— Lecture on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XIX

1. A brief mechanical digression

Def. A symplectic manifold (M, ω) is a smooth manifold equipped with a closed, non degenerate 2-form ω, i.e. $\omega \in Z^2(M)$ $(d\omega = 0)$ and any matrix w_{ij} representing the form in local coordinates is non singular (enough to check this in one coord. system)

$$\boxed{\begin{aligned}\omega &= \sum_{i<j} w_{ij}\, dx^i \wedge dx^j \quad (i<j) \\ &= \tfrac{1}{2}\, w_{ij}\, dx^i \wedge dx^j\end{aligned}}$$

Einstein

$\Omega = (w_{ij})$ is then antisymmetric and non singular
$\Omega = -\Omega^T$, $\det \Omega \neq 0$

Any T^*M (cotangent bundle) carries a natural symplectic structure

Notice that, in finite dimension, $\dim M = 2m$ (even)

since
$$\det \Omega = \det(-\Omega^T) = (-1)^n \det \Omega^T = (-1)^n \det \Omega$$

$\Rightarrow$ $\det \Omega = 0$ if n is odd, hence Ω would be singular.

Def. A vector field X on M $(X \in \mathfrak{X}(M))$ is called symplectic if it preserves the symplectic form, i.e.

$$\boxed{\mathcal{L}_X \omega = 0}$$

Application of Cartan's formula yields

$$d\, i_X \omega + i_X d\omega = 0 \quad \Rightarrow \quad \boxed{d\, i_X \omega = 0}$$

$\Rightarrow$ $i_X \omega$ is closed $(i_X \omega \in Z^1(M))$

The Poincaré lemma then shows that $i_X \omega$ is <u>locally exact</u>. If $i_X \omega$ is indeed exact, X is called a <u>Hamiltonian</u> vector field, and there exists $\lambda_X \in \mathscr{C}^\infty(M)$, determined up to a constant (+), such that

(+) Assume M connected

$$\boxed{i_X \omega = d\lambda_X}$$

λ_X is called a Hamiltonian pertaining to X

Conversely, starting from $\lambda \in \mathscr{C}^\infty(M)$, its symplectic gradient X_λ is the (unique) Hamiltonian vector field X_λ determined by

$$\boxed{i_{X_\lambda} \omega = d\lambda}$$

(existence is guaranteed by virtue of the non-degeneracy of ω)

Let us compute this explicitly

$$i_X \omega = dH \qquad \text{given } H \in \mathscr{C}^\infty(M)$$

$$dH = \frac{\partial H}{\partial x^j} dx^j \qquad X = \xi^i \frac{\partial}{\partial x^i}$$

$$\underbrace{\underbrace{\left(\frac{1}{2} \omega_{ij}\, dx^i \wedge dx^j\right)}_{\omega}(X, Y)}_{(i_X \omega)(Y)} = dH(Y)$$

"test v. field"

$$(\iota_X \omega)(Y) =$$

$$\frac{1}{2}\, w_{ij}\left[dx^i(X)\, dx^j(Y) - dx^i(Y)\, dx^j(X) \right]$$

$$= \frac{1}{2}\, w_{ij}\left[\underbrace{dx^i\left(\xi^k \frac{\partial}{\partial x^k}\right)}^{\xi^i} dx^j(Y) - dx^i(Y)\, dx^j\left(\xi^k \frac{\partial}{\partial x^k}\right) \right]$$

$$= \frac{1}{2}\, w_{ij}\left[\xi^i\, dx^j(Y) - \xi^j\, dx^i(Y) \right]$$

exchange — you can, since a summation is involved

$$= \frac{1}{2}\left(\omega_{ij}\, \xi^i - \underbrace{\omega_{ji}}_{= -\omega_{ij}} \xi^i \right) dx^j(Y)$$

$$= \cancel{\tfrac{1}{2}} \cdot 2 \cdot w_{ij}\, \xi^i \cdot dx^j(Y)$$

$\Rightarrow$ (dropping Y) $\qquad w_{ij}\, \xi^i\, dx^j = \dfrac{\partial H}{\partial x^j}\, dx^j$

$$\overset{\Omega}{\downarrow}$$

$$w_{ij}\, \xi^i = \frac{\partial H}{\partial x^j}$$

$$(\Omega^T)_{ji}\, \xi^i = \frac{\partial H}{\partial x^j}$$

$$\Rightarrow \qquad \Omega^T \xi = \nabla H$$

"gradient" abuse of terminology

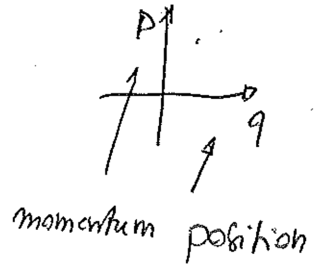
$$\boxed{\xi = \Omega^{-T}\, \nabla H}$$

This gives X in terms of H

(Ω is invertible)

Let us treat the simplest case

$$(M, \omega) = (\mathbb{R}^2, \ \omega = dq \wedge dp)$$



$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Given $H \in \mathcal{C}^\infty(\mathbb{R}^2)$,

$$\Omega^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\Omega \qquad \Omega^{-T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega$$

$X = X_H$ is given by

$$X_H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} +\frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

check $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$$\Rightarrow \quad X_H = +\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

As an exercise, let us, retrospectively, compute

$$i_{X_H}\omega = (dq \wedge dp)\left(+\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}, \ \cdot\right)$$

$$= (dq \wedge dp)\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q}, \cdot\right) - (dq \wedge dp)\left(\frac{\partial H}{\partial q}\frac{\partial}{\partial p}, \cdot\right)$$

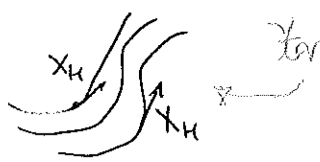
⚠

$$= \frac{\partial H}{\partial p}\, dp + \frac{\partial H}{\partial q}\, dq$$

$$= dH \qquad ✓$$

☆ (M, ω, H) : Hamiltonian system

Given $H \in \mathcal{H}(M)$, the integral curves of X_H (Hamiltonian flow) are solutions of the so-called Hamilton's equations (they yield trajectories in phase space)



Hamiltonian flow

$$\boxed{\dot{c}(t) = X_H(c(t))}$$

In our simple example one has

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \qquad (\underset{\mathbb{R}^2}{\underset{\parallel}{M}}, \underset{dq\wedge dp}{\underset{\parallel}{\omega}}, H)$$

Observe that $\mathcal{L}_X H = \qquad (X = X_H)$

$$= X(H) = dH(X) = \left(\frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp, \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \right)$$

$$= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0$$

$\Rightarrow$ H is constant along trajectories (energy conservation)

(true in general: $dH(X_H) = (i_{X_H}\omega)(X_H) = \omega(X_H, X_H) = 0$)

Also, given $\lambda, \mu \in \mathcal{C}^\infty(\mathbb{R}^2)$, we find, successively

$$\omega(X_\lambda, X_\mu) = (dq\wedge dp)\left(\frac{\partial \lambda}{\partial p}\frac{\partial}{\partial q} - \frac{\partial \lambda}{\partial q}\frac{\partial}{\partial p}, \frac{\partial \mu}{\partial p}\frac{\partial}{\partial q} - \frac{\partial \mu}{\partial q}\frac{\partial}{\partial p} \right)$$

$$= -\frac{\partial \lambda}{\partial p}\frac{\partial \mu}{\partial q} + \frac{\partial \lambda}{\partial q}\frac{\partial \mu}{\partial p} = \frac{\partial \lambda}{\partial q}\frac{\partial \mu}{\partial p} - \frac{\partial \lambda}{\partial p}\frac{\partial \mu}{\partial q} = \{\lambda, \mu\}$$

Poisson brackets

(on a general symplectic manifold, Poisson brackets are introduced via the above formula)

Notice that, given $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, $f = f(q,p)$ "classical observable" $\equiv f(q(t), p(t))$

one has

$$\dot{f} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} = \{f, H\}$$

$\boxed{\dot{f} = \{f, H\}}$ also called Hamilton's equation

Given a Hamiltonian system (M, ω, H),

integral of motion

a first integral is an observable ($f \in C^\infty(M)$) which Poisson-commutes with H: $\{f, H\} = 0$

This entails that $\dot{f} = 0$ along the motion trajectories, i.e. f is constant thereon (whence the name).

From the Jacobi identity $\{f,\{g,H\}\} + \{g\{H,f\}\} + \{H,\{f,g\}\} = 0$ it follows that, if f and g are first integrals, so is $\{f,g\}$

(in the general case, the Jacobi identity above stems from the closure of ω).

Two first integrals f and g are said to be in involution if they Poisson-commute: $\{f,g\} = 0$

This is equivalent to $-\omega(X_f, X_g) = 0$, and also to $[X_f, X_g] = 0$.

commuting flows

The interesting case is when f is not a function of g ...

Indeed, one finds (exercise, in $\mathbb{R}^2$) $[X_\lambda, X_\mu] = -X_{\{\lambda,\mu\}}$

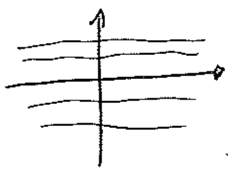
(one has a Lie-algebra anti-isomorphism, but we stop our discussion here)

Examples. $(\mathbb{R}^2, dq\wedge dp)$ $\quad X = \frac{\partial}{\partial q}$ (translations along the q axis)

$$i_X \omega = (dq \wedge dp)\left(\frac{\partial}{\partial q}, \cdot\right) = dp \qquad \lambda_X = p \ (+c)$$

momentum which is conserved along motion trajectories

Hamilton: $\begin{cases} \dot{q} = 1 \\ \dot{p} = 0 \end{cases} \qquad \begin{array}{l} q = q_0 + t \\ p = p_0 \end{array}$

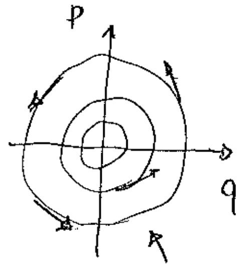


translation-invariance

$(\mathbb{R}^2, dq \wedge dp)$ $\qquad X = q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}$ $\rightsquigarrow$ generates rotations

$$\underbrace{(dq \wedge dp)\left(q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}, \cdot \right)}_{i_X \omega} = -q\,dq - p\,dp = d\Big(\underbrace{-\frac{p^2+q^2}{2}}_{\lambda_X (+c)}\Big)$$

Hamilton: $\begin{cases} \dot{q} = -p \\ \dot{p} = q \end{cases}$



motion trajectories: circles centred at 0.

$\lambda_{-X} = \frac{p^2+q^2}{2}$

Hamiltonian of the harmonic oscillator

Variant

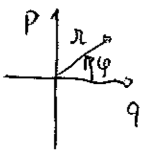
$\Omega^T \xi = \nabla H$ find H:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -p \\ q \end{pmatrix} = \begin{pmatrix} -q \\ -p \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \Rightarrow H = -\frac{1}{2}(p^2+q^2)(+c)$$

Notice that in these examples $X_\lambda \perp \operatorname{grad} \lambda$ (as it should be)

(use euclidean metric $ds^2 = dq^2 + dp^2$ on $\mathbb{R}^2$)

Let us take $H = \frac{1}{2}(q^2+p^2)$ and let us pass to polar coordinates: $H = \frac{1}{2} r^2$



$$\omega = \underbrace{dq \wedge dp}_{} = r\,dr \wedge d\varphi = dH \wedge d\varphi \equiv \underbrace{dI \wedge d\varphi}_{}$$

symplectic (canonical) variables $\rightarrow$

first integral

$H \equiv I$

I: "action variable"

φ: "angle variable"

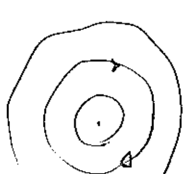
Hamilton: $\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = p \\ \dot{p} = -\frac{\partial H}{\partial q} = -q \end{cases}$ solutions: circles...

$\ddot{q} = \dot{p} = -q, \quad \ddot{q} + q = 0$

$q(t) = A \cos t + B \sin t$

In terms of (I, φ): $\;$ & trivially solved

$$\begin{cases} \dot{I} = \frac{\partial H}{\partial \varphi} = \frac{\partial I}{\partial \varphi} = 0 \\ \dot{\varphi} = -\frac{\partial H}{\partial I} = -\frac{\partial I}{\partial I} = -1 \end{cases} \qquad \begin{aligned} I &= c \\ \varphi &= -t + c \end{aligned}$$



$\rightsquigarrow$ lagrangian torus

$\rightsquigarrow$ integrable systems