Examples

In $\mathbb{R}^3$ | 1. $X = \frac{\partial}{\partial x}$ $\quad \omega = a\, dx \wedge dy \quad a \in \mathcal{C}^\infty(\mathbb{R}^3)$

$$d\omega = da \wedge dx \wedge dy = \frac{\partial a}{\partial z} dz \wedge dx \wedge dy = \frac{\partial a}{\partial z} dx \wedge dy \wedge dz$$

$$\boxed{i_X\, d\omega = \frac{\partial a}{\partial z} dy \wedge dz}$$

$$(i_X d\omega\, (Y,Z) = \underbrace{\left(\frac{\partial a}{\partial z} dx \wedge dy \wedge dz \right)}_{} (\underset{\frac{\partial}{\partial x}}{X}, Y, Z) =$$

running through $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$= \frac{\partial a}{\partial z} \begin{vmatrix} \overset{1}{dx(X)} & dx(Y) & dx(Z) \\ \underset{0}{dy(X)} & \cdot & \cdot \\ \underset{0}{dz(X)} & \cdot & \cdot \end{vmatrix} = \frac{\partial a}{\partial z} \begin{vmatrix} dy(Y) & dy(Z) \\ dz(Y) & dz(Z) \end{vmatrix}$$

$$= \frac{\partial a}{\partial z} (dy \wedge dz)(Y, Z)$$

One can proceed directly from the definition

$$\frac{\partial a}{\partial z} (dx \wedge dy \wedge dz)(X, \overset{Y}{\bullet}, \overset{Z}{\diamond}) =$$

$$= \frac{\partial a}{\partial z} \Big(\overset{1}{dx(\underset{\frac{\partial}{\partial x}}{X})}\, dy(\bullet)\, dz(\diamond) - \underbrace{dx(X)}_{1}\, dy(\diamond)(\bullet) + \dots \Big)$$

all other terms are 0 since $dy(\underset{\frac{\partial}{\partial x}}{X}) = dz(X) = 0$

$$= \frac{\partial a}{\partial z} dy \wedge dz \,(\bullet, \diamond)$$

Still in another way:

$$i_X\left(\frac{\partial a}{\partial z} dx \wedge dy \wedge dz\right) = \frac{\partial a}{\partial z} i_X(dx \wedge dy \wedge dz)$$

$$= \frac{\partial a}{\partial z} (\underset{\substack{dx(\frac{\partial}{\partial x}) \\ \| \\ 1}}{i_X dx}) \wedge dy \wedge dz - dx \wedge \underbrace{i_X dy}_{dy(X) = X(y) = \frac{\partial y}{\partial x} = 0} \wedge dz + \underset{=0}{\text{other terms}})$$

$$= \frac{\partial a}{\partial z} dy \wedge dz$$

$$i_X \omega = a\, i_X (dx \wedge dy) = a\, dy$$

$X = \frac{\partial}{\partial x}$ (proceed as before..)

$$d(i_X \omega) = d(a\, dy) = da \wedge dy = \frac{\partial a}{\partial x} dx \wedge dy + \frac{\partial a}{\partial z} dz \wedge dy$$

$$= \frac{\partial a}{\partial x} dx \wedge dy - \frac{\partial a}{\partial z} dy \wedge dz$$

Therefore $(d\, i_X + i_X d)\, \omega = \frac{\partial a}{\partial x} dx \wedge dy$

Compute $\mathcal{L}_X \omega$ directly:

Leibniz

$$\mathcal{L}_X (a\, dx \wedge dy) = \underbrace{(\mathcal{L}_X a)}_{X(a) = \frac{\partial a}{\partial x}} dx \wedge dy + a \overbrace{\mathcal{L}_X dx}^{d \mathcal{L}_{\frac{\partial}{\partial x}} x = d1 = 0} \wedge dy + a\, dx \wedge \underbrace{\mathcal{L}_X dy}_{=0}$$

$$= \frac{\partial a}{\partial x} dx \wedge dy \quad \checkmark$$

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2. $X = \frac{\partial}{\partial x} \qquad \omega = a\, dy \wedge dz$

$$d\omega = da \wedge dy \wedge dz = \frac{\partial a}{\partial x} dx \wedge dy \wedge dz$$

$$i_X d\omega = \ldots = \frac{\partial a}{\partial x} dy \wedge dz$$

$$i_X \omega = \ldots 0 \qquad (a\, dy \wedge dz)(X, \cdot) = a[\overset{0}{dy(X)}\, dz(\cdot) - dy(\cdot)\, \underset{0}{dz(X)}]$$

$$d\, i_X \omega = 0 \qquad (d i_X + i d_X)\, \omega = \frac{\partial a}{\partial x} dy \wedge dz$$

$$\mathcal{L}_X (a\, dy \wedge dz) = \mathcal{L}_X a \cdot dy \wedge dz + \ldots 0 \ldots = \frac{\partial a}{\partial x} dy \wedge dz. \quad \checkmark$$

4. Let (M, g) be a Riemannian manifold and $X \in \mathcal{X}(M)$. Compute $\mathcal{L}_X g$. Work locally:

$X = \xi^i \frac{\partial}{\partial x^i}$

(we cannot use Cartan's formula, g is not a 2-form; in fact, it is a (symmetric, positive definite) (0,2)-tensor field)

$$\mathcal{L}_X (g_{ij} dx^i dx^j) =$$

$$= \underbrace{\mathcal{L}_X(g_{ij})}_{= X(g_{ij})} dx^i dx^j + g_{ij} \underbrace{(\mathcal{L}_X dx^i)}_{= d\mathcal{L}_X x^i = d\, X(x^i) = d\xi^i} dx^j + g_{ij} dx^i \underbrace{\mathcal{L}_X dx^j}_{= d\xi^j}$$

$$= X(g_{ij}) dx^i dx^j + g_{ij}\, d\xi^i dx^j + g_{ij}\, dx^i d\xi^j$$

one wants ij

$$= \xi^k \frac{\partial}{\partial x^k} g_{ij}\, dx^i dx^j + g_{ij} \frac{\partial \xi^i}{\partial x^k} dx^k dx^j + g_{ij}\, dx^i \frac{\partial \xi^j}{\partial x^l} dx^l$$

so relabel $k \to i$, $i \to k$ relabel: $l \to j$, $j \to k$

$$= \xi^k \frac{\partial g_{ij}}{\partial x^k} dx^i dx^j + g_{kj} \frac{\partial \xi^k}{\partial x^i} dx^i dx^k + g_{ik} \frac{\partial \xi^k}{\partial x^j} dx^i dx^j$$

$$= \left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} \right) dx^i dx^j$$

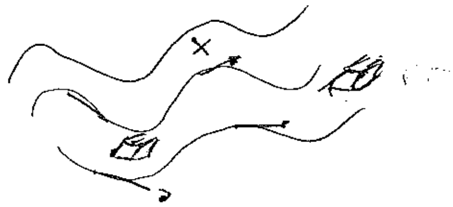
X is a Killing vector field (or, simply, X is Killing) if $\mathcal{L}_X g = 0$ ($\Rightarrow X$ generates isometries of (M, g))

3. in $\mathbb{R}^3$ $\quad \omega = dx \wedge dy \wedge dz$ standard volume form

$$\text{let } X = \underbrace{\alpha \frac{\partial}{\partial x}}_{X_1} + \underbrace{\beta \frac{\partial}{\partial y}}_{X_2} + \underbrace{\gamma \frac{\partial}{\partial z}}_{X_3} \qquad \alpha, \beta, \gamma \in \mathcal{C}^\infty(\mathbb{R}^3)$$

Compute $\quad \mathcal{L}_X \omega = \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) dx \wedge dy \wedge dz$

(computes variation of volume elements) $\qquad$ divergence of X $\quad$ ⚠ a metric is involved



$\mathcal{L}_X \omega = 0$ if $\operatorname{div} X = 0$ $\quad X$ solenoidal (or divergence-free)

defines volume-preserving flows

$\cong$ incompressible flows

This holds on Riemannian manifolds

Let us perform the calculation; directly:

$$\mathcal{L}_X (dx \wedge dy \wedge dz) = \mathcal{L}_X dx \wedge dy \wedge dz + dx \wedge \mathcal{L}_X dy \wedge dz + dx \wedge dy \wedge \mathcal{L}_X dz$$

$$= d(\mathcal{L}_X x) \wedge dy \wedge dz + \text{similar terms}$$

$$= d(X(x)) \wedge dy \wedge dz + \dots \qquad \boxed{X(x) = .. = \alpha}$$

$$= d\alpha \wedge dy \wedge dz + \dots$$

$$= \frac{\partial \alpha}{\partial x} dx \wedge dy \wedge dz + \dots \qquad \checkmark$$

We can use Cartan:
$$\underbrace{d\omega = 0}_{\text{a 4-form on } \mathbb{R}^3} \implies \mathcal{L}_X \omega = d\, i_X \omega$$

$$\underbrace{i_{X_1}}_{\alpha \frac{\partial}{\partial x}} \omega = .. = \alpha\, dy \wedge dz \implies d\, i_{X_1} \omega = \frac{\partial \alpha}{\partial x} dx \wedge dy \wedge dz$$

Summing up, we achieve the conclusion. $\checkmark$

For instance, in $(\mathbb{R}^n, g = \Sigma (dx^i)^2)$

$$g_{ij} = \delta_{ij}$$

The Killing condition becomes:

$$\left(\delta_{kj} \frac{\partial \xi^k}{\partial x^i} + \delta_{ik} \frac{\partial \xi^k}{\partial x^j} \right) = 0$$

i.e.

$$\boxed{\frac{\partial \xi^j}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} = 0}$$

Constant vector fields and generators of rotations in the plane (i,j), namely $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$,

are examples of Killing fields.