2D Discrete Fourier Transform (DFT)

Outline

- Circular and linear convolutions
- 2D DFT
- 2D DCT
- Properties
- Other formulations
- Examples

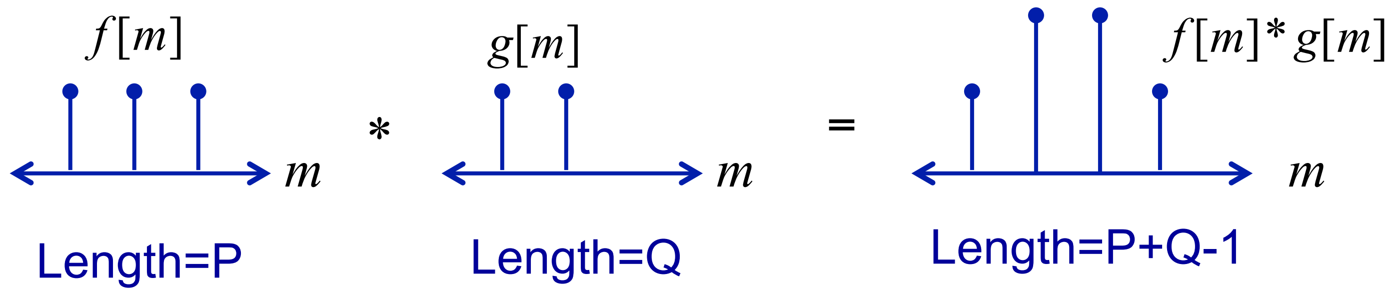
Circular convolution

- Finite length signals (N_0 samples) $\rightarrow$ circular or periodic convolution
  - the summation is over 1 period
  - the result is a N_0 period sequence

$$\boxed{c[k] = f[k] \otimes g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n]}$$

- The circular convolution is equivalent to the linear convolution of the zero-padded equal length sequences

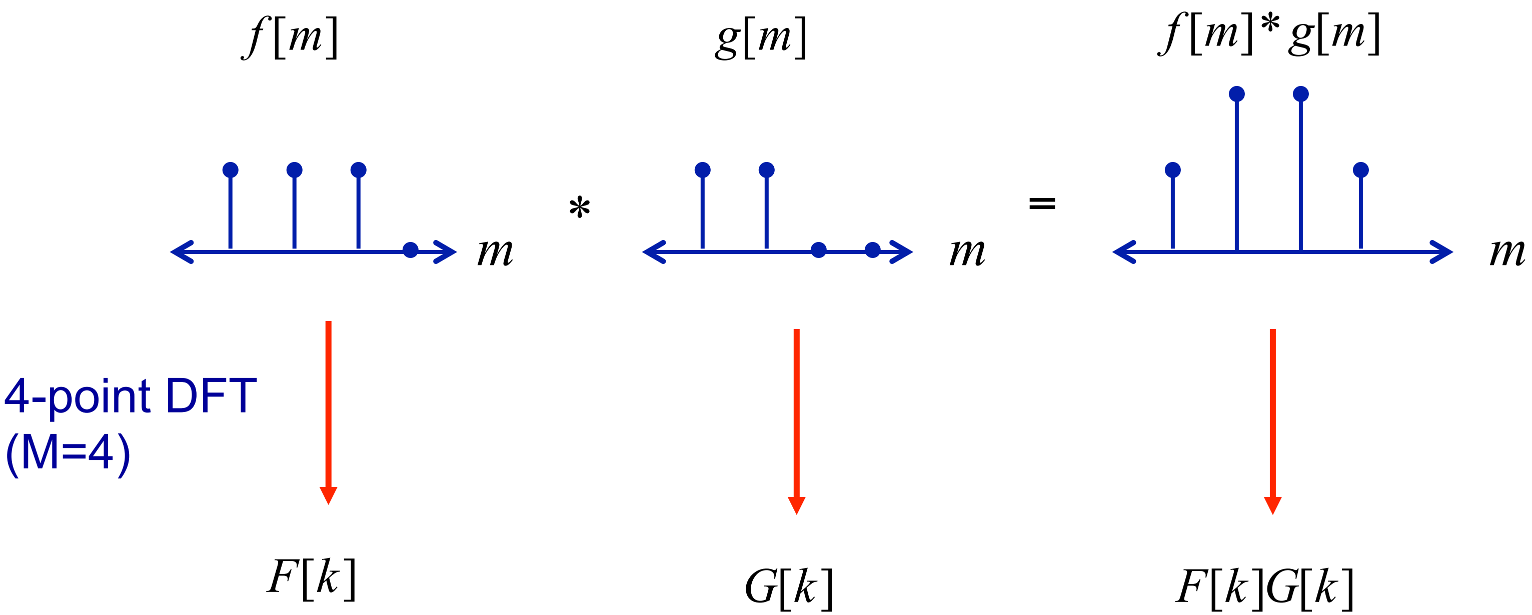
$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$



For the convolution property to hold, M must be *greater than or equal* to P+Q-1.

Convolution

- Zero padding $\qquad f[m]*g[m] \Leftrightarrow F[k]G[k]$



In words

- Given 2 sequences of length N and M, let y[k] be their linear convolution

$$y[k] = f[k] * h[k] = \sum_{n=-\infty}^{+\infty} f[n]h[k-n]$$

- y[k] is also equal to the circular convolution of the two suitably zero padded sequences making them consist of the same number of samples

$$c[k] = f[k] \otimes h[k] = \sum_{n=0}^{N_0-1} f[n]h[k-n]$$

$$N_0 = N_f + N_h - 1 : \text{ length of the zero-padded seq}$$

- In this way, the linear convolution between two sequences having a different length (filtering) can be computed by the DFT (which rests on the circular convolution)
  - The procedure is the following
    - Pad f[n] with N_h-1 zeros and h[n] with N_f-1 zeros
    - Find Y[r] as the product of F[r] and H[r] (which are the DFTs of the corresponding zero-padded signals)
    - Find the inverse DFT of Y[r]
- **Allows to perform linear filtering using DFT**

2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size MxN

or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.
- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a sampled version of the 2D DTFT

2D Discrete Fourier Transform (2D DFT)

- 2D Fourier (discrete time) Transform (DTFT) [Gonzalez]

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

a-periodic signal
periodic transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

periodized signal
periodic and **sampled** transform

2D DFT can be regarded as a sampled version of 2D DTFT.

2D DFT: Periodicity

- A [M,N] point DFT is periodic with period [M,N]
  - Proof

$$F[k,l]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$F[k+M,l+N]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k+M}{M}m+\frac{l+N}{N}n\right)}$$

$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\cancelto{1}{e^{-j2\pi\left(\frac{M}{M}m+\frac{N}{N}n\right)}}$$

$$=F[k,l]$$

(In what follows: spatial coordinates=k,l, frequency coordinates: u,v)

2D DFT: Periodicity

- Periodicity

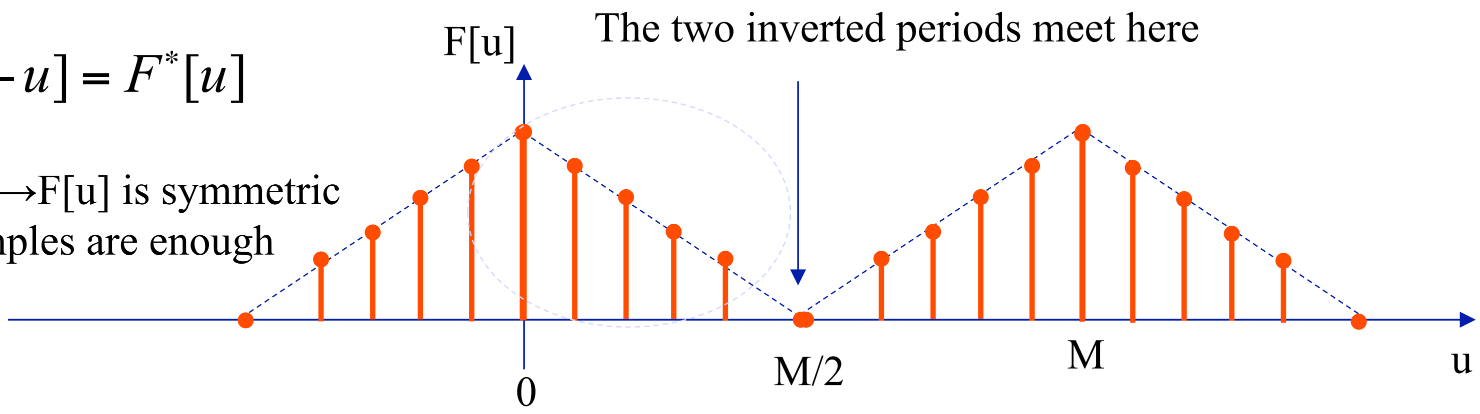
$$F[u,v] = F[u+mM,v] = F[u,v+nN] = F[u+mM,v+nN]$$

$$f[k,l] = f[k+mM,l] = f[k,l+nN] = f[k+mM,l+nN]$$

- This has important consequences on the implementation and energy compaction property
  - 1D

$$F[N-u] = F^*[u]$$

f[k] real→F[u] is symmetric
M/2 samples are enough



Periodicity: 1D

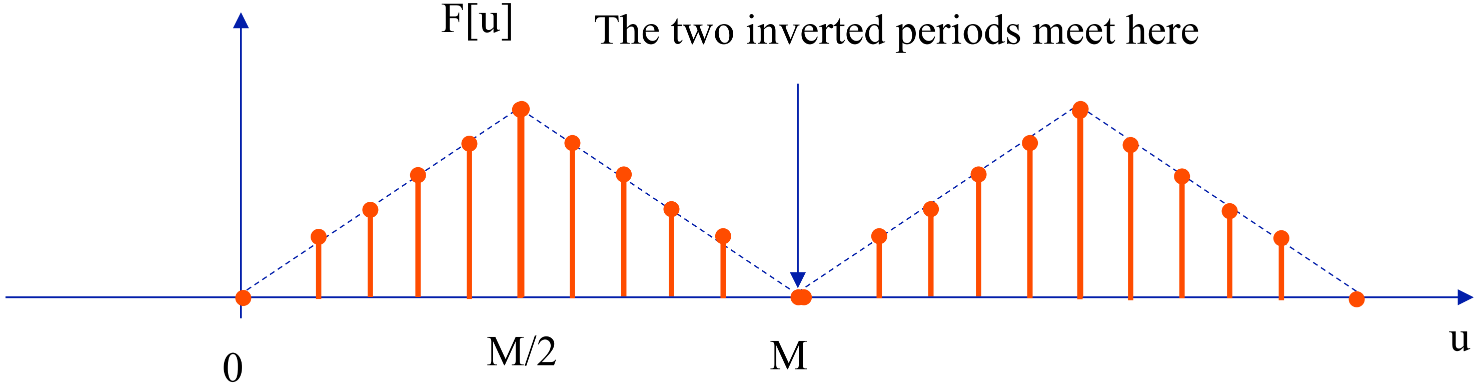
$$f[k] \Leftrightarrow F[u]$$

$$f[k]e^{j2\pi\frac{u_0 k}{M}} \Leftrightarrow F[u-u_0]$$

$$u_0 = \frac{M}{2} \rightarrow e^{j2\pi\frac{u_0 k}{M}} = e^{j2\pi\frac{Mk}{2M}} = e^{j\pi k} = (-1)^k$$

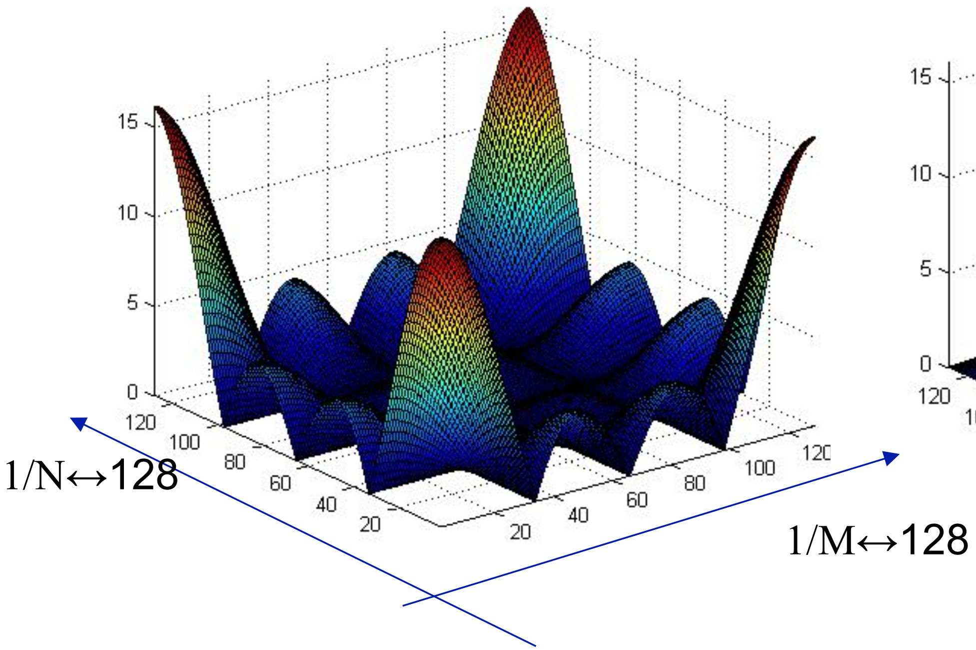
$$(-1)^k f[k] \Leftrightarrow F[u-\frac{M}{2}]$$

changing the sign of every other sample of the DFT puts F[0] at the center of the interval [0,M]

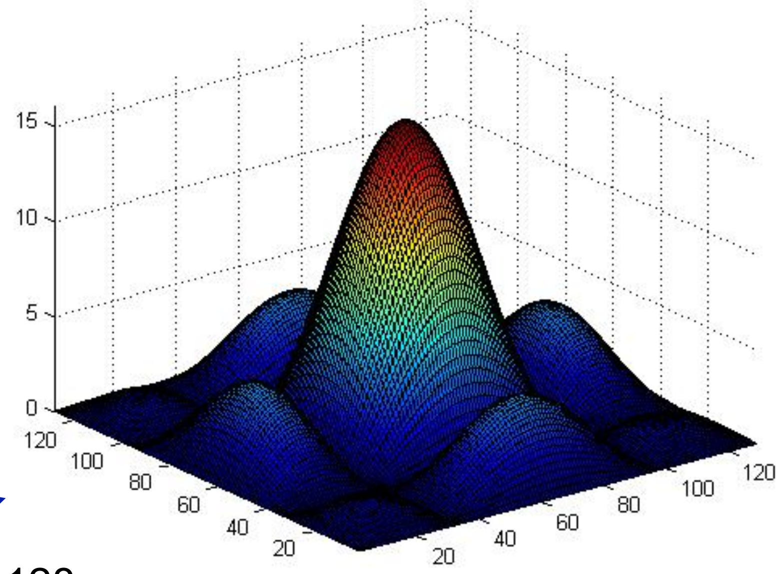


It is more practical to have one complete period positioned in [0, M-1]

Periodicity in 2D

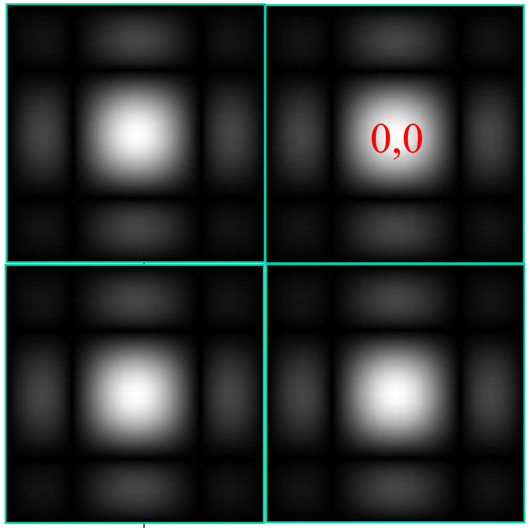


I 4 semiperiodi si incontrano ai vertici

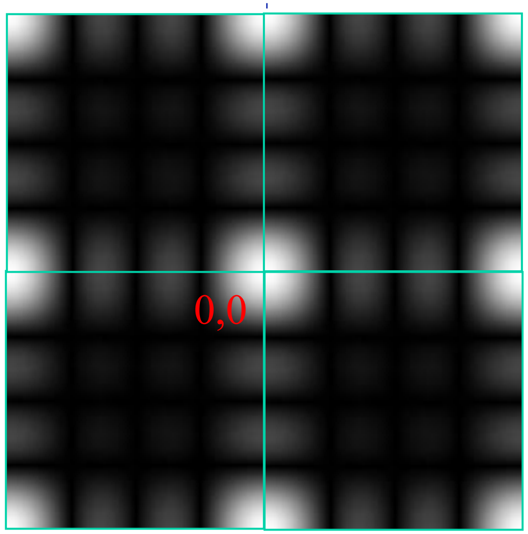


I 4 semiperiodi si incontrano al centro

Periodicity



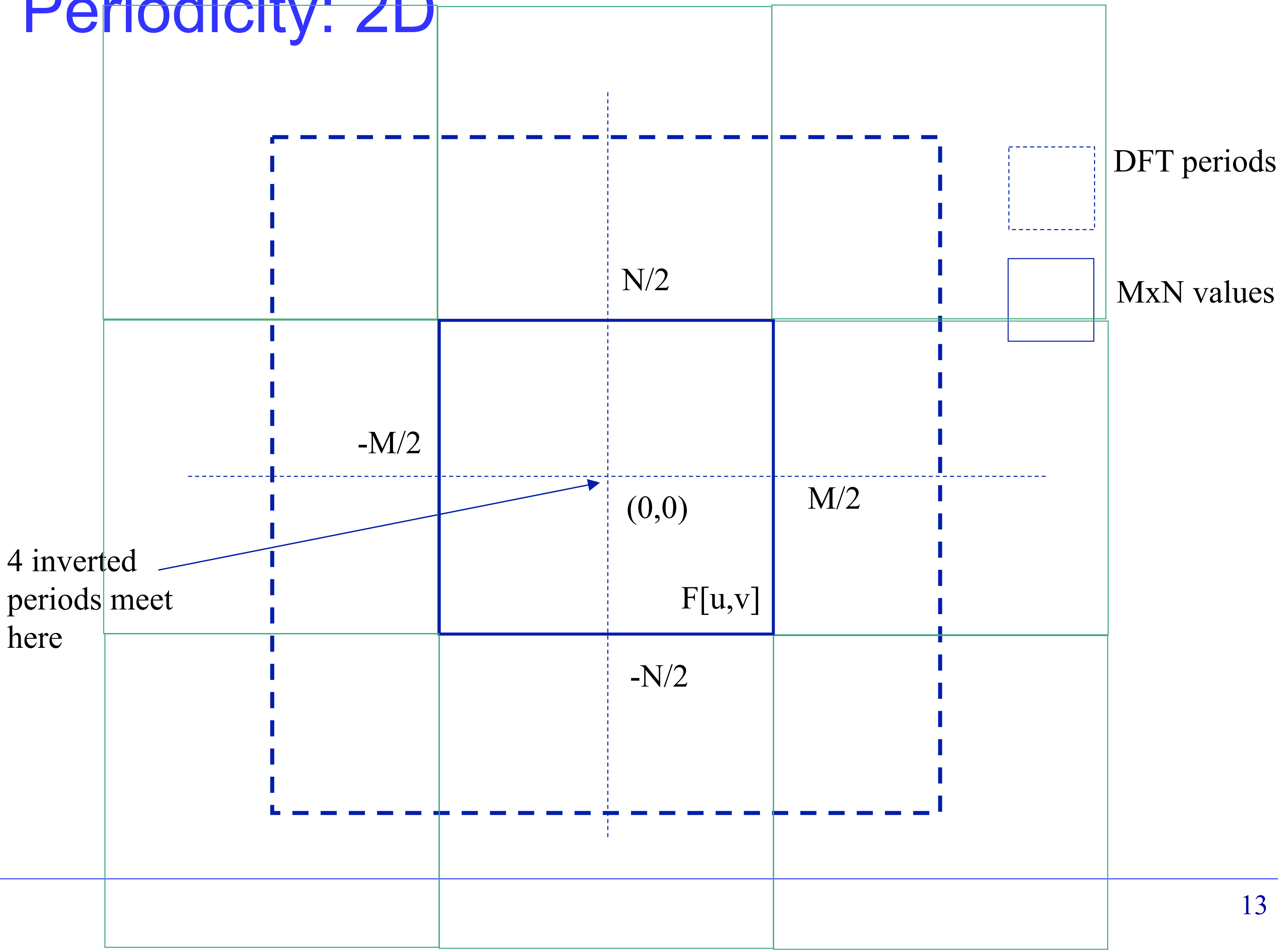
fftshift(fft2)



fft2

0,127=1/M,1/N

Periodicity: 2D

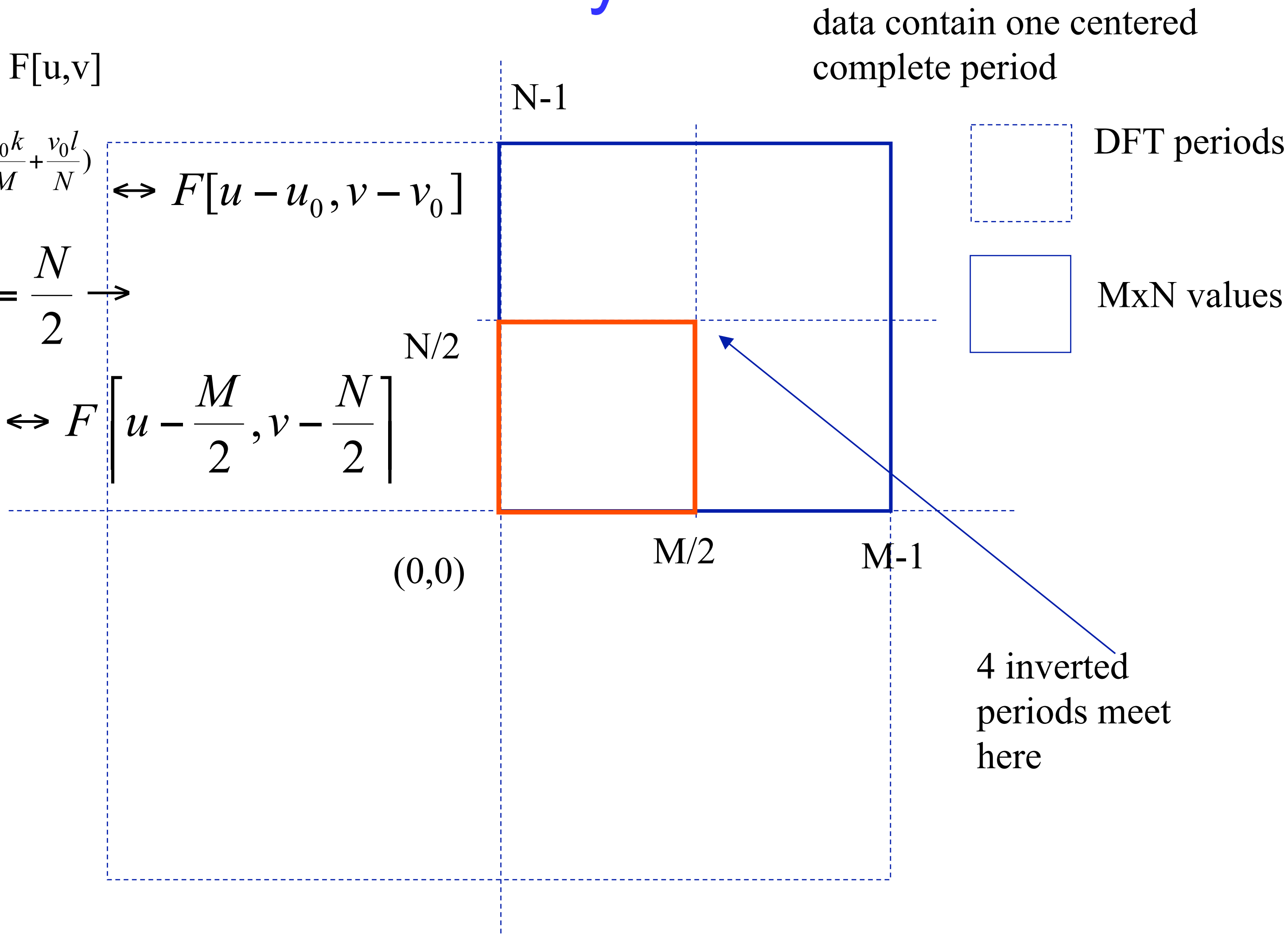


Periodicity: 2D

$$f[k,l]e^{j2\pi(\frac{u_0 k}{M}+\frac{v_0 l}{N})} \Leftrightarrow F[u-u_0, v-v_0]$$

$$u_0 = \frac{M}{2}, v_0 = \frac{N}{2} \rightarrow$$

$$(-1)^{k+l} f[k] \Leftrightarrow F\left[u-\frac{M}{2}, v-\frac{N}{2}\right]$$



Angle and phase spectra

$$F[u,v] = |F[u,v]| e^{j\Phi[u,v]}$$

$$|F[u,v]| = \left[\mathrm{Re}\{F[u,v]\}^2 + \mathrm{Im}\{F[u,v]\}^2 \right]^{1/2}$$ modulus (amplitude spectrum)

$$\Phi[u,v] = \arctan\left[\frac{\mathrm{Im}\{F[u,v]\}}{\mathrm{Re}\{F[u,v]\}}\right]$$ phase

$$P[u,v] = |F[u,v]|^2$$ power spectrum

For a real function

$$F[-u,-v] = F^*[u,v]$$ conjugate symmetric with respect to the origin

$$|F[-u,-v]| = |F[u,v]|$$

$$\Phi[-u,-v] = -\Phi[u,v]$$

Translation and rotation

$$f[m,n]e^{j2\pi\left(\frac{p}{M}k+\frac{q}{N}l\right)} \leftrightarrow F\left[k-p,l-q\right]$$

$$f\left[m-p,n-q\right] \leftrightarrow F\left[u,v\right]^{-j2\pi\left(\frac{p}{M}k+\frac{q}{N}l\right)}$$

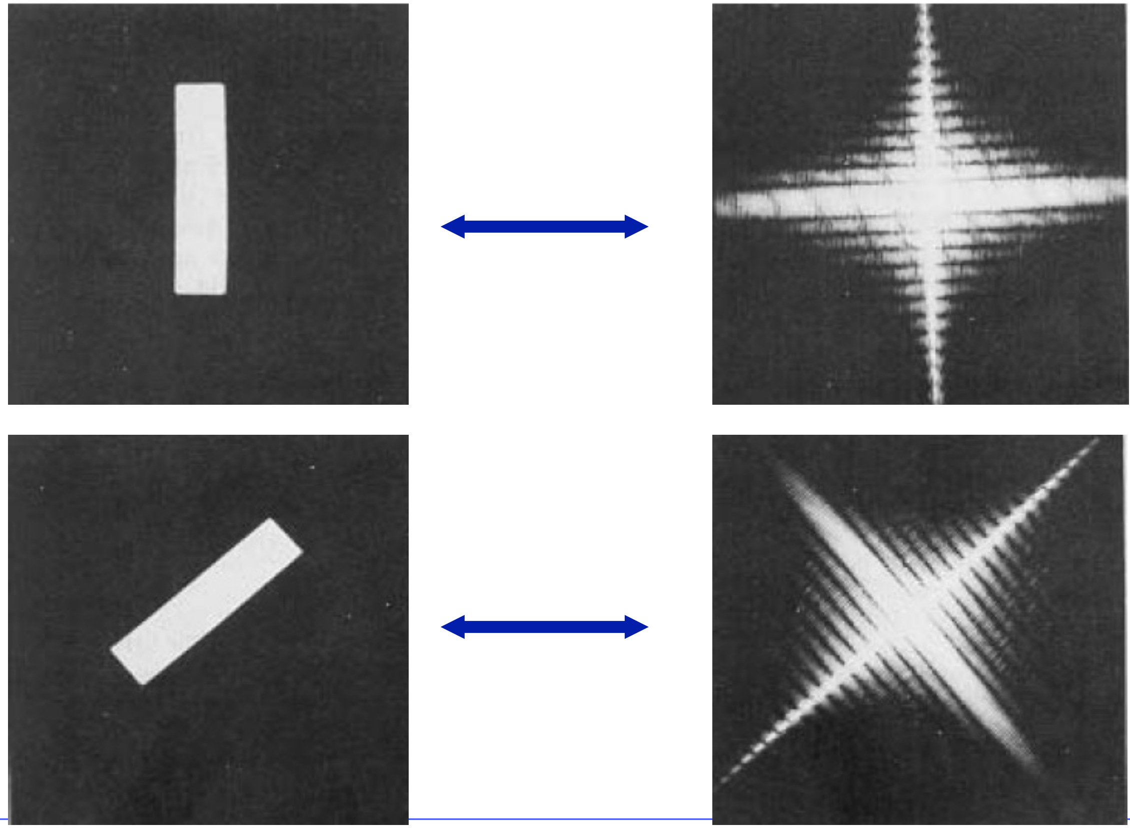
$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \qquad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

$$f\left[r,\vartheta+\vartheta_0\right] \leftrightarrow F\left[\omega,\varphi+\vartheta_0\right]$$

Rotations in spatial domain correspond equal rotations in Fourier domain

DFT Properties: (5) Rotation

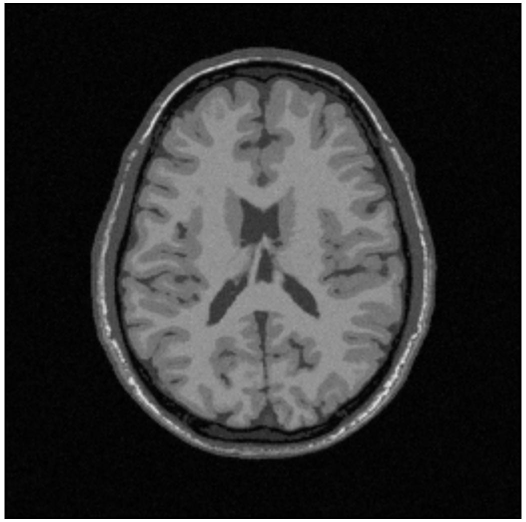
- Rotating f(x,y) by θ rotates F(u,v) by θ

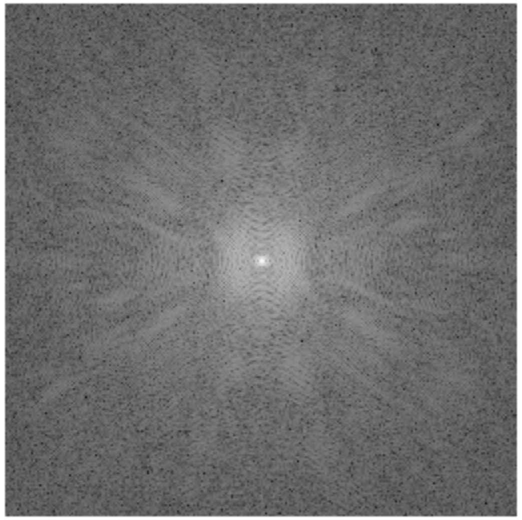


mean value

$$F[0,0] = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f[n,m]$$

DC coefficient





Separability

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

- inverse transform

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

  - Because the transform kernels are separable and symmetric, the two dimensional transforms can be computed as sequential row and column one-dimensional transforms.
  - The basis functions of the transform are complex exponentials that may be decomposed into sine and cosine components.

2D DFT: summary

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

2D DFT: summary

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

2D DFT: summary

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

2D DFT: summary

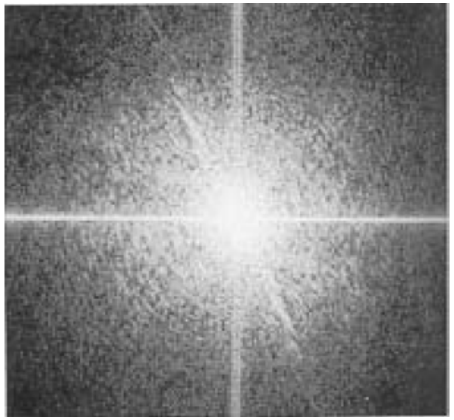
Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

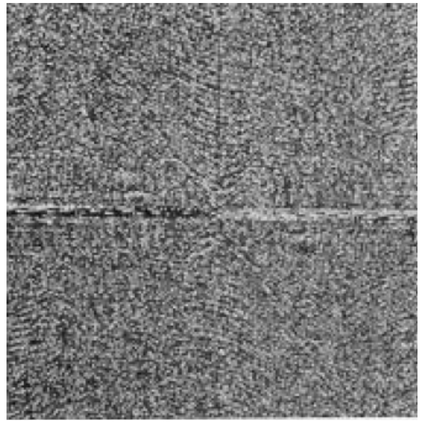
† Assumes that functions have been extended by zero padding.

Magnitude and Phase of DFT

- What is more important?



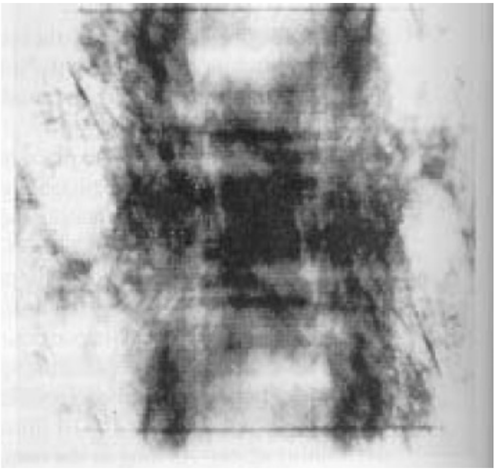
magnitude



phase

<u>Hint:</u> use inverse DFT to reconstruct the image using magnitude or phase only information

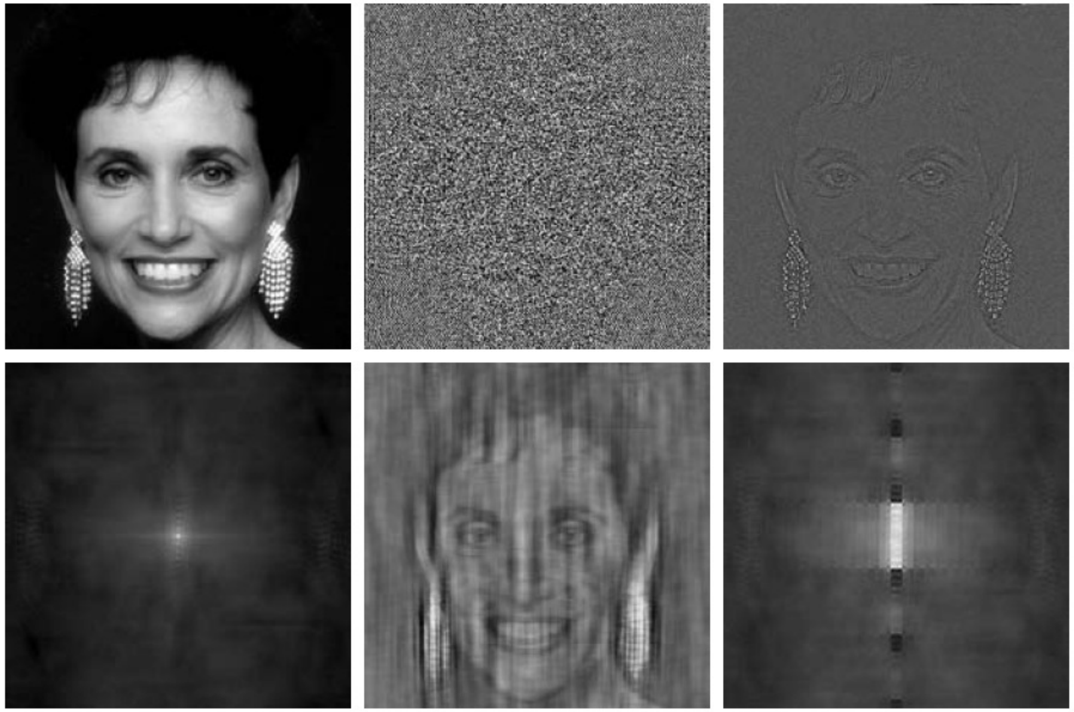
Magnitude and Phase of DFT (cont'd)



Reconstructed image using
magnitude only
(i.e., magnitude determines the
contribution of each component!)

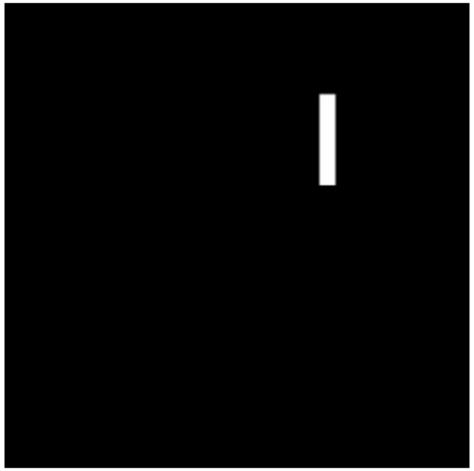
Reconstructed image using
phase only
(i.e., phase determines
which components are present!)

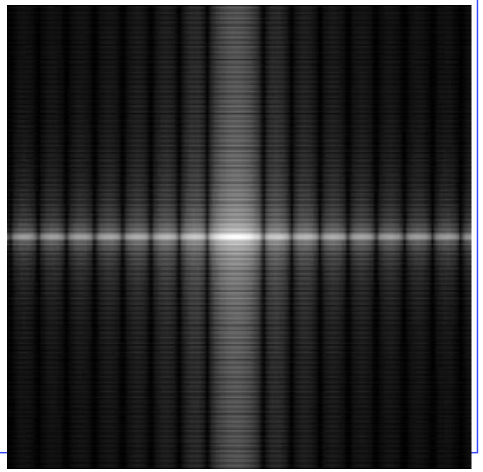
Magnitude and Phase of DFT (cont’d)



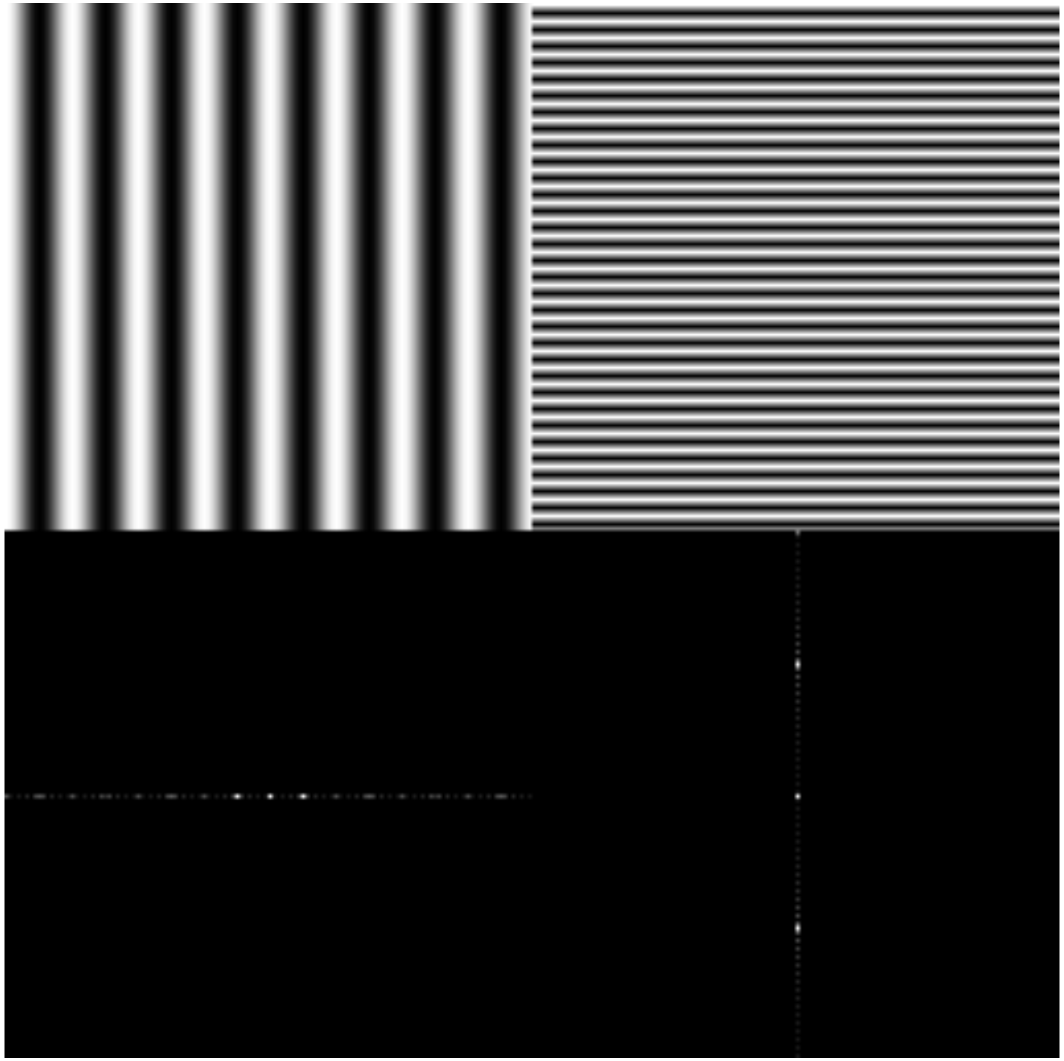
a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

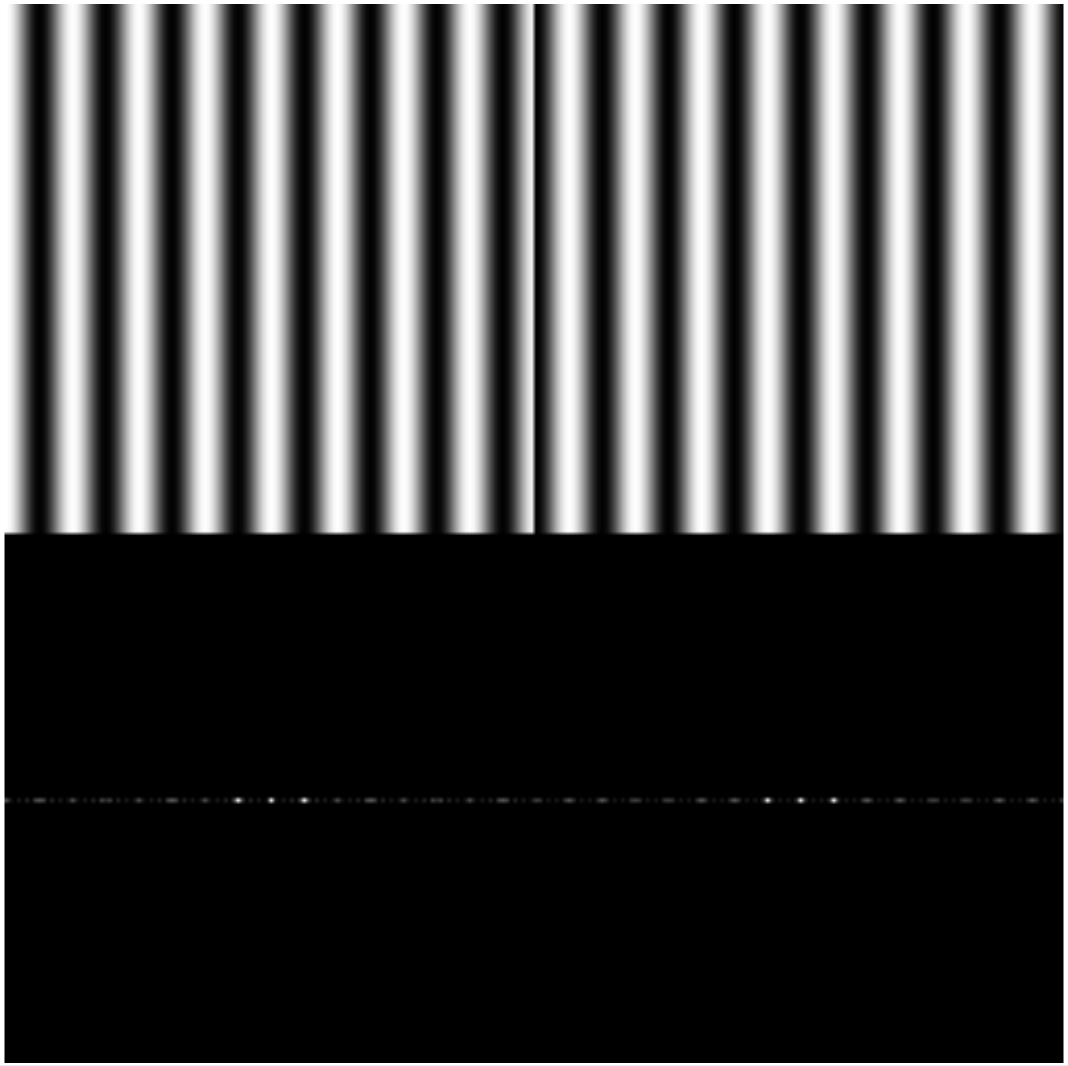




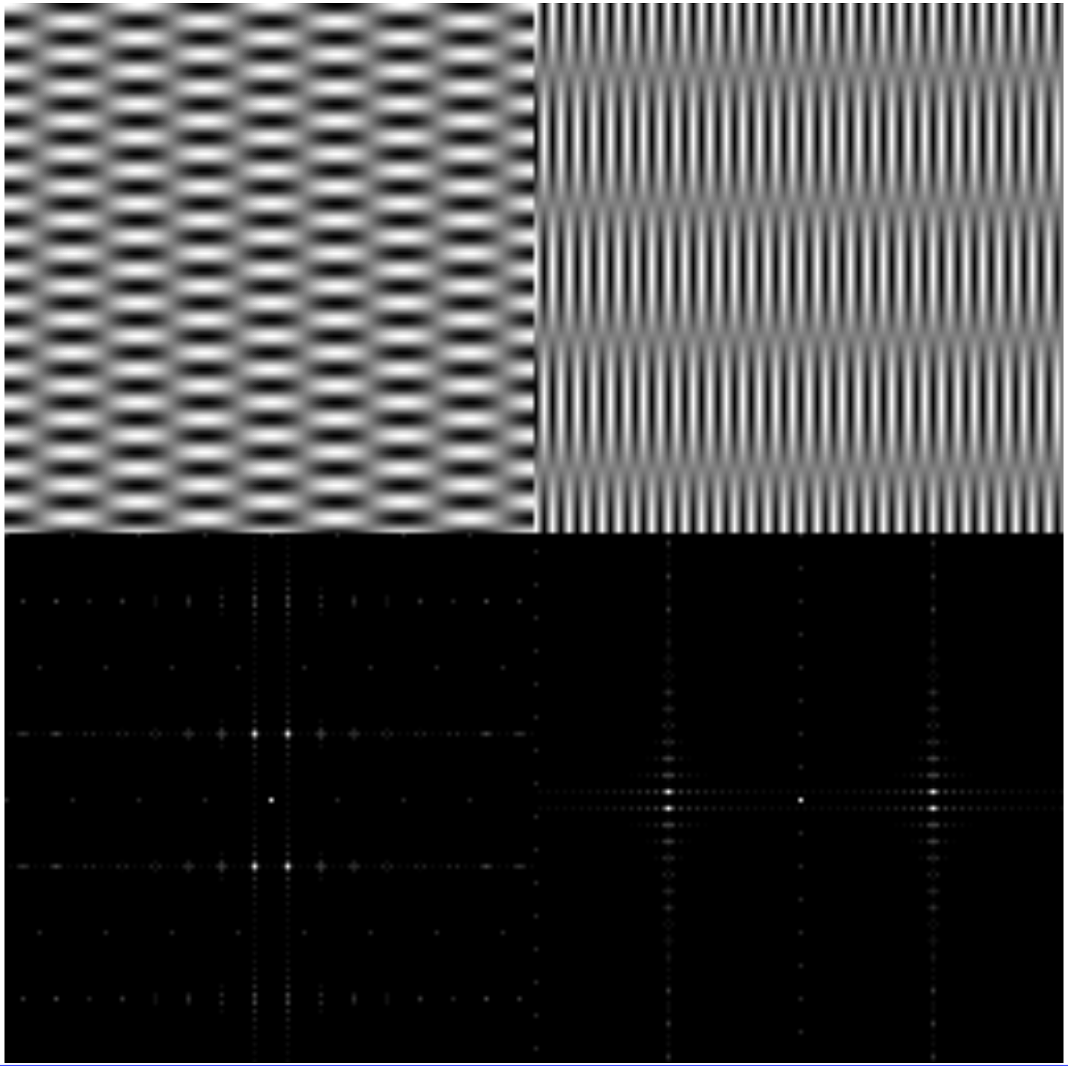
Ex. 1



Ex. 2

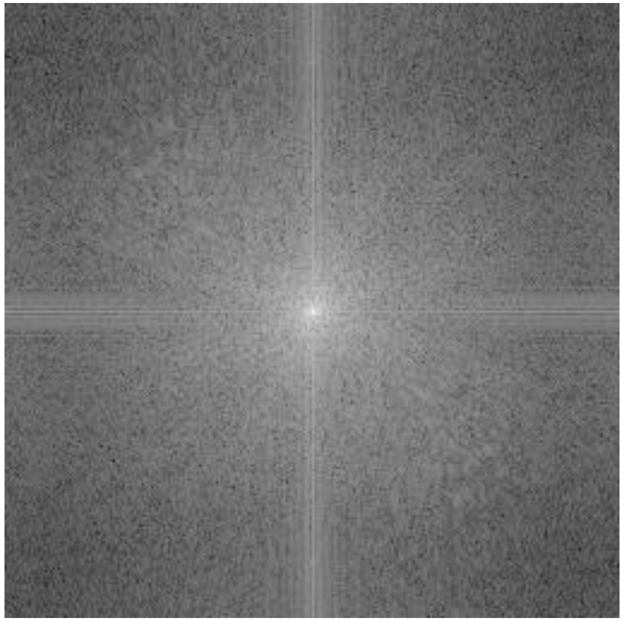


Ex. 3



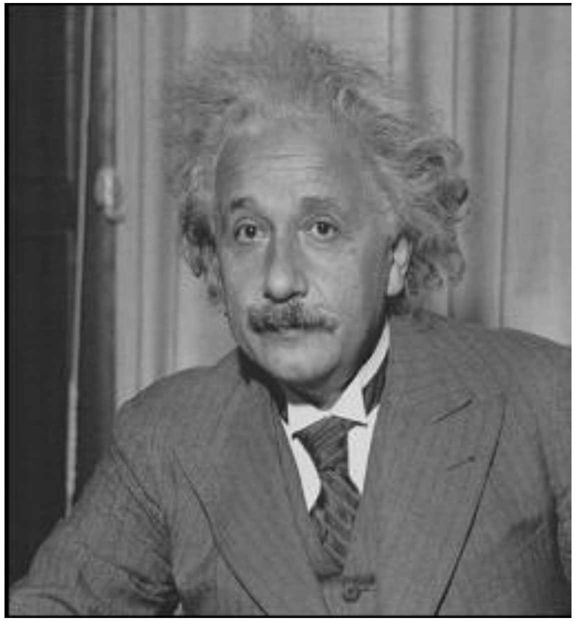
Magnitudes

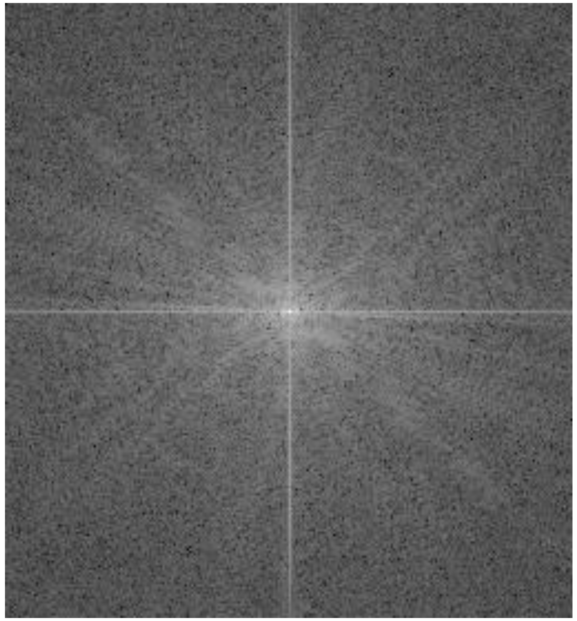
Margherita Hack



log amplitude of the spectrum

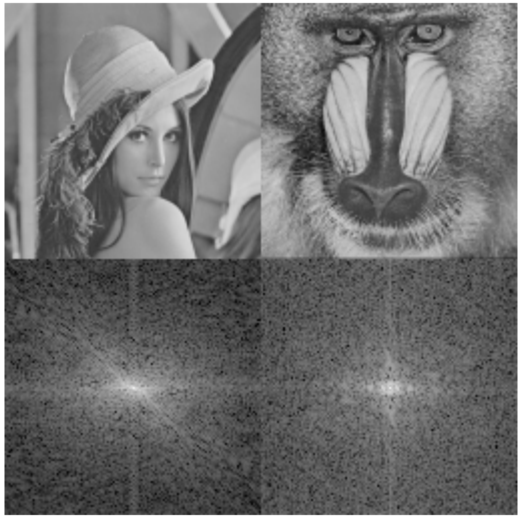
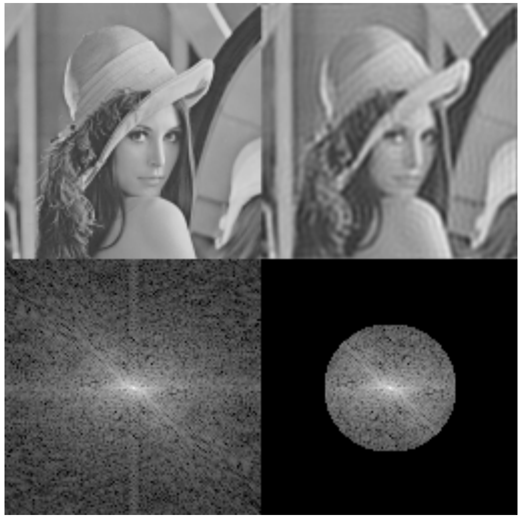
Einstein

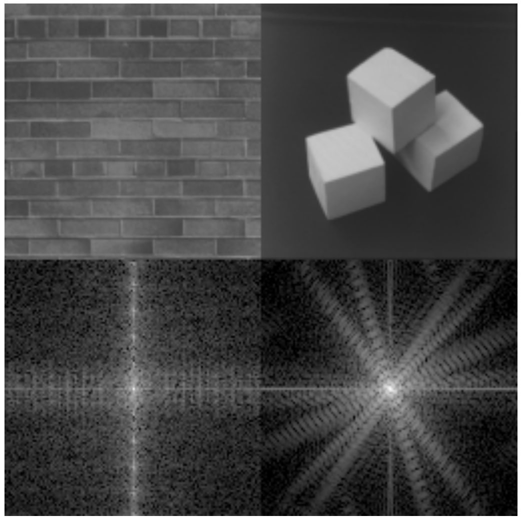
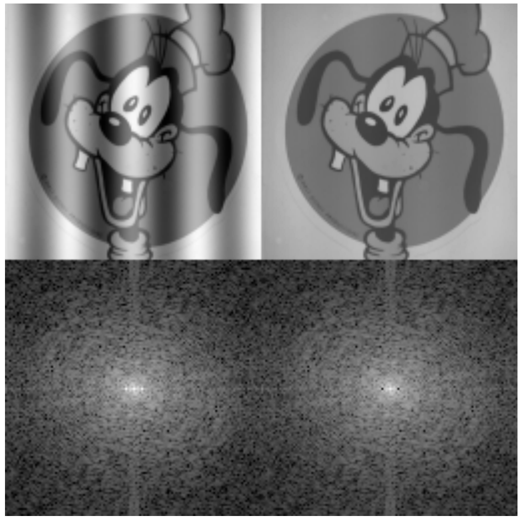




log amplitude of the spectrum

Examples

other formulations

2D Discrete Fourier Transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $l = 0,1,\ldots,N-1$

$k = 0,1,\ldots,M-1$

- Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- It is also possible to define DFT as follows

$$F[k,l] = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0, 1, \ldots, M-1$
$l = 0, 1, \ldots, N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- Or, as follows

$$F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D DFT

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$\mathcal{F}(u, v) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} F(j, k) \exp\left\{\frac{-2\pi i}{N}(uj + vk)\right\}$$

- inverse transform

$$F(j, k) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \mathcal{F}(u, v) \exp\left\{\frac{2\pi i}{N}(uj + vk)\right\}$$

2D DCT

Discrete Cosine Transform

2D DCT

- based on most common form for 1D DCT

$$C(u)=\alpha(u)\sum_{x=0}^{N-1} f(x)\cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

u,x=0,1,…, N-1

$$f(x)=\sum_{u=0}^{N-1}\alpha(u)C(u)\cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

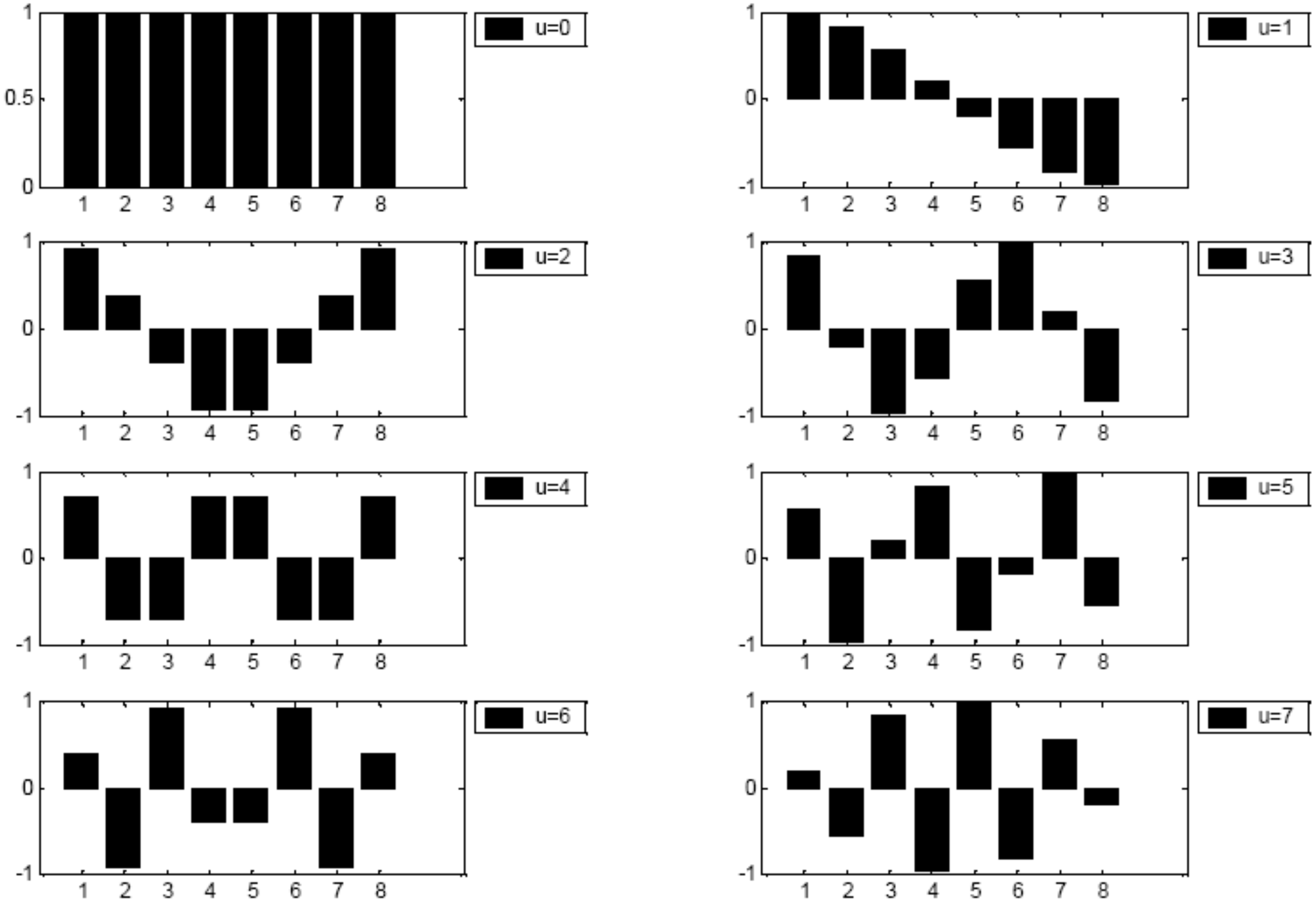
$$\alpha(u)=\begin{cases}\sqrt{\frac{1}{N}} & for \quad u=0\\ \sqrt{\frac{2}{N}} & for \quad u\neq 0.\end{cases}$$

$$C(u=0)=\sqrt{\frac{1}{N}}\sum_{x=0}^{N-1} f(x).$$

"mean" value

1D basis functions

Figure 1



Cosine basis functions are orthogonal

2D DCT

- Corresponding 2D formulation

direct $$C(u,v)=\alpha(u)\alpha(v)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

u,v=0,1,...., N-1

$$\alpha(u)=\begin{cases}\sqrt{\frac{1}{N}} & \text{for} \quad u=0\\ \sqrt{\frac{2}{N}} & \text{for} \quad u\neq 0.\end{cases}$$

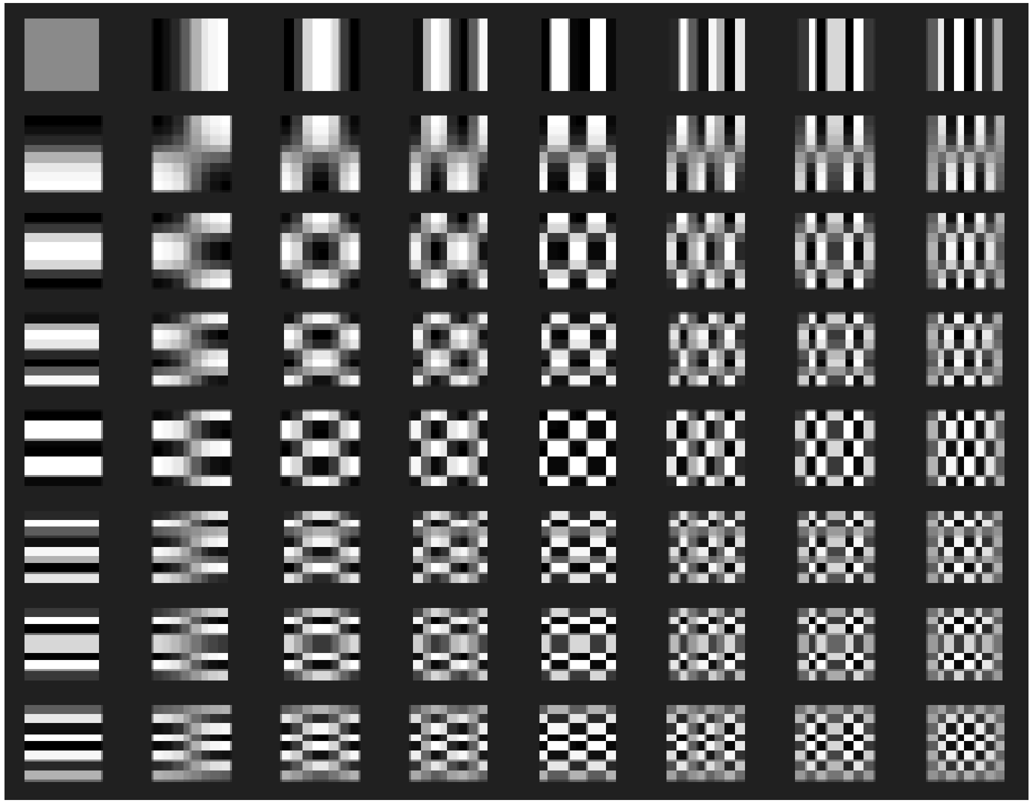
inverse $$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\alpha(u)\alpha(v)C(u,v)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

2D basis functions

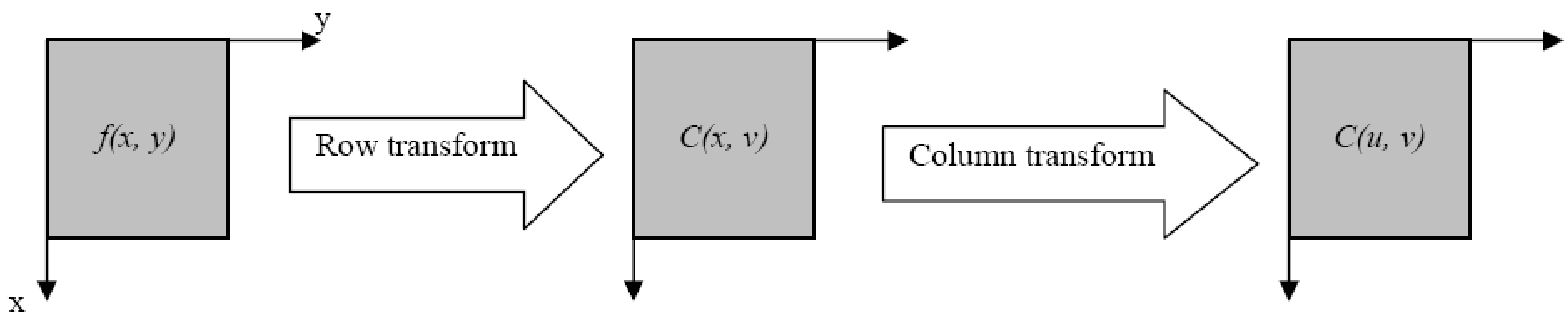
- The 2-D basis functions can be generated by multiplying the horizontally oriented 1-D basis functions (shown in Figure 1) with vertically oriented set of the same functions.
- The basis functions for $N = 8$ are shown in Figure 2.
  - The basis functions exhibit a progressive increase in frequency both in the vertical and horizontal direction.
  - The top left basis function assumes a constant value and is referred to as the *DC coefficient*.

2D DCT basis functions

Figure 2



Separability



The inverse of a multi-dimensional DCT is just a separable product of the inverse(s) of the corresponding one-dimensional DCT , e.g. the one-dimensional inverses applied along one dimension at a time

Separability

- Symmetry
  - Another look at the row and column operations reveals that these operations are functionally identical. Such a transformation is called a *symmetric transformation*.
  - A separable and symmetric transform can be expressed in the form

$$T = AfA$$

  - where A is a NxN symmetric transformation matrix which entries a(i,j) are given by

$$a(i,j) = \alpha(j)\sum_{j=0}^{N-1}\cos\left[\frac{\pi(2j+1)i}{2N}\right],$$

    - This is an extremely useful property since it implies that the transformation matrix can be pre computed offline and then applied to the image thereby providing orders of magnitude improvement in computation efficiency.

Computational efficiency

- Computational efficiency
  - Inverse transform

$$f = A^{-1}TA^{-1}.$$

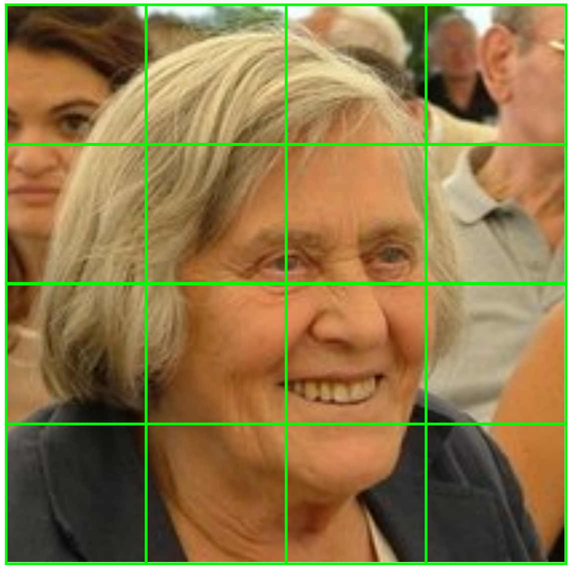
  - DCT basis functions are orthogonal. Thus, the inverse transformation matrix of A is equal to its transpose i.e. $A^{-1} = A^{T}$. This property renders some reduction in the pre-computation complexity.

Block-based implementation

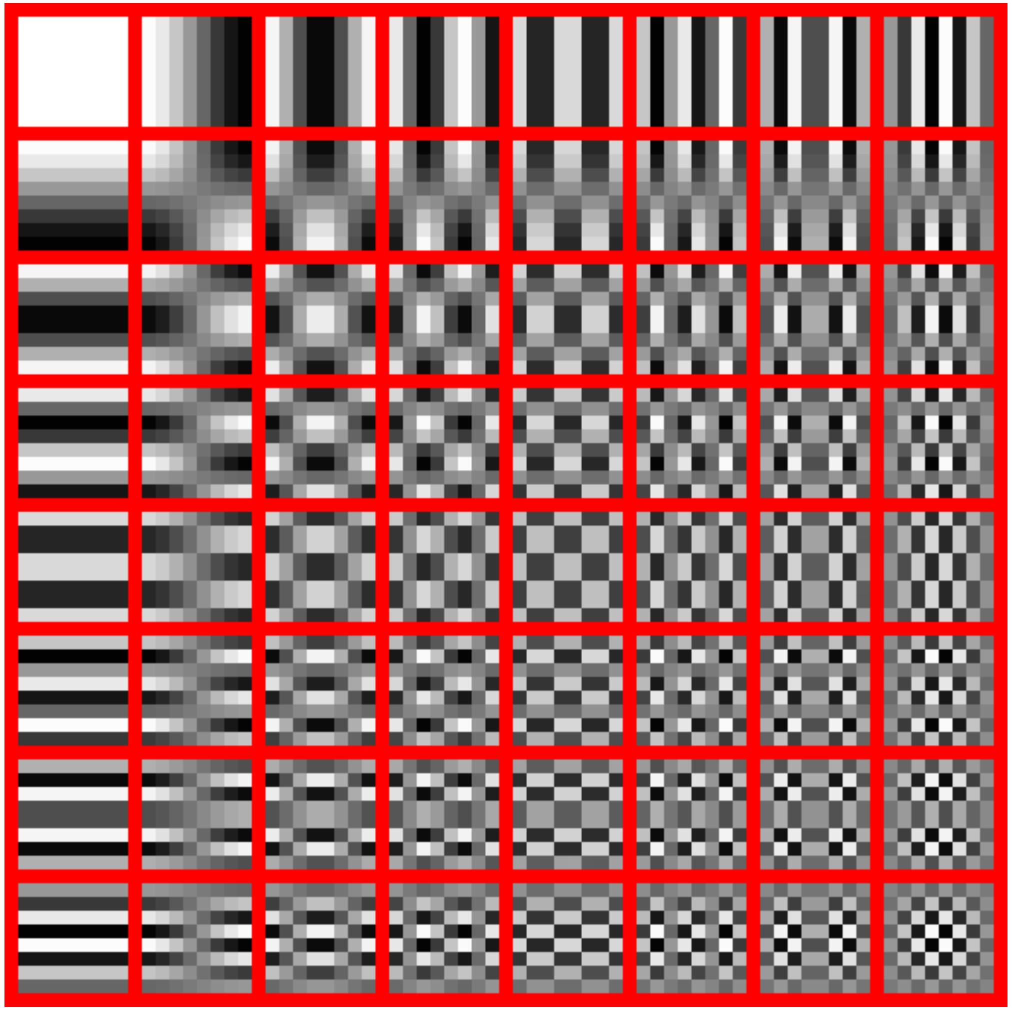
Block-based transform

Block size
N=M=8

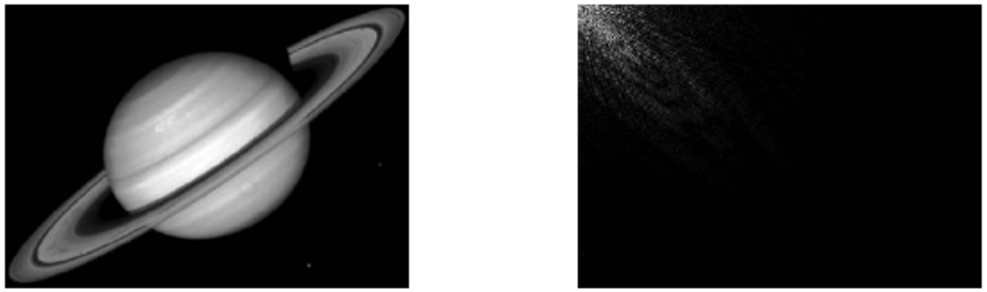
The source data (8x8) is transformed to a linear combination of these 64 frequency squares.



Basis function

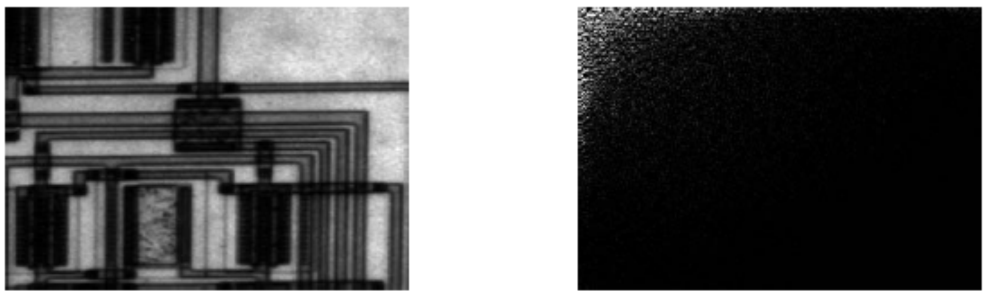


Energy compaction



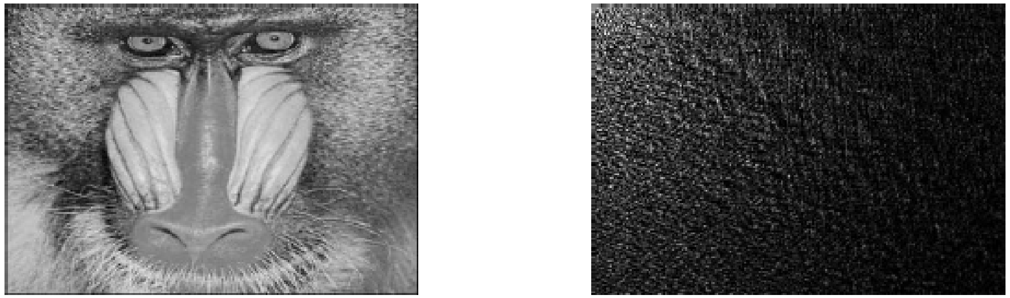
(a)

(b)

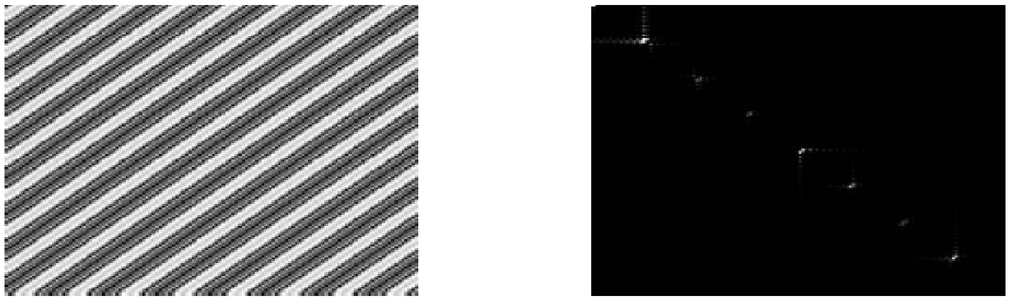


Energy compaction

(d)



(e)



Appendix

- Eulero's formula

$$A(j,k;u,v) = \exp\left\{\frac{-2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} - i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$

$$B(j,k;u,v) = \exp\left\{\frac{2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} + i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$