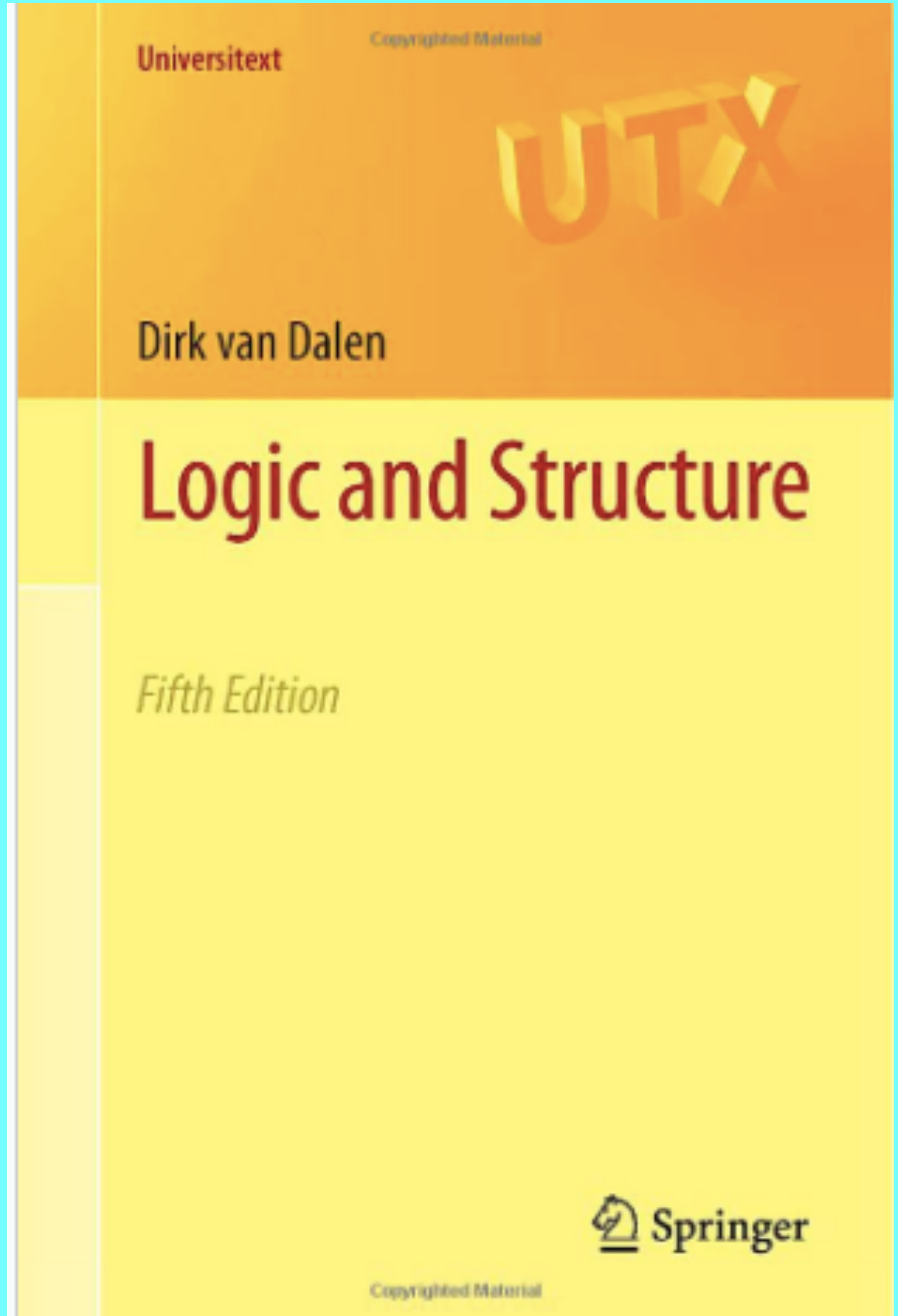
Propositional Logic

Libro di Testo



Lettura aggiuntiva

language of propositional logic

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\wedge, \vee, \rightarrow, \neg, \leftrightarrow, \perp,$

(iii) auxiliary symbols : (,).

$AT = \{p_0, p_1, p_2, \ldots,\} \cup \{\perp\}$

$\wedge$	**and**
$\vee$	**or**
$\rightarrow$	**if ...., then ...**
$\neg$	**not**
$\leftrightarrow$	**iff**
$\perp$	**falsity**

The set PROP of propositions is the smallest set X with the properties

(i) $p_i \in X (i \in N)$, $\perp \in X$,

(ii) $\phi, \psi \in X \Rightarrow (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi) \in X$,

(iii) $\phi \in X \Rightarrow (\neg \phi) \in X$.

PROP is well defined? (PROP $\neq \varnothing$?)

The set PROP of propositions is the **smallest** set X with the properties

(i) $p_i \in X (i \in N)$, $\bot \in X$,

(ii) $\phi, \psi \in X \Rightarrow (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi), (\phi \leftrightarrow \psi) \in X$,

(iii) $\phi \in X \Rightarrow (\neg \phi) \in X$.

$\neg \to \bot \notin \text{PROP}$

Suppose $\neg \to \bot \in \text{PROP}$.

$\mathbf{Y} = \text{PROP} - \{\neg \to \bot\}$ also satisfies (i), (ii) and (iii).

- $\bot, p_i \in \mathbf{Y}$.

- $\phi, \psi \in \mathbf{Y} \Rightarrow \phi, \psi \in \text{PROP} \Rightarrow (\phi \circ \psi) \in \text{PROP}$.

  $(\phi \circ \psi) \neq \neg \to \bot \Rightarrow (\phi \circ \psi) \in \mathbf{Y}$.

- $\phi \in \mathbf{Y} \Rightarrow \phi \in \text{PROP} \Rightarrow (\neg \phi) \in \text{PROP}$.

  $(\neg \phi) \neq \neg \to \bot \Rightarrow (\neg \phi) \in \mathbf{Y}$.

- PROP is not the smallest set satisfying (i), (ii) and (iii)!!! **impossible**

Theorem

Let h: $\mathbb{N} \times A \to A$ and $c \in A$.

There exist one and only one function $f : \mathbb{N} \to A$ t.c.:

1. $f(0)=c$
2. $\forall n \in \mathbb{N}$, $f(n+1)=h(n,f(n))$

the proof is difficult

$\square \in \{\wedge, \vee, \to\}$

Theorem 1.1.6 (Definition by Recursion) *Let mappings $H_\square : A^2 \to A$ and $H_\neg : A \to A$ be given and let H_{at} be a mapping from the set of atoms into A, then there exists exactly one mapping $F : PROP \to A$ such that*

$$\begin{cases} F(\varphi) & = H_{at}(\varphi) \text{ for } \varphi \text{ atomic,} \\ F((\varphi \square \psi)) & = H_\square(F(\varphi), F(\psi)), \\ F((\neg\varphi)) & = H_\neg(F(\varphi)). \end{cases}$$

exercise

Theorem 1.1.3 (Induction Principle)

Let A be a property, then $A(\phi)$ holds for all $\phi \in PROP$ if

(i) $A(p_i)$, for all i,and $A(\perp)$,

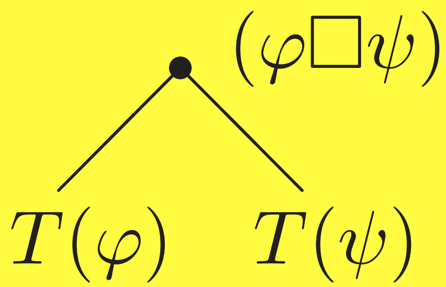
(ii) $A(\phi), A(\psi) \Rightarrow A((\phi \rightarrow \psi))$,

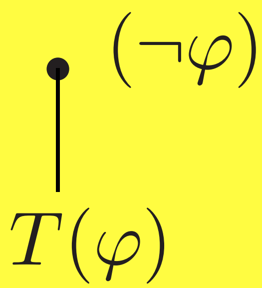
(iii) $A(\phi), A(\psi) \Rightarrow A((\phi \wedge \psi))$,

(iv) $A(\phi), A(\psi) \Rightarrow A((\phi \vee \psi))$,

(v) $A(\phi) \Rightarrow A((\neg \phi))$.

$T(\varphi) \quad = \quad$  for atomic φ

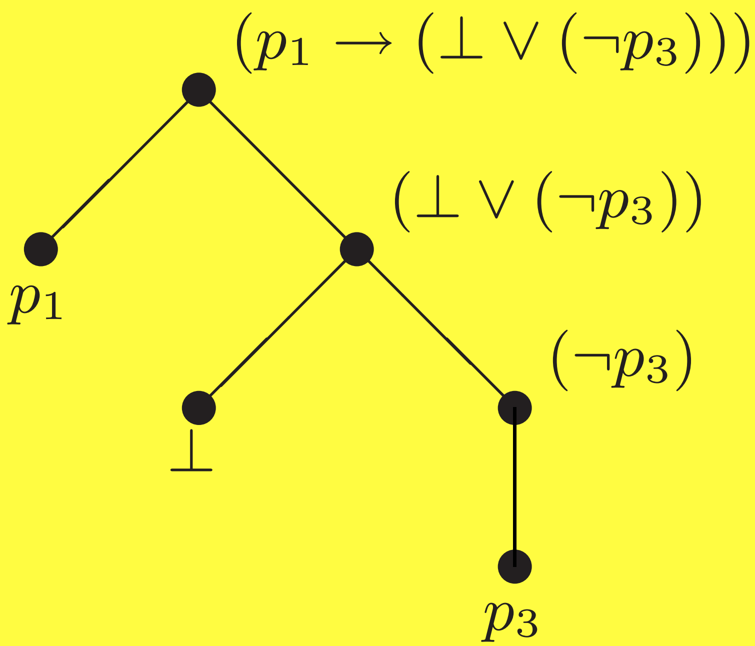
$T((\varphi \square \psi)) \quad = \quad$ 

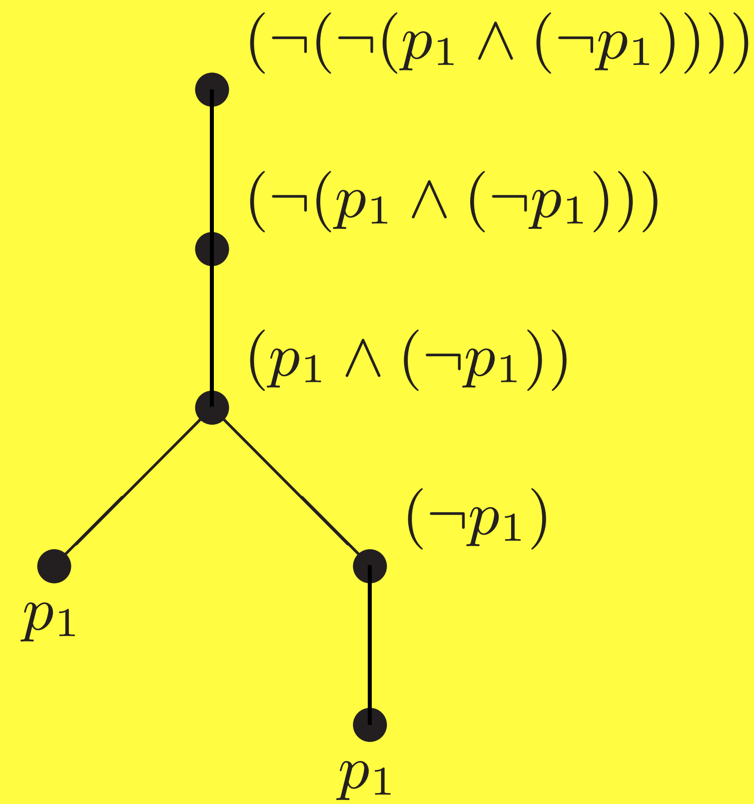
$T((\neg\varphi)) \quad = \quad$ 

Examples. $T\big((p_1 \to (\perp \vee (\neg p_3)))\big)$; $T\big(\neg(\neg(p_1 \wedge (\neg p_1)))\big)$

?

Examples. $T\big((p_1 \to (\bot \vee (\neg p_3)))\big)$; $T\big(\neg(\neg(p_1 \wedge (\neg p_1)))\big)$





SEMANTICS

truth table

$\wedge$	0	1
0	0	0
1	0	1

Definition 1

A mapping $v : PROP \to \{0, 1\}$ is a valuation if

$v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$,

$v(\phi \vee \psi) = \max(v(\phi), v(\psi))$,

$v(\phi \to \psi)=0 \Leftrightarrow v(\phi)=1$ and $v(\psi)=0$,

$v(\phi \leftrightarrow \psi)=1 \Leftrightarrow v(\phi)=v(\psi)$,

$v(\neg\phi) = 1 - v(\phi)$

$v(\perp) = 0$.

Definition 2

A mapping $v : PROP \to \{0, 1\}$ is a valuation if

$v(\phi \wedge \psi) = 1 \Leftrightarrow v(\phi)=1$ and $v(\psi)=1$

$v(\phi \vee \psi) =1 \Leftrightarrow v(\phi)=1$ or $v(\psi)=1$

$v(\phi \to \psi)=1 \Leftrightarrow v(\phi)=0$ or $v(\psi)=1$,

$v(\phi \leftrightarrow \psi)=1 \Leftrightarrow v(\phi)=v(\psi)$,

$v(\neg\phi) = 1 \Leftrightarrow v(\phi)=0$

$v(\perp) = 0$.

the two definitions are equivalent

Theorem

v: **AT**$\rightarrow$ {0, 1} s.t. $v(\perp) = 0$ (assignment for atoms)

$\Rightarrow$

there exists a unique valuation **$[\cdot]_v$:PROP$\rightarrow$\{0,1\}**

such that $[\phi]_v = v(\phi)$ for each $\phi \in$ **AT**

Lemma If v, w are two assignments for atoms s.t. for all p_i occurring in ϕ, $v(p_i) = w(p_i)$, then $[\phi]_v = [\phi]_w$.

Definition

- ϕ is a **tautology** if $[\phi]_v = 1$ for all valuations v,
- $\models \phi$ stands for 'ϕ is a tautology',
- let Γ be a set of propositions,

$\Gamma \models \phi$ iff for all v: $([\psi]_v = 1$ for all $\psi \in \Gamma) \Rightarrow [\phi]_v = 1$.

SUBSTITUTION

$$\varphi[\psi/p] = \begin{cases} \psi \text{ if } \varphi = p \\ \varphi \text{ if } \varphi =/= p \text{ if } \varphi \text{ atomic} \end{cases}$$

$(\phi_1 \square \phi_2)[\psi/p] = (\phi_1[\psi/p] \square \phi_2[\psi/p])$

$(\neg\phi)[\psi/p] = (\neg\phi[\psi/p])$

Substitution Theorem

- If $\models \phi_1 \leftrightarrow \phi_2$, then $\models \psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]$, where p is an atom.
- $[\phi_1 \leftrightarrow \phi_2]_v \leq [\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]]_v$
- $\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p])$

tautologies

➡ $(\phi \vee \psi) \vee \sigma \leftrightarrow \phi \vee (\psi \vee \sigma)$ $(\phi \wedge \psi) \wedge \sigma \leftrightarrow \phi \wedge (\psi \wedge \sigma)$

associativity

➡ $\phi \vee \psi \leftrightarrow \psi \vee \phi$ $\phi \wedge \psi \leftrightarrow \psi \wedge \phi$

commutativity

➡ $\phi \vee (\psi \wedge \sigma) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \sigma)$ $\phi \wedge (\psi \vee \sigma) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \sigma)$

distributivity

➡ $\neg(\phi \vee \psi) \leftrightarrow \neg\phi \wedge \neg\psi$ $\neg(\phi \wedge \psi) \leftrightarrow \neg\phi \vee \neg\psi$

De Morgan's laws

➡ $\phi \vee \phi \leftrightarrow \phi$ $\phi \wedge \phi \leftrightarrow \phi$

idempotency

➡ $\neg\neg\phi \leftrightarrow \phi$

double negation law

De Morgan's law: $[\neg(\phi \vee \psi)]=1 \Leftrightarrow [\phi \vee \psi]=0 \Leftrightarrow [\phi]=[\psi]=0 \Leftrightarrow [\neg\phi]=[\neg\psi]=1 \Leftrightarrow [\neg\phi \wedge \neg\psi] = 1$.

So $[\neg(\phi \vee \psi)] = [\neg\phi \wedge \neg\psi]$ for all valuations, i.e $\models \neg(\phi \vee \psi) \leftrightarrow \neg\phi \wedge \neg\psi$.

$\models (\varphi \leftrightarrow \psi) \leftrightarrow (\varphi \to \psi) \wedge (\psi \to \varphi)$

$\models (\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$

$\models \varphi \vee \psi \leftrightarrow (\neg\varphi \to \psi)$

$\models \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$

$\models \varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$

$\models \neg\varphi \leftrightarrow (\varphi \to \bot)$,

$\models \bot \leftrightarrow \varphi \wedge \neg\varphi$.

$\approx\ \subseteq$ PROPxPROP : $\phi \approx \psi$ iff $\models \phi \leftrightarrow \psi$.

exercise $\approx$ is an equivalence relation on PROP