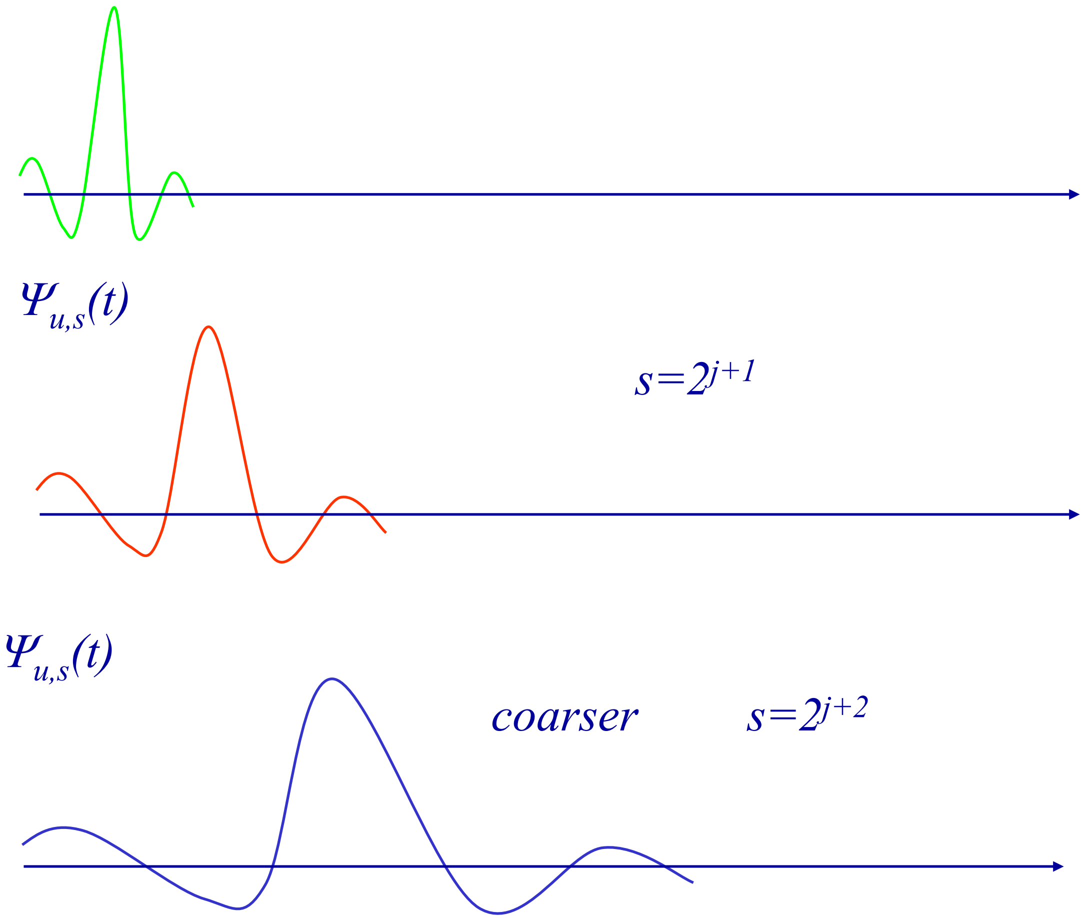
Wavelet transform



Wavelet transform



Changing the *scale*

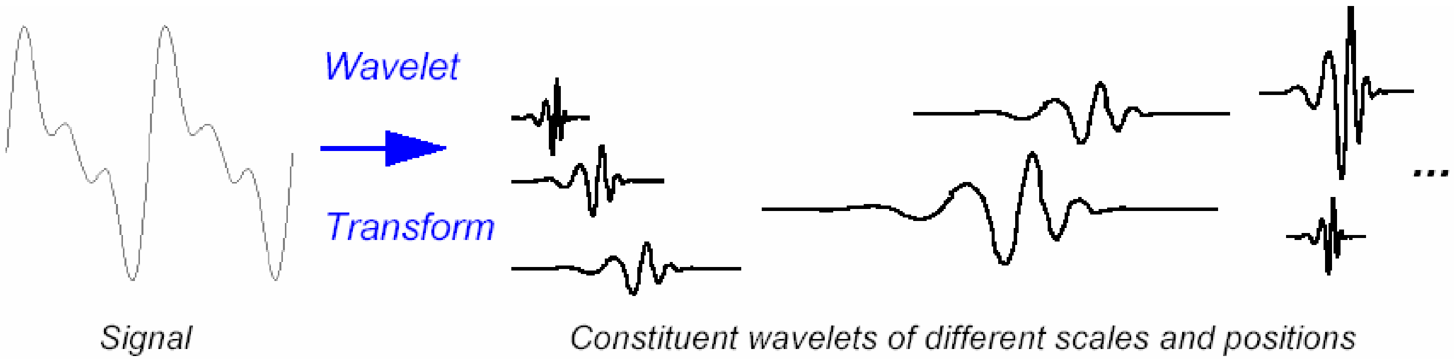


multiresolution

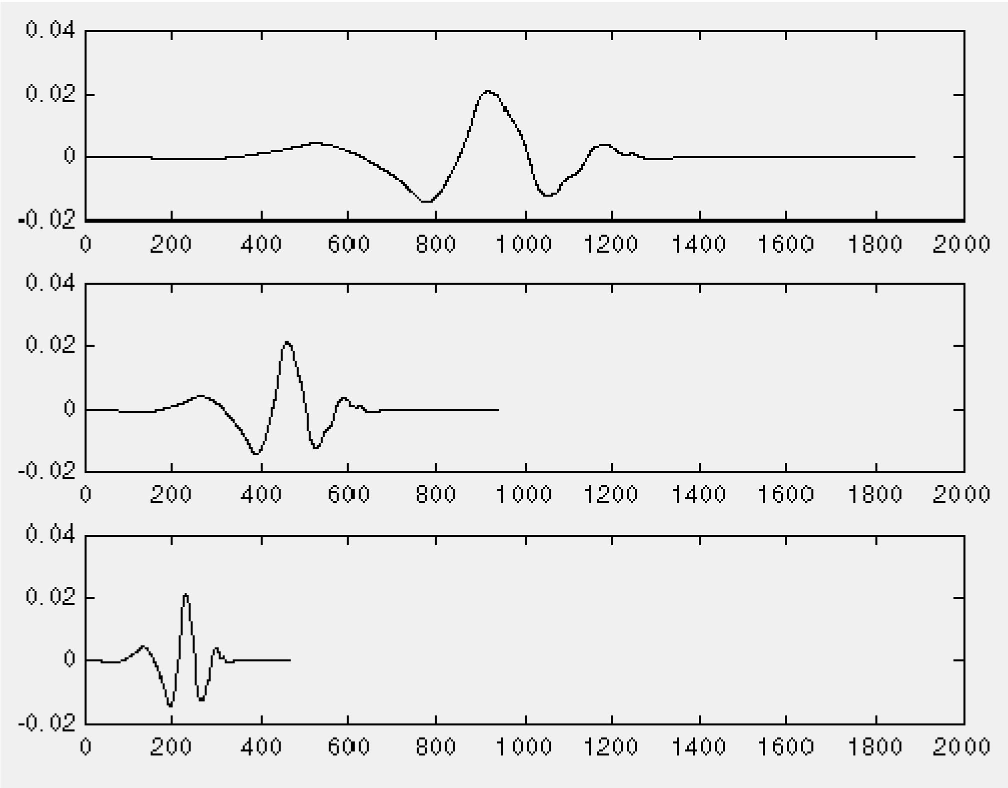
Fourier versus Wavelets

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$C(scale, position) = \int_{-\infty}^{\infty} f(t) \psi(scale, position, t) dt$$



Scaling

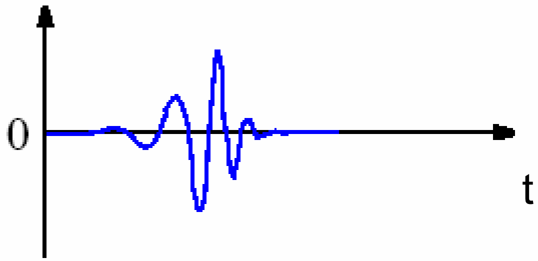


$$f(t) = \psi(t) \quad ; \quad a = 1$$

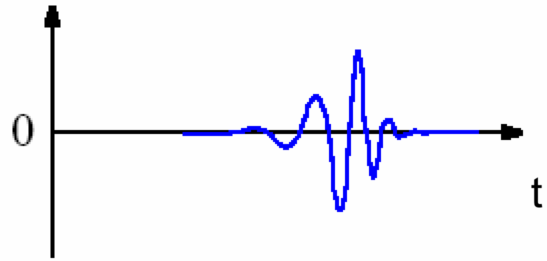
$$f(t) = \psi(2t) \quad ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \quad ; \quad a = \frac{1}{4}$$

Shifting



Wavelet function
$\psi(t)$



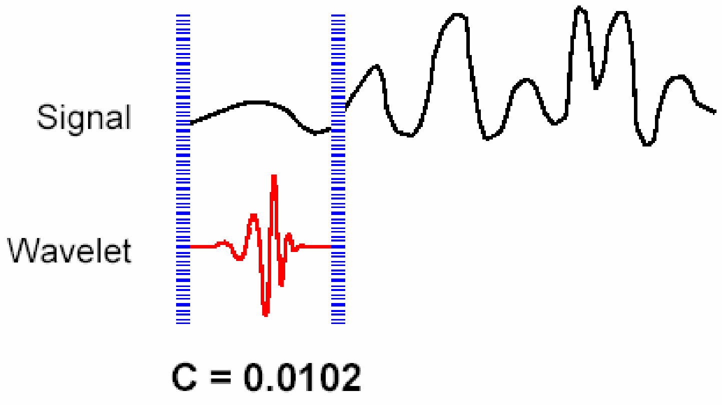
Shifted wavelet function
$\psi(t-k)$

Recipe

1 Take a wavelet and compare it to a section at the start of the original signal.

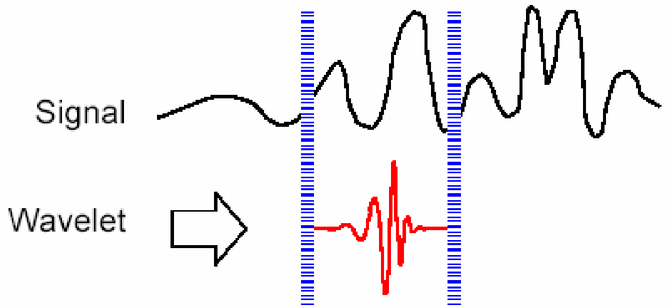
2 Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one, C may be interpreted as a correlation coefficient.

Note that the results will depend on the shape of the wavelet you choose.

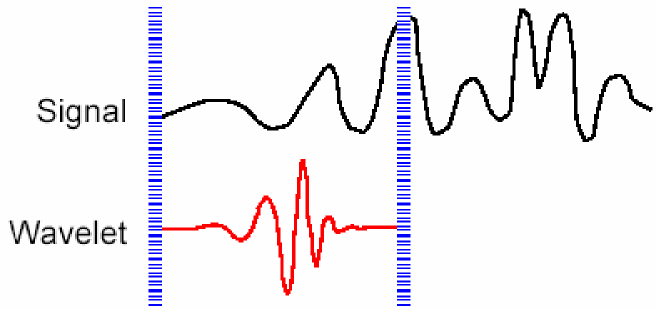


Recipe

3 Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



4 Scale (stretch) the wavelet and repeat steps 1 through 3.



C = 0.2247

5 Repeat steps 1 through 4 for all scales.

Wavelet Zoom

- WT at position u and scale s measures the local correlation between the signal and the wavelet

Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:

(small) • Low scale $a \Rightarrow$ Compressed wavelet $\Rightarrow$ Rapidly changing details $\Rightarrow$ High frequency ω .

(large) • High scale $a \Rightarrow$ Stretched wavelet $\Rightarrow$ Slowly changing, coarse features $\Rightarrow$ Low frequency ω .

Frequency domain

- Parseval $$Wf(u,s) = \int_{-\infty}^{+\infty} f(t)\psi^*{}_{u,s}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\Psi^*{}_{u,s}(\omega)d\omega$$

  The wavelet coefficients Wf(u,s) depend on the values of f(t) (and F(ω)) in the time-frequency region where the energy of the corresponding wavelet function (respectively, its transform) is concentrated

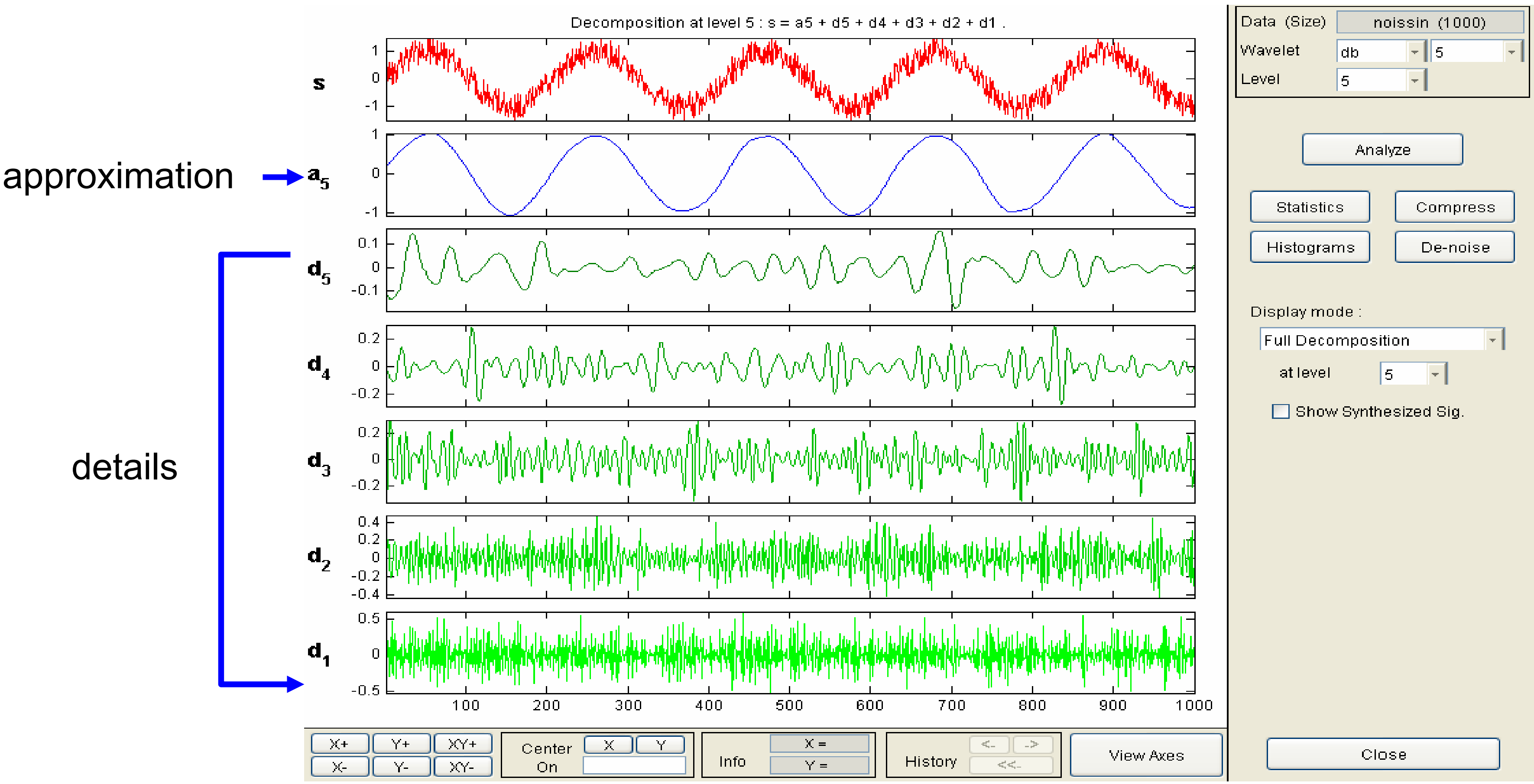
  - *time/frequency localization*
  - The *position and scale* of high amplitude coefficients allow to characterize the *temporal evolution* of the signal

  - Time domain signals (1D) : Temporal evolution
  - Spatial domain signals (2D) : Localize and characterize spatial singularities

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \Leftrightarrow \Psi_{u,s}(\omega) = \sqrt{s}\,\Psi(s\omega)e^{-j\omega s}$$

Stratching in time ↔ Shrinking in frequency (and viceversa)

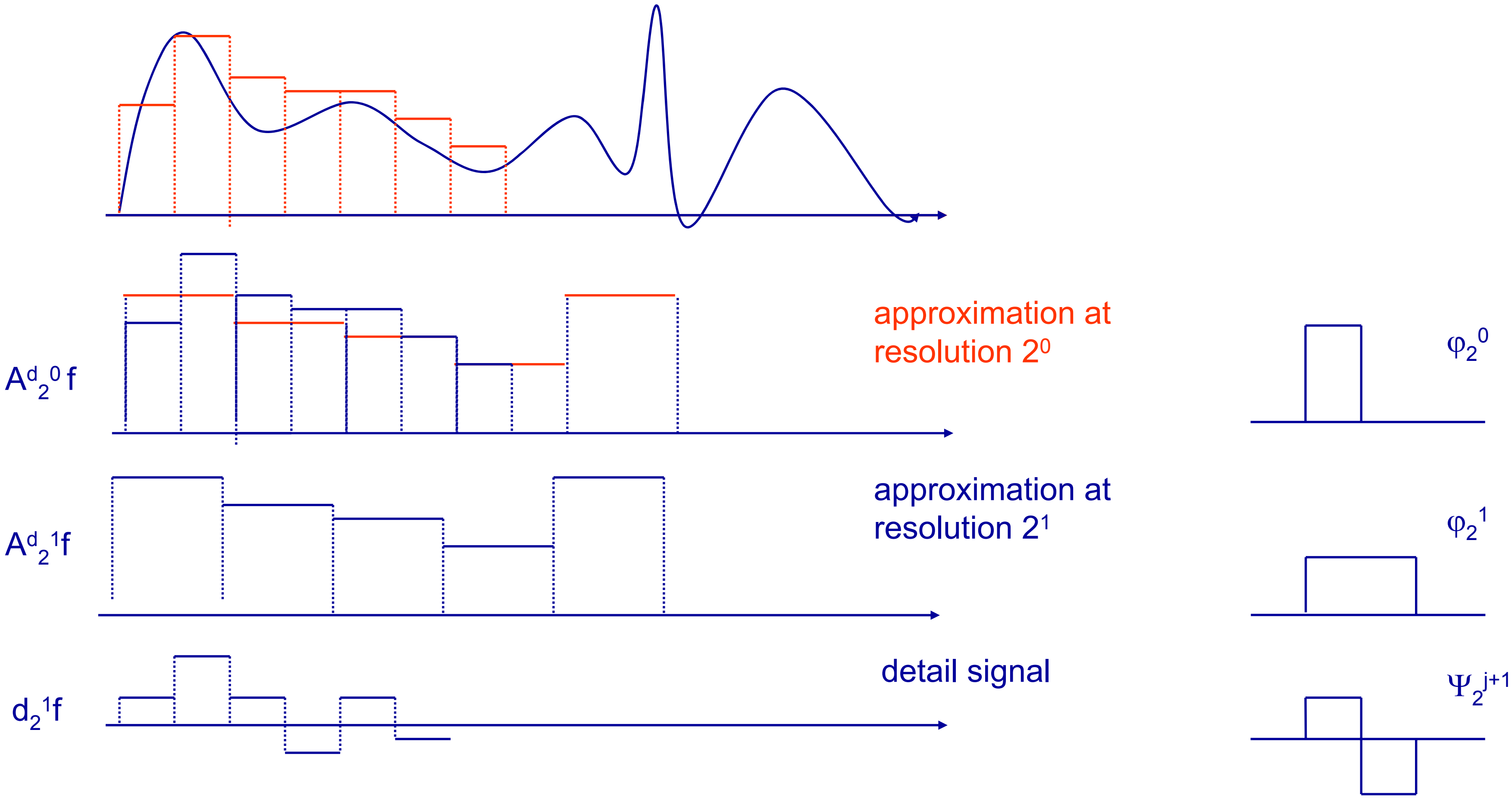
Example



Wavelet representation = approximation + details

approximation ↔ scaling function
details ↔ wavelets

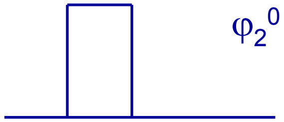
A different perspective



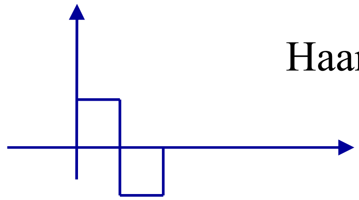
$A^d_{2^j} f = A^d_{2^{j+1}} f + d_{2^{j+1}} f$

Haar pyramid [Haar 1910]

Haar basis function



φ_2^0



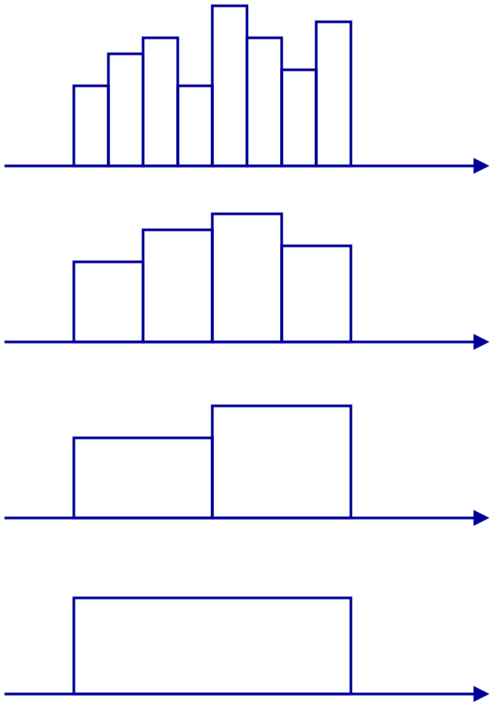
Haar wavelet

sig_0

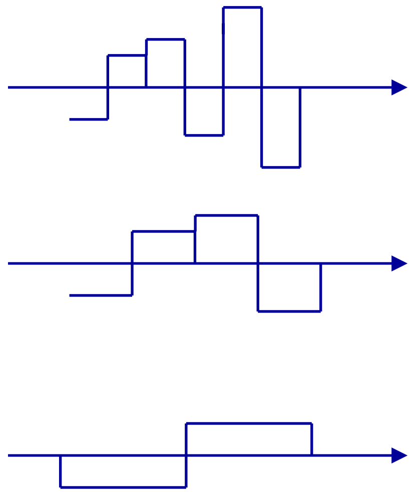
sig_1

sig_2

sig_3

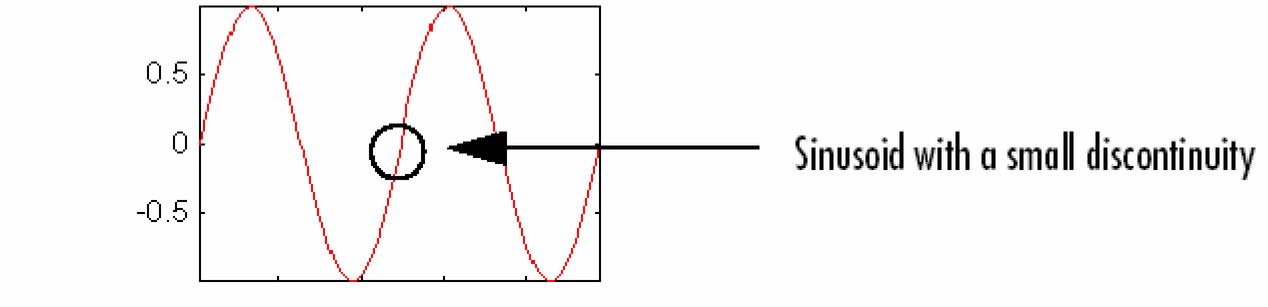


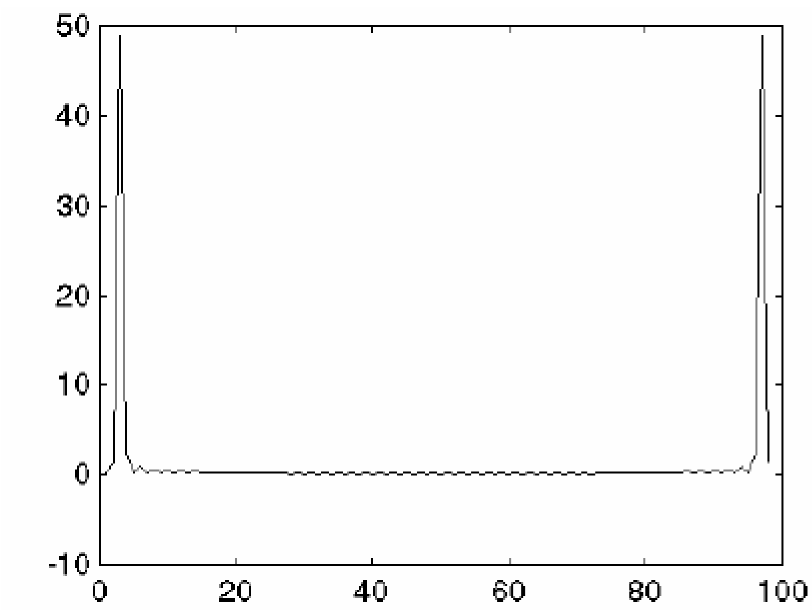
details



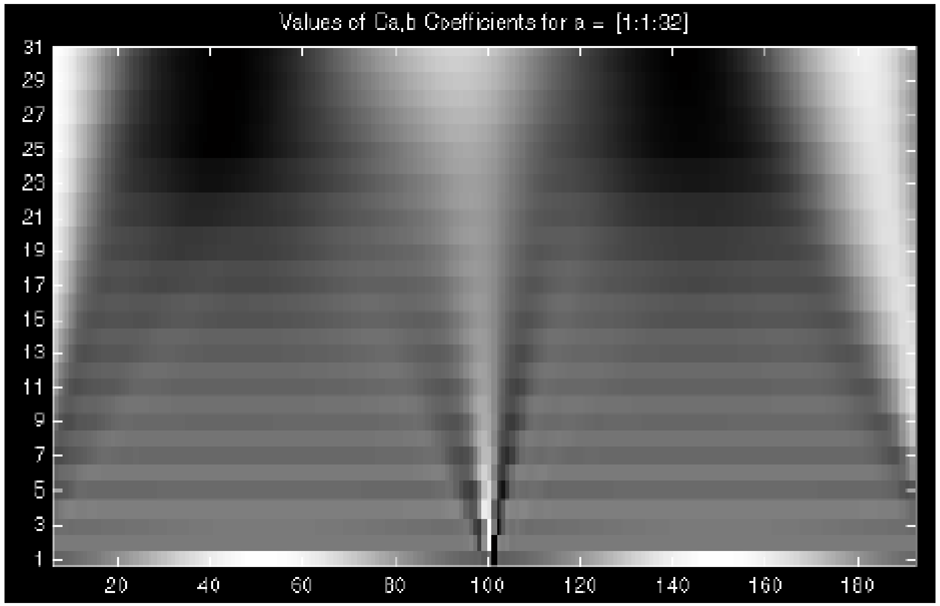
signal=approximation at scale n + details at scales 1 to n

What wavelets can do?



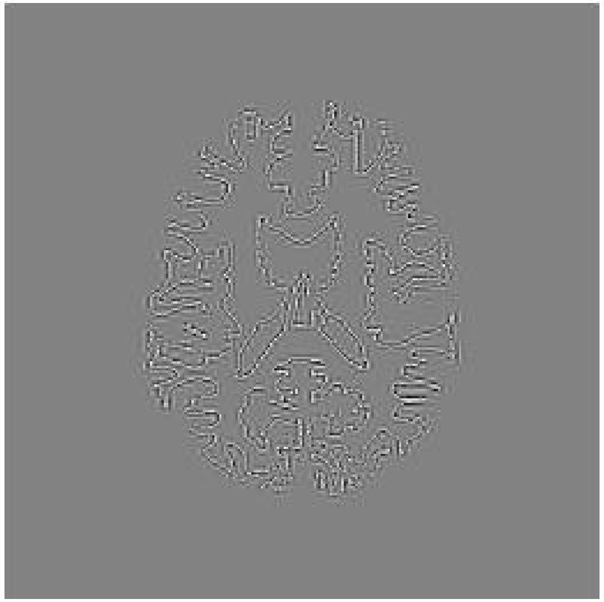


Fourier Coefficients



Wavelet Coefficients

Multiscale edge detection



Real wavelets: example

- The wavelet transform was calculated using a Mexican hat wavelet

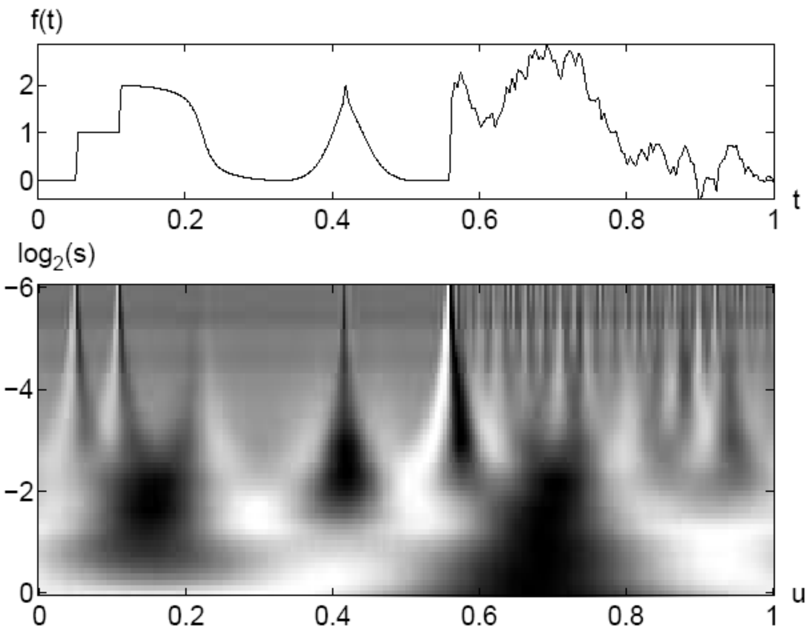


Fig. 4.7. A Wavelet Tour of Signal Processing, 3rd ed. Real wavelet transform $Wf(u,s)$ computed with a Mexican hat wavelet The vertical axis represents $\log_2 s$. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients.

Wavelets and linear filtering

- The WT can be rewritten as a convolution product and thus the transform can be interpreted as a linear filtering operation

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt = f * \bar{\psi}_s(u)$$

$$\bar{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi^* \left(\frac{-t}{s} \right)$$

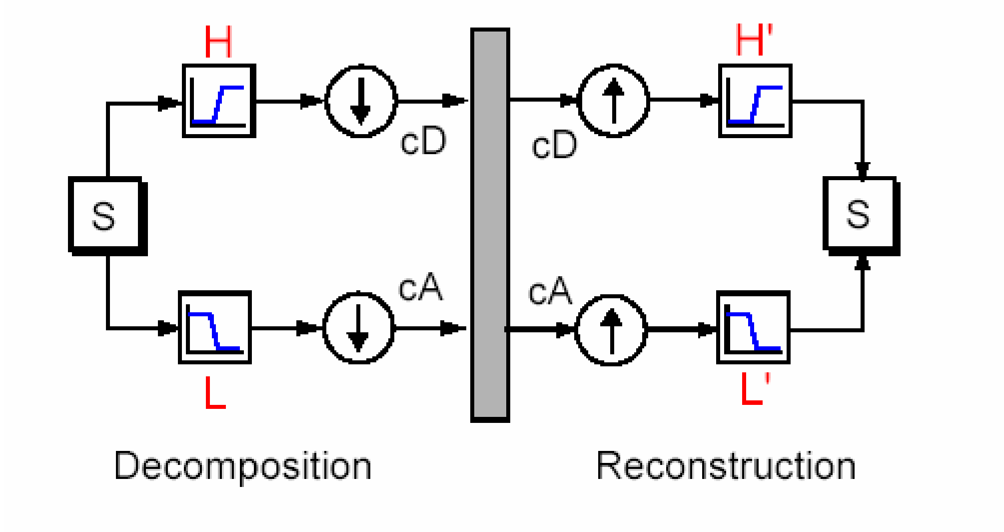
$$\hat{\bar{\psi}}_s(\omega) = \sqrt{s} \hat{\psi}^*(s\omega)$$

$$\hat{\psi}(0) = 0$$

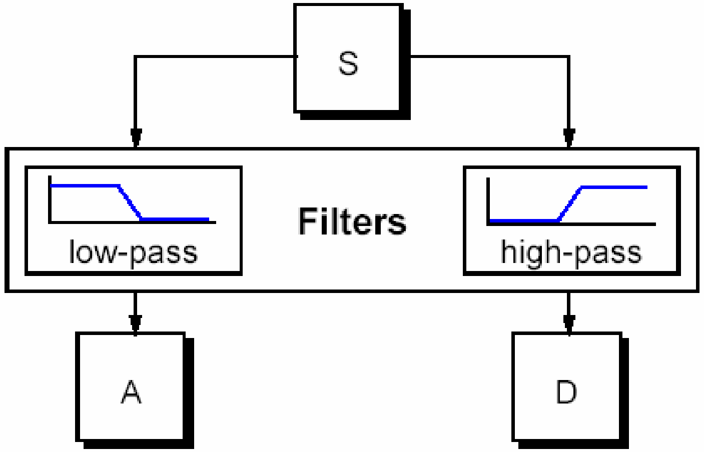
$\rightarrow$ band-pass filter

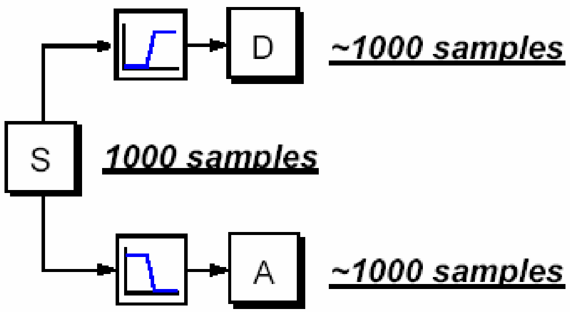
Wavelets & filterbanks

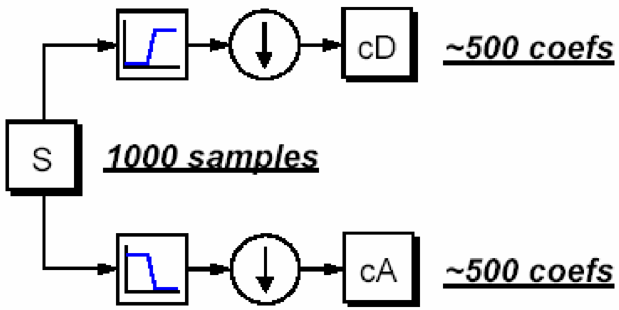
Quadrature Mirror Filter (QMF)



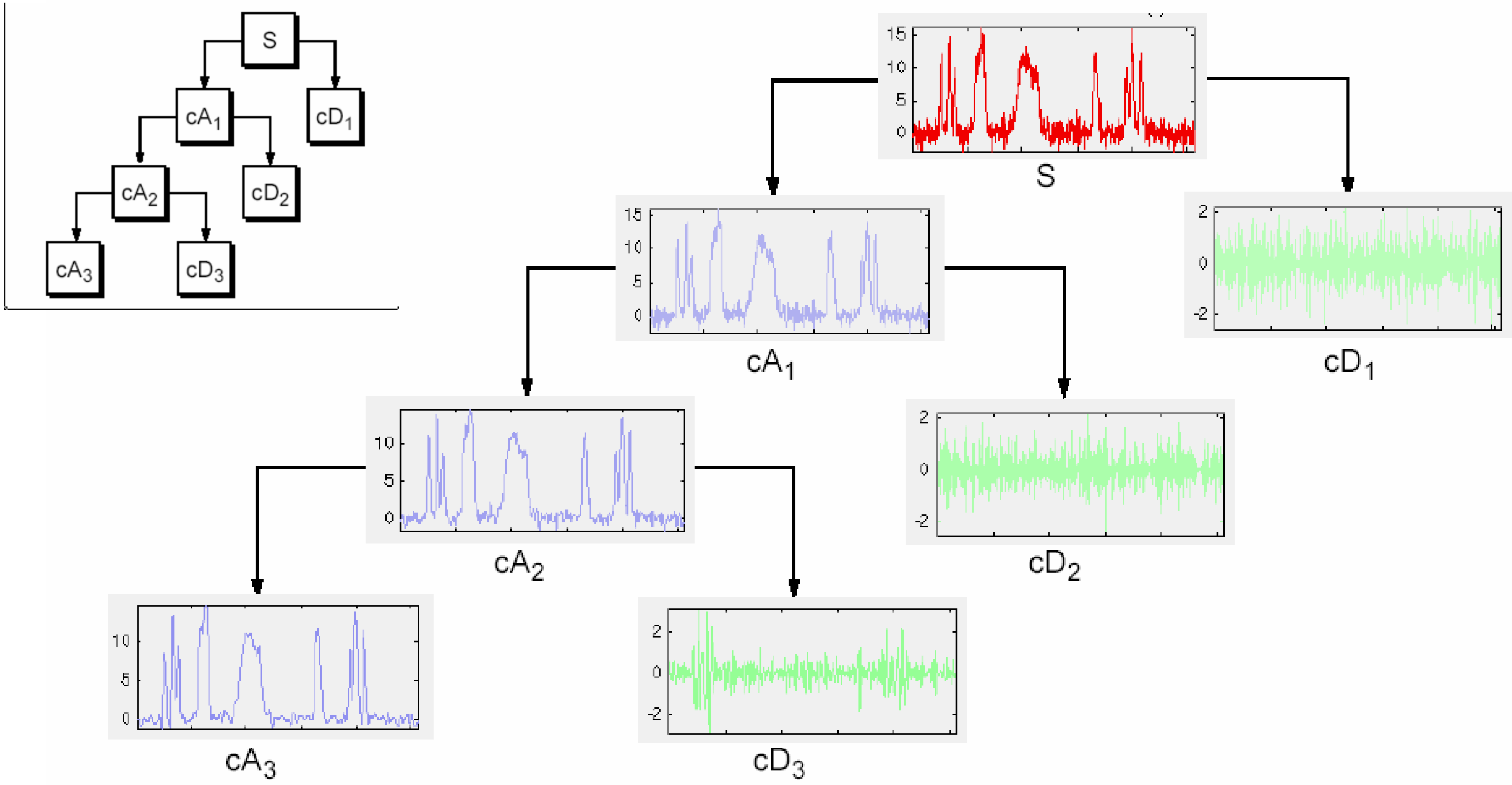
Analysis or decomposition



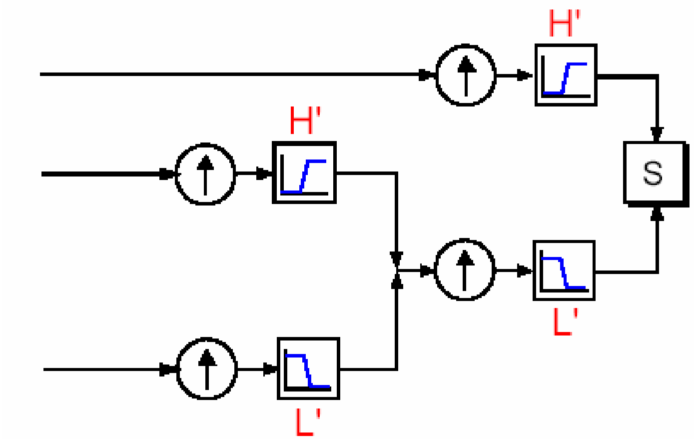




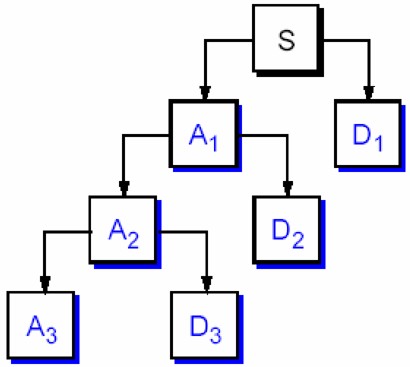
Analysis or decomposition



Synthesis or reconstruction

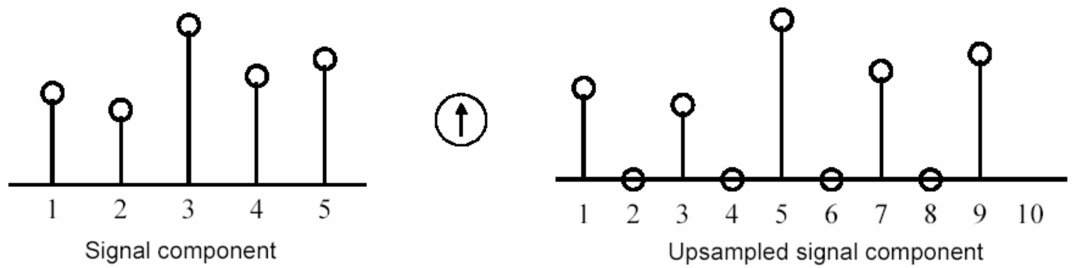


Reconstructed Signal Components

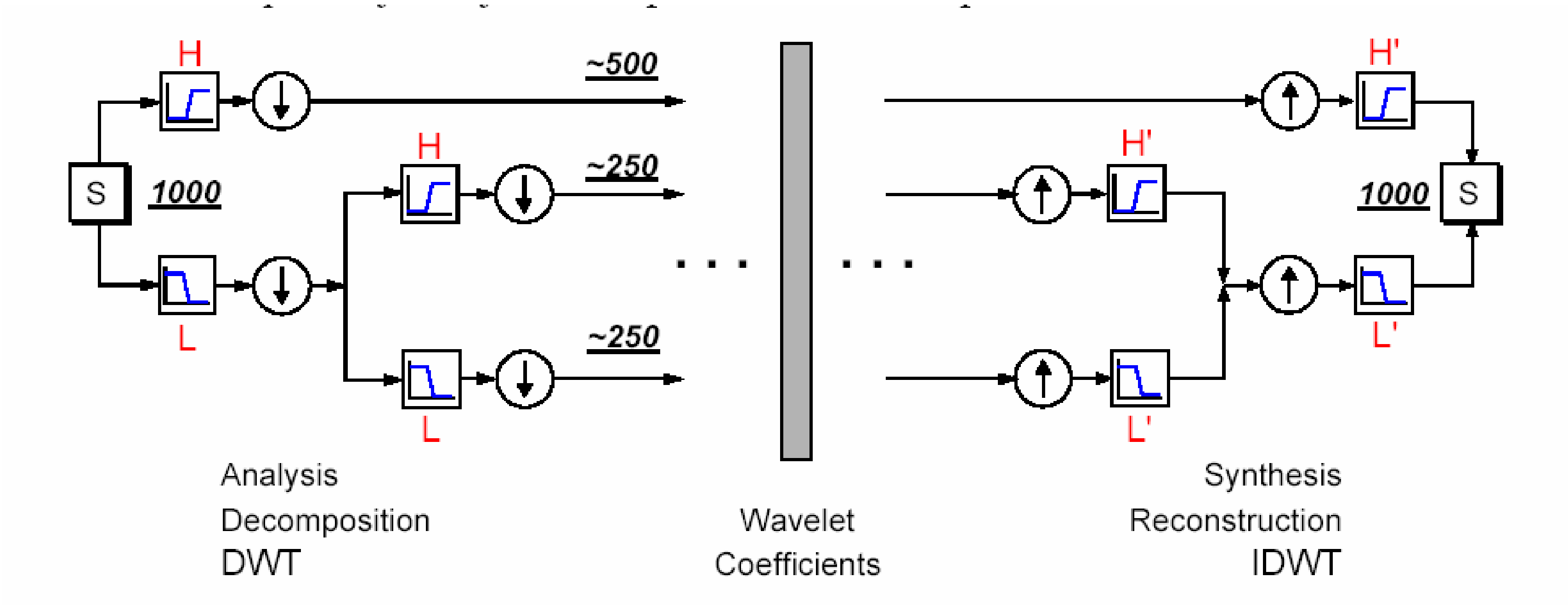


$$S = A_1 + D_1$$
$$= A_2 + D_2 + D_1$$
$$= A_3 + D_3 + D_2 + D_1$$

upsampling

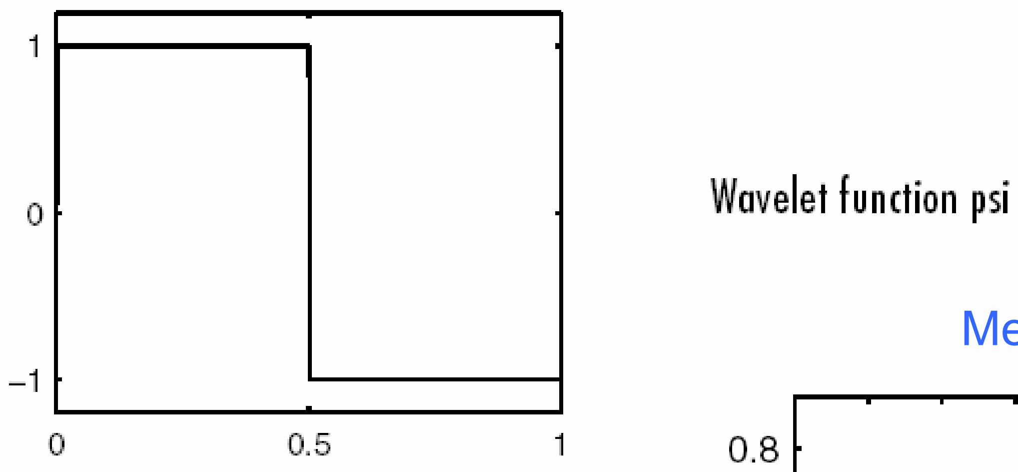


Multi-scale analysis

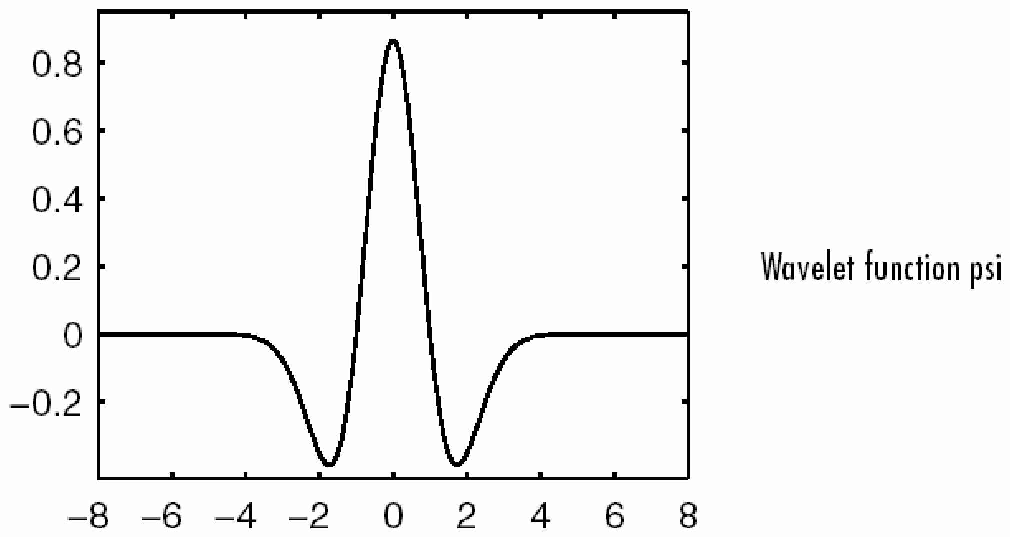


Famous wavelets

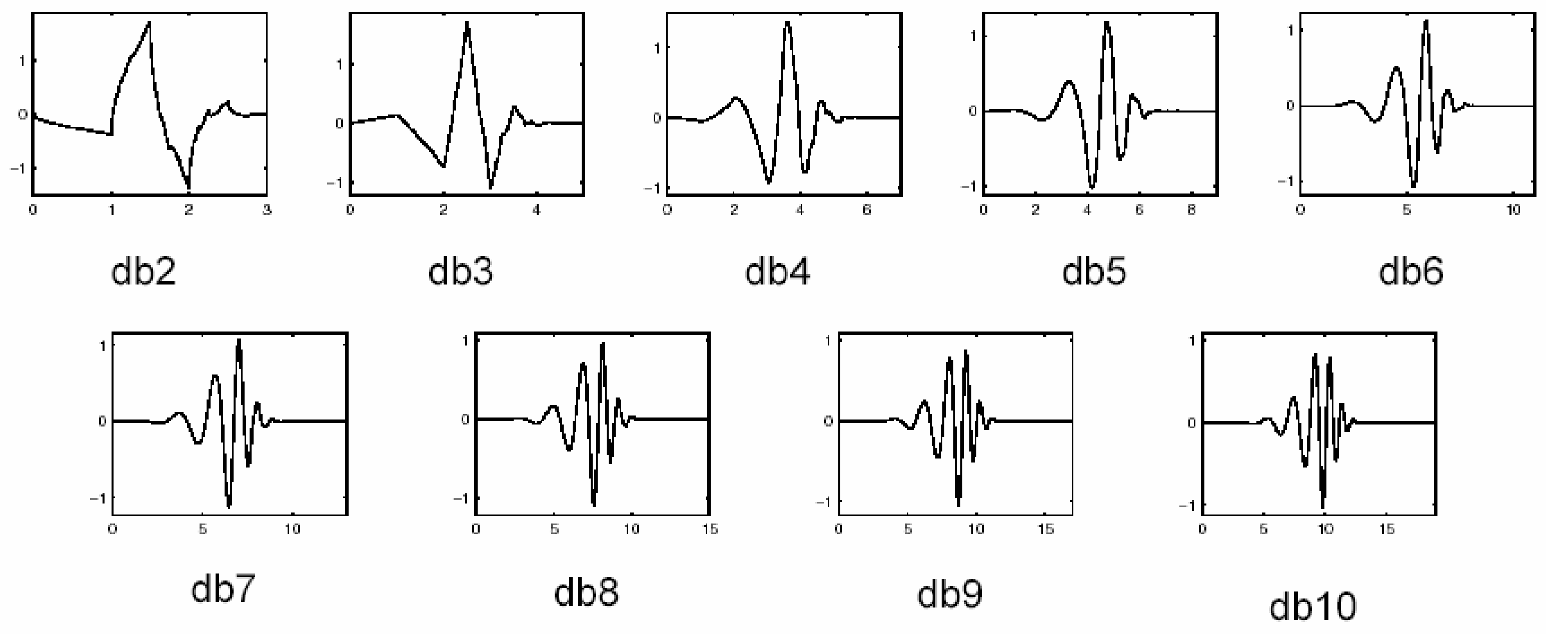
Haar



Mexican hat

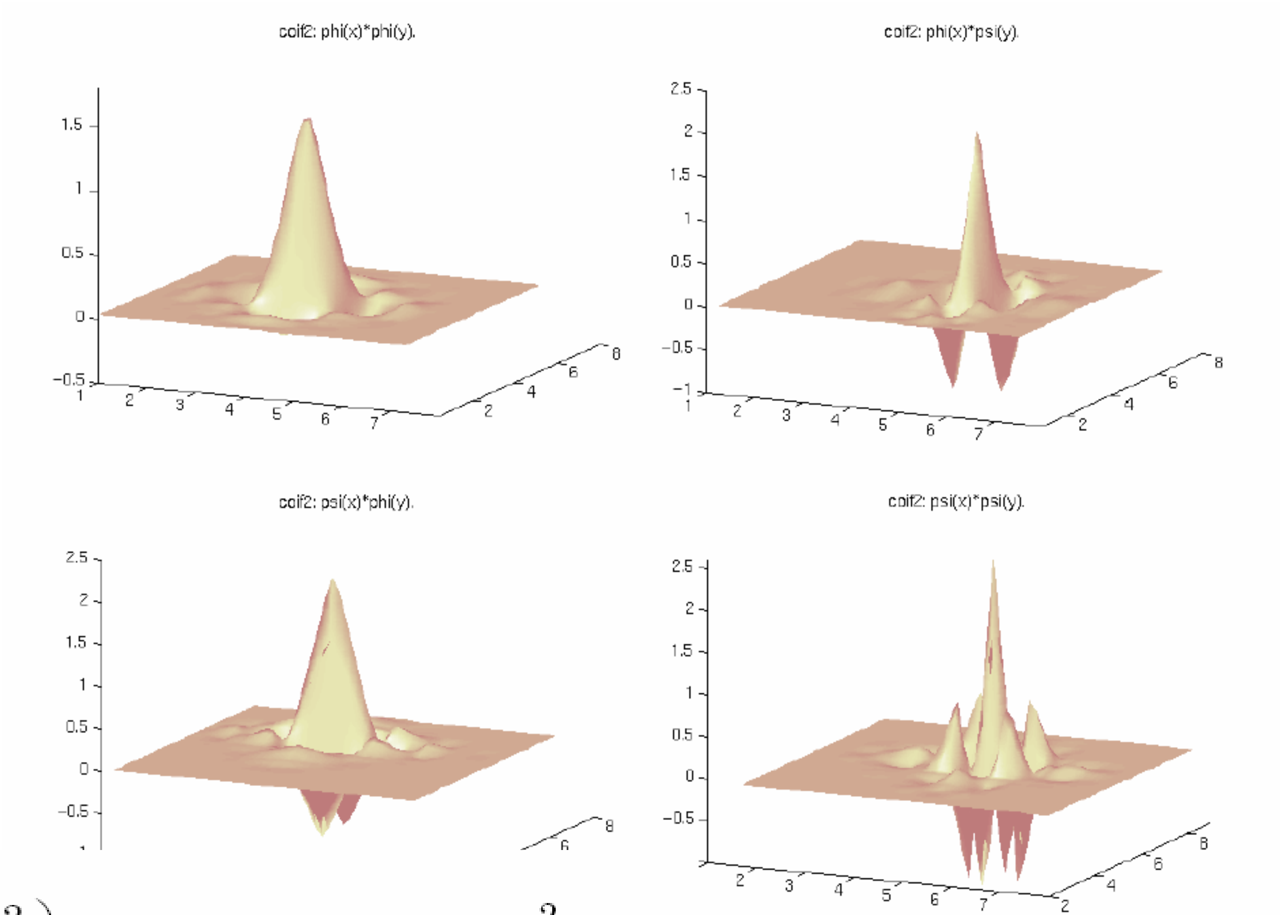


Daubechie's



Bi-dimensional wavelets

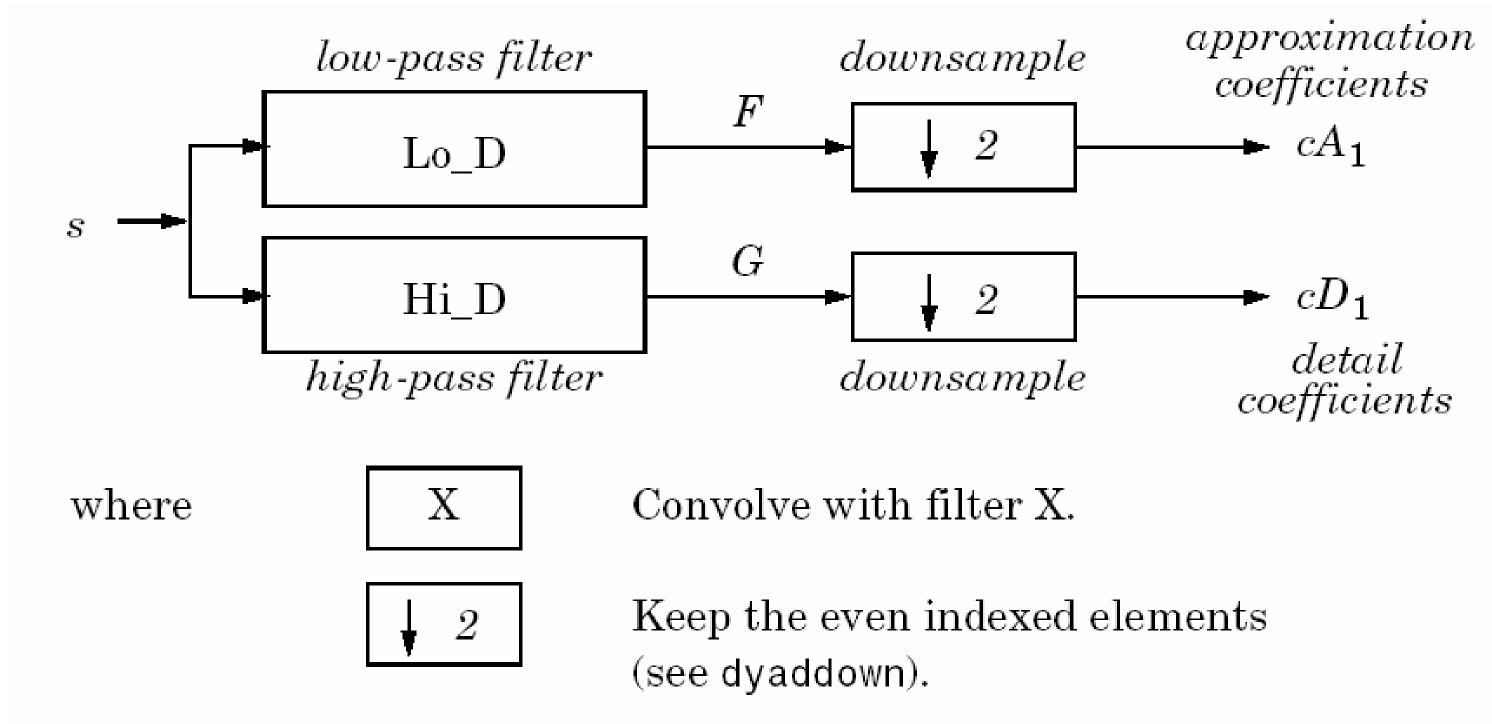
$$\varphi(x,y)=\varphi(x)\varphi(y)$$
$$\psi^1(x,y)=\varphi(x)\psi(y)$$
$$\psi^2(x,y)=\psi(x)\varphi(y)$$
$$\psi^3(x,y)=\psi(x)\psi(y)$$



$$\frac{1}{\sqrt{a_1 a_2}}\psi\left(\frac{x_1-b_1}{a_1},\frac{x_2-b_2}{a_2}\right) \text{ where } (x=(x_1,x_2)\in R^2)$$

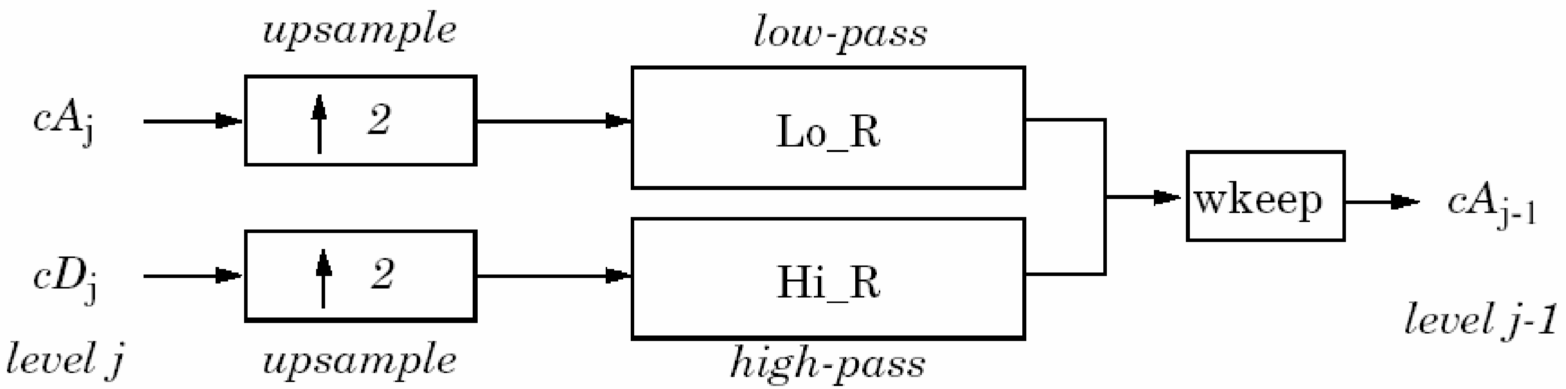
Fast wavelet transform algorithm (DWT)

Decomposition step



Fast wavelet transform algorithm (DWT)

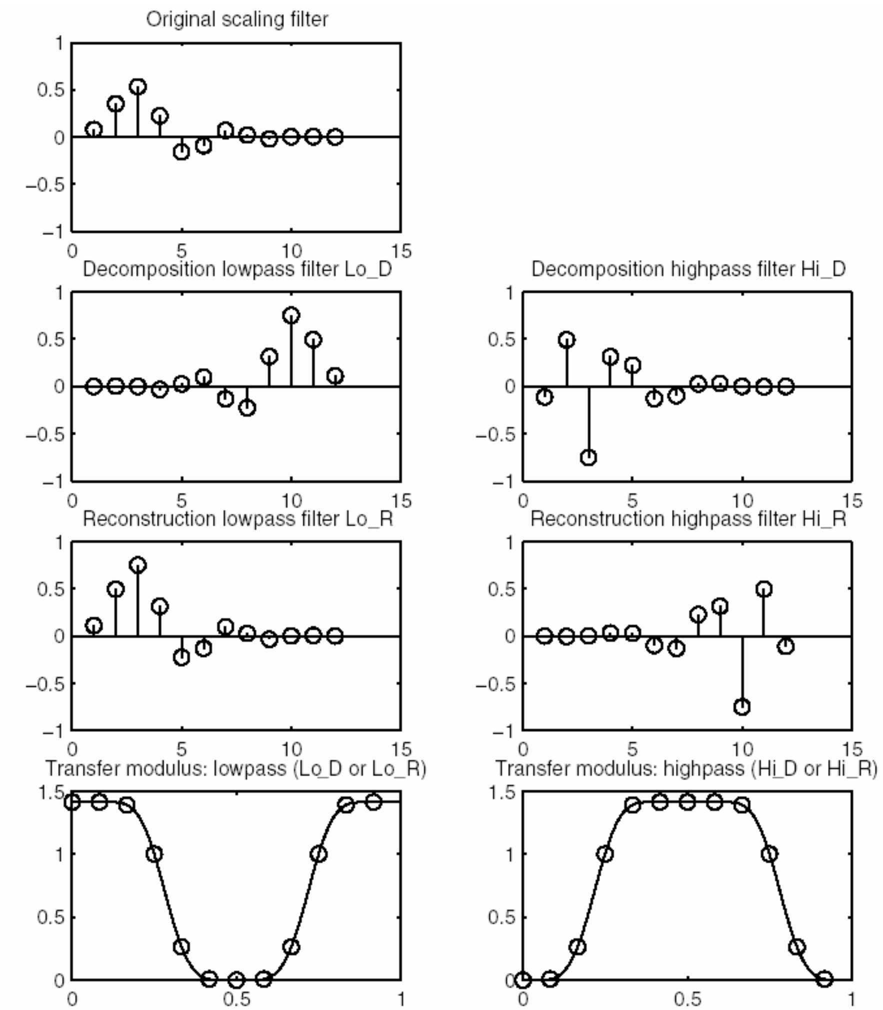
Reconstruction Step



where

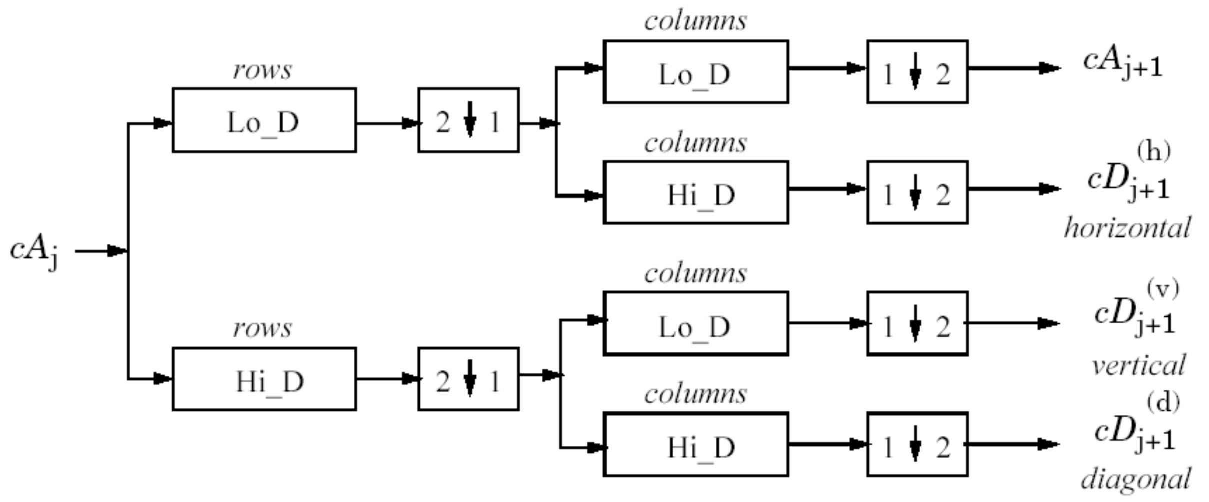
$\uparrow 2$	Insert zeros at odd-indexed elements.
X	Convolve with filter X.
wkeep	Take the central part of U with the convenient length.

Filters


Original scaling filter
Decomposition lowpass filter Lo_D
Decomposition highpass filter Hi_D
Reconstruction lowpass filter Lo_R
Reconstruction highpass filter Hi_R
Transfer modulus: lowpass (Lo_D or Lo_R)
Transfer modulus: highpass (Hi_D or Hi_R)

Fast DWT for images

Decomposition Step



where $\boxed{2 \downarrow 1}$ Downsample columns: keep the even indexed columns.

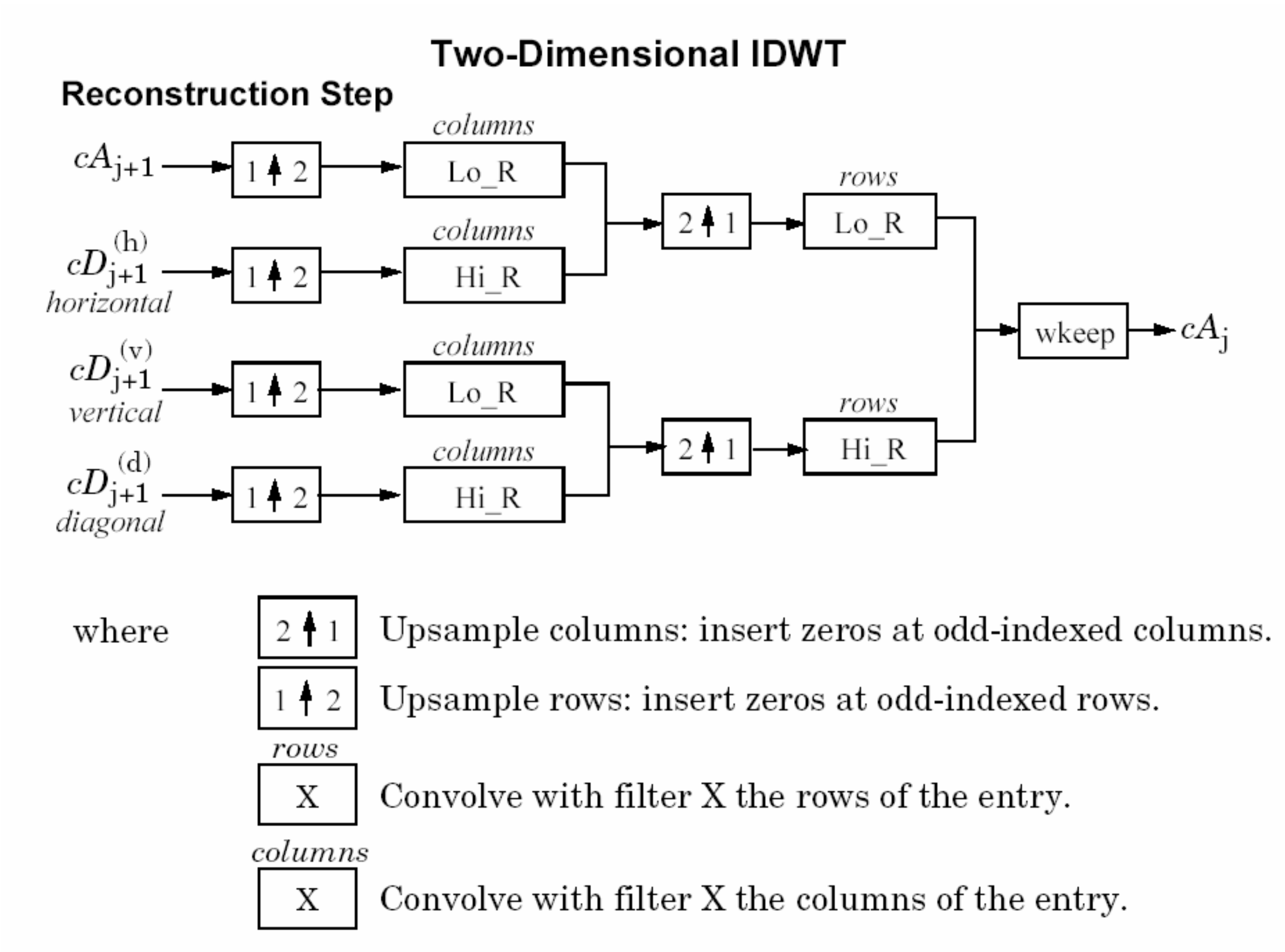
$\boxed{1 \downarrow 2}$ Downsample rows: keep the even indexed rows.

rows $\boxed{X}$ Convolve with filter X the rows of the entry.

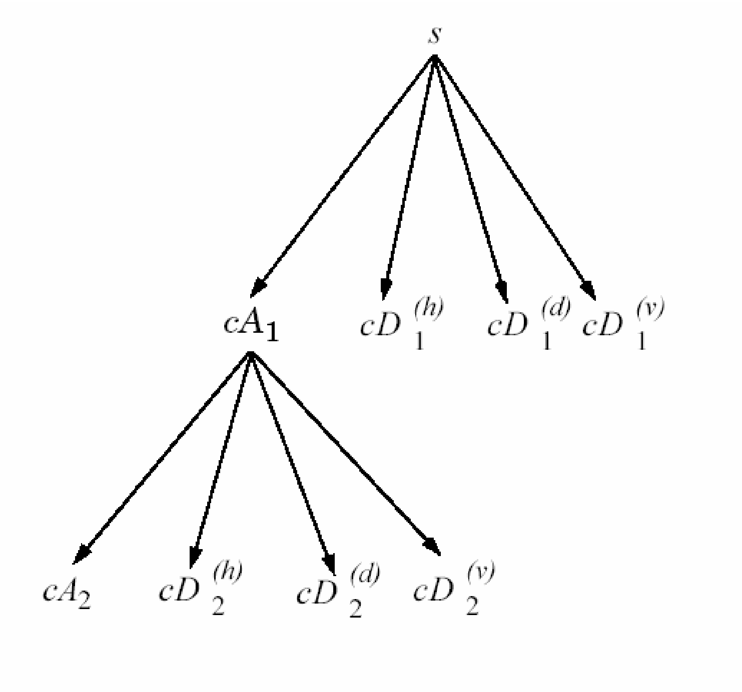
columns $\boxed{X}$ Convolve with filter X the columns of the entry.

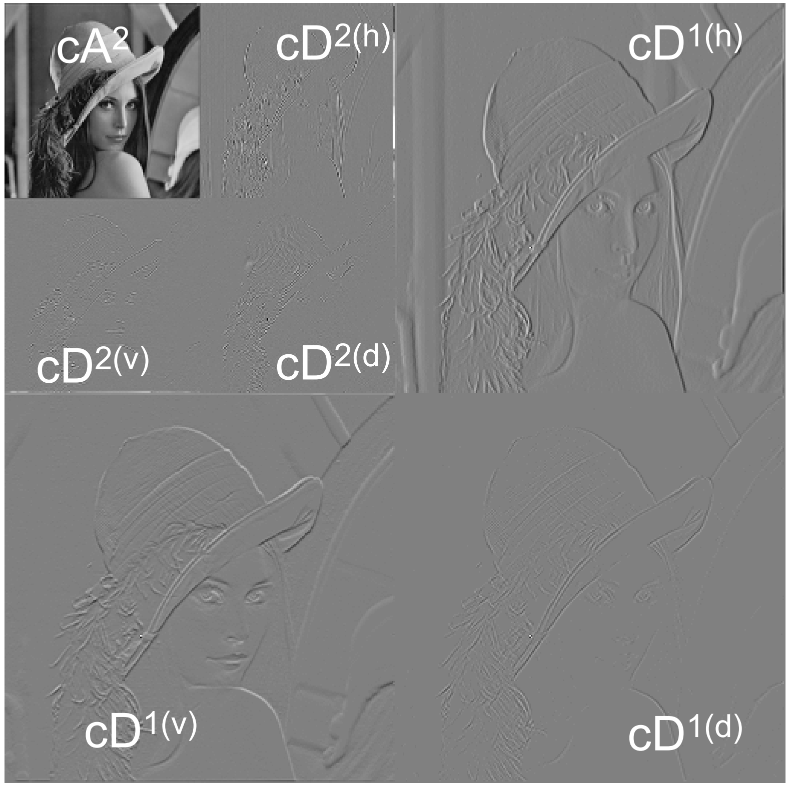
Initialization $CA_0 = s$ for the decomposition initialization.

Fast DWT for images



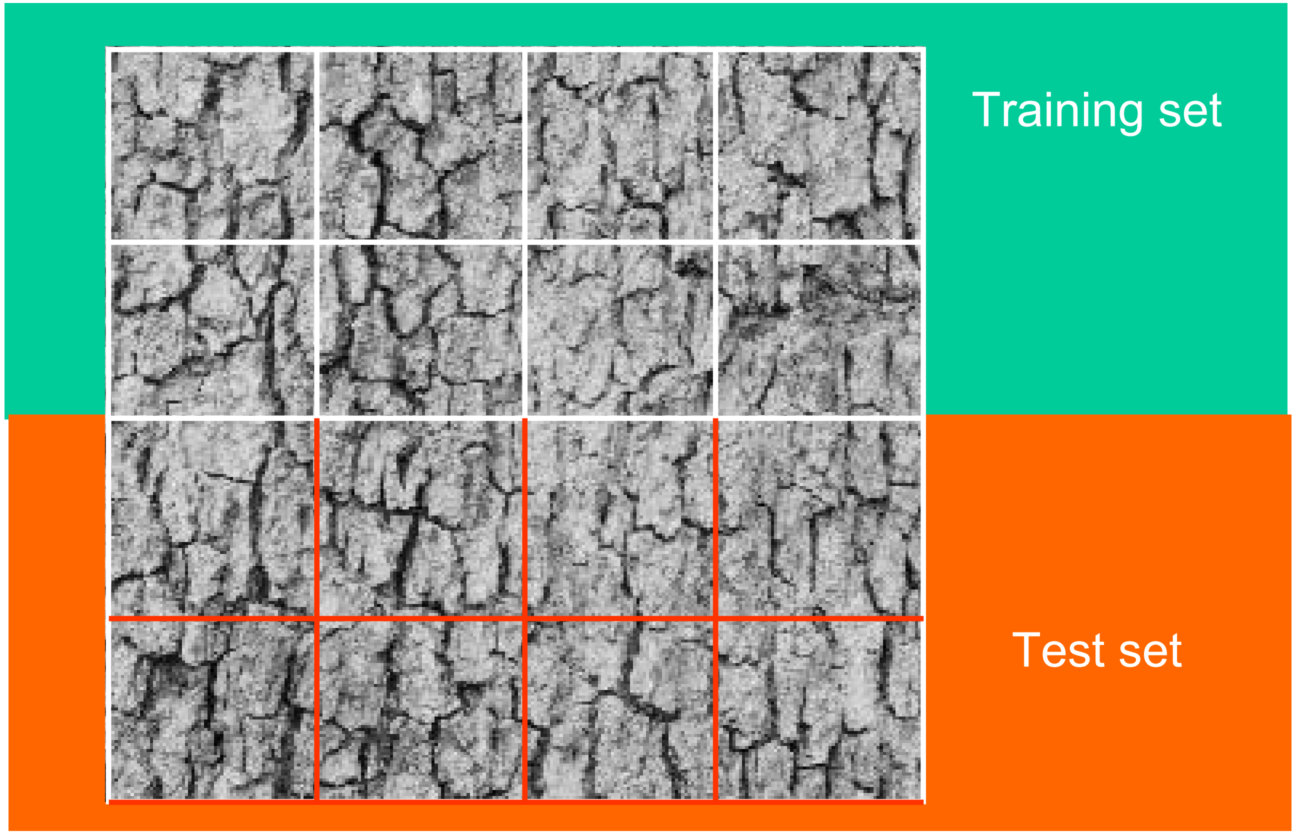
Subband structure for images





FV extraction

- Step 1: create independent texture instances



Feature extraction

- Step 2: extract features to form *feature vectors*

Intensity image

↓

Transformation

↓

For each sub-image

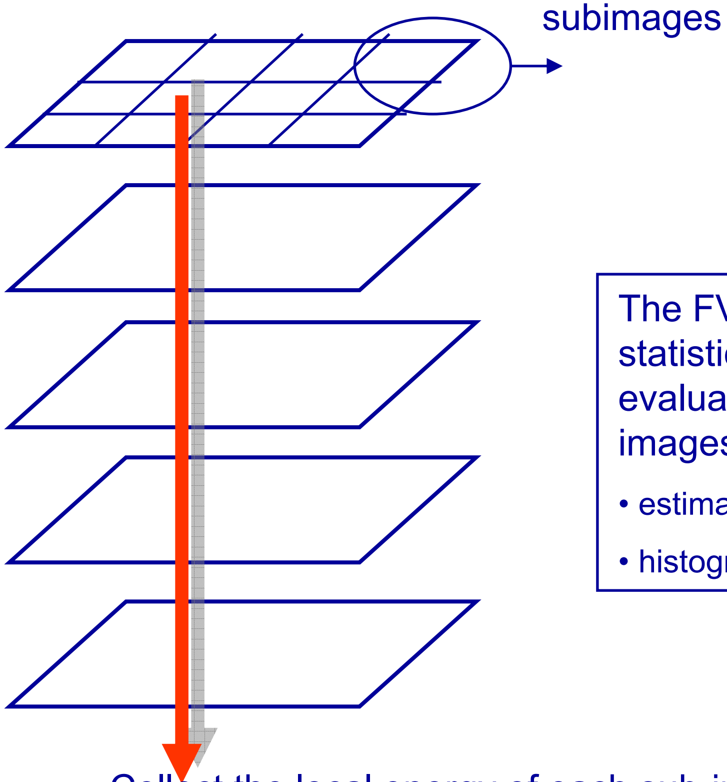
For each subband

Calculate the local energy (variance)

Fill the corresponding position in the FV

↓

One FV for each sub-image ⇒ Classification algorithm

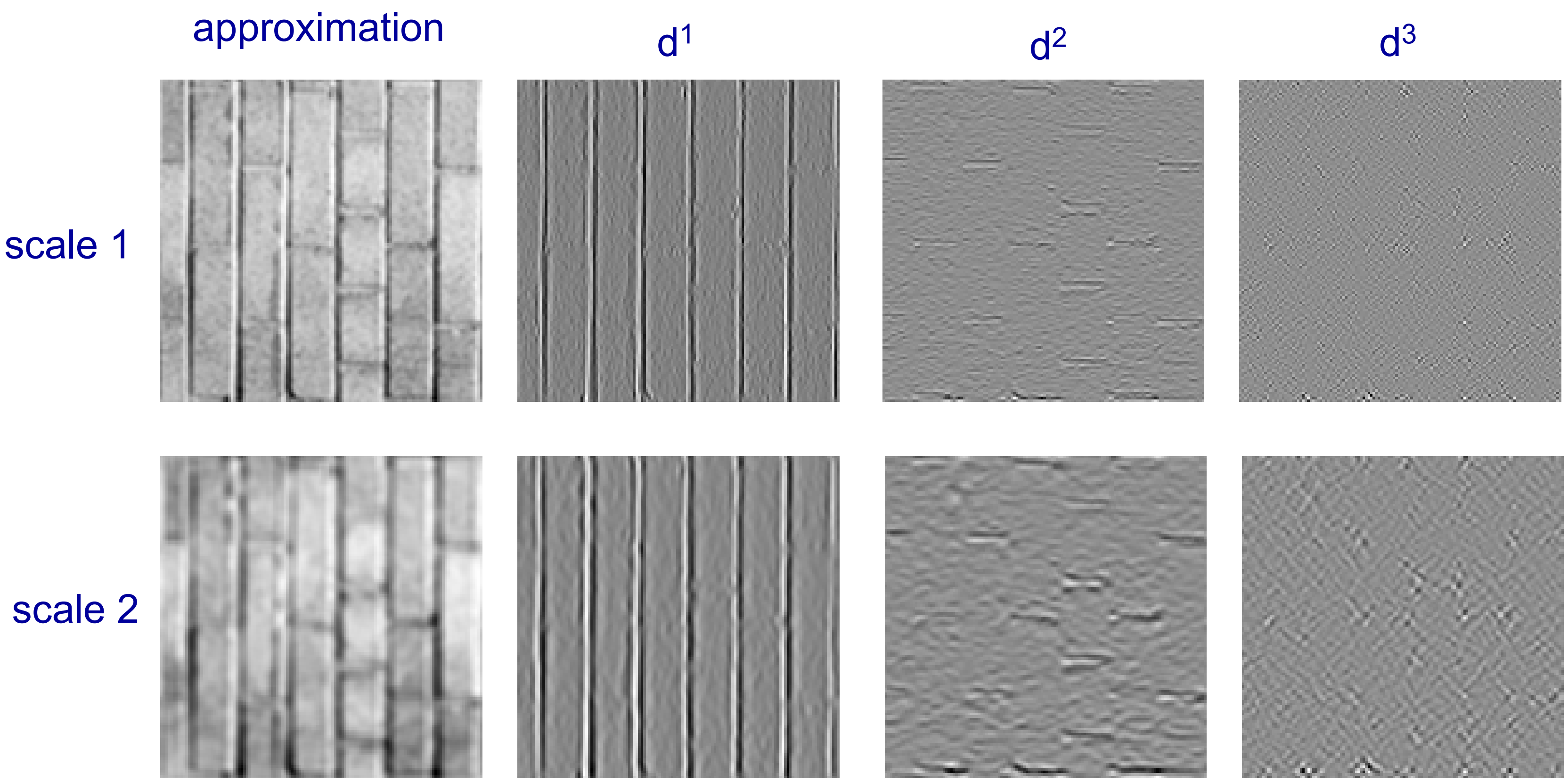


The FVs contain some statistical parameters evaluated on the subband images

- estimates of local variances
- histograms

⇒ Collect the local energy of each sub-image in the different subbands in a vector

Building the FV



Building the FV

elements of FV_1 of texture 1
elements of FV_2 of texture 1