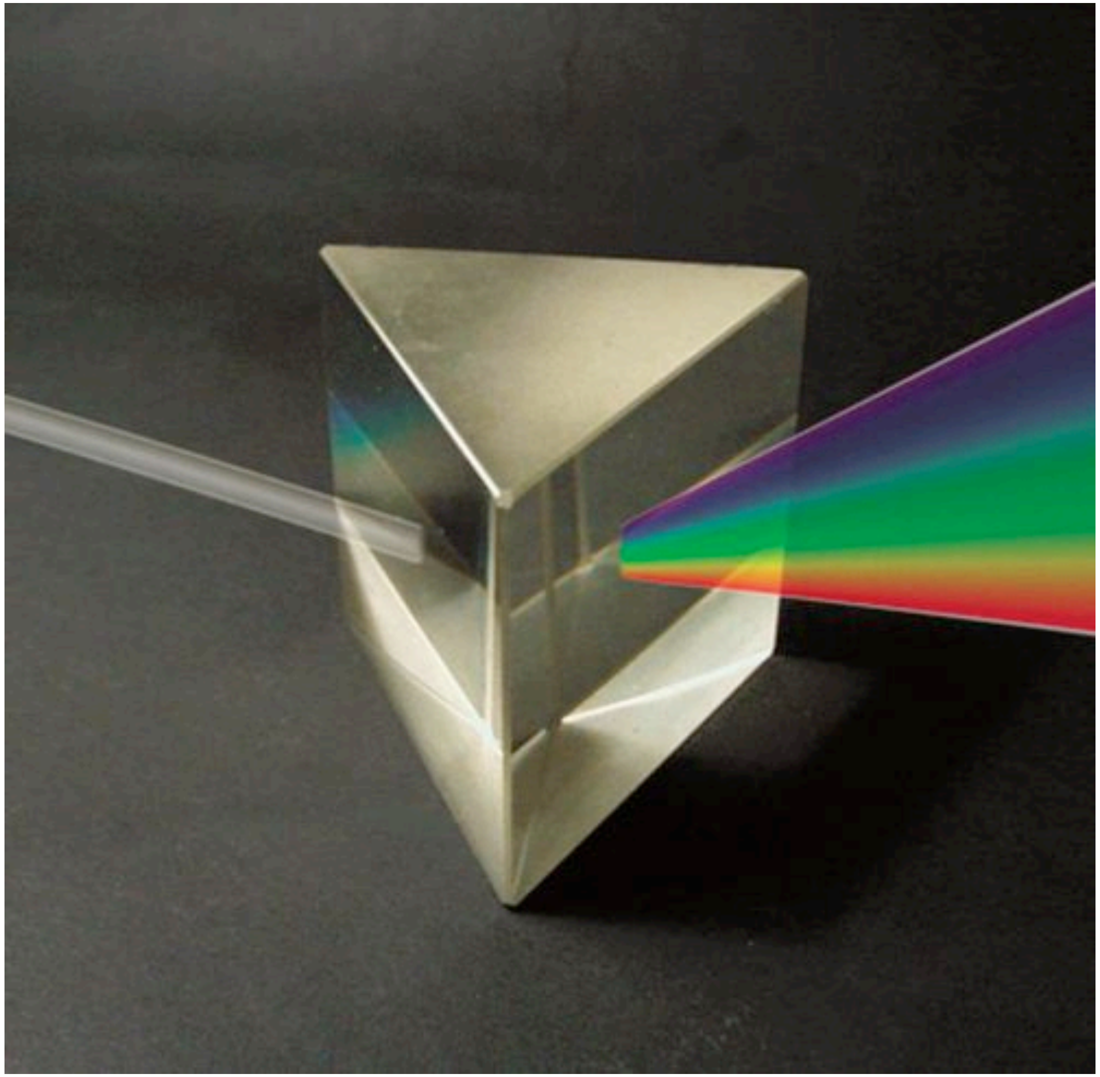
Fourier Transform

The idea

A signal can be interpreted as en electromagnetic wave. This consists of lights of different “color”, or frequency, that can be split apart usign an optic prism. Each component is a “monochromatic” light with sinusoidal shape.

Following this analogy, each signal can be decomposed into its “sinusoidal” components which represent its “colors”.

Of course these components in general do not correspond to visible monochromatic light. However, they give an idea of how fast are the changes of the signal.



Contents

- Signals as functions (1D, 2D)
  - Tools
- Continuous Time Fourier Transform (CTFT)
- Discrete Time Fourier Transform (DTFT)
- Discrete Fourier Transform (DFT)
- Discrete Cosine Transform (DCT)
- Sampling theorem

Fourier Transform

- Different formulations for the different classes of signals
  - Summary table: Fourier transforms with various combinations of continuous/ discrete time and frequency variables.
  - Notations:
    - CTFT: continuous time FT: t is real and f real (f=ω) (CT, CF)
    - DTFT: Discrete Time FT: t is discrete (t=n), f is real (f=ω) (DT, CF)
    - CTFS: CT Fourier Series (summation synthesis): t is real AND the function is periodic, f is discrete (f=k), (CT, DF)
    - DTFS: DT Fourier Series (summation synthesis): t=n AND the function is periodic, f discrete (f=k), (DT, DF)
    - P: periodical signals
    - T: sampling period
    - ω_s: sampling frequency ($\omega_s=2\pi/T$)
    - For DTFT: $T=1 \rightarrow \omega_s=2\pi$
- *This is a hint for those who are interested in a more exhaustive theoretical approach*

Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Mathematical Background: Complex Numbers

- A complex number **x** is of the form:

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

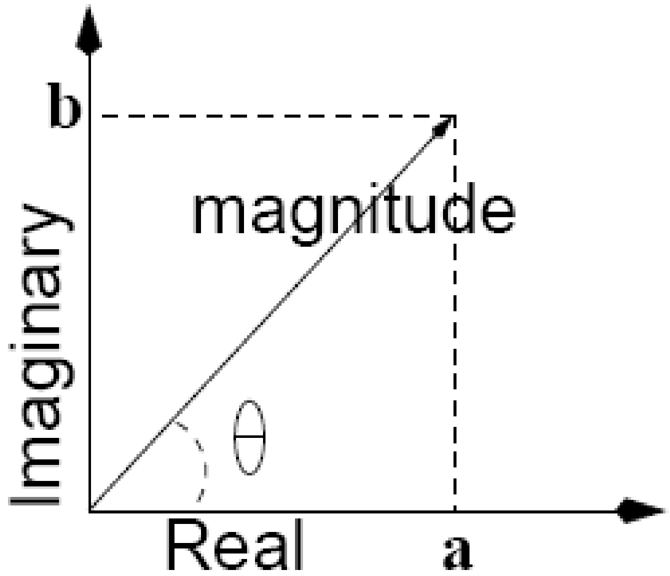
a: **real part**, b: **imaginary** part

- Addition $(a + jb) + (c + jd) = (a + c) + j(b + d)$

- Multiplication $(a + jb) \,.\, (c + jd) = (ac - bd) + j(ad + bc)$

Mathematical Background: Complex Numbers (cont' d)

- Magnitude-Phase (i.e.,vector) representation



Magnitude:

Phase:

$$|x| = \sqrt{a^2 + b^2}$$

$$\phi(x) = \tan^{-1}(b/a)$$

Phase – Magnitude notation:

$$x = |x|e^{j\phi(x)}$$

Mathematical Background: Complex Numbers (cont'd)

- Multiplication using magnitude-phase representation

$$xy = |x|e^{j\phi(x)} \,.\, |y|e^{j\phi(y)} = |x|\ |y|\ e^{j(\phi(x)+\phi(y))}$$

- Complex conjugate

$$x^* = a - jb$$

- Properties

$$\begin{aligned} &|x| = |x^*| \\ &\phi(x) = -\phi(x^*) \\ &xx^* = |x|^2 \end{aligned}$$

Mathematical Background: Complex Numbers (cont' d)

- Euler' s formula

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

- Properties

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2i}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Mathematical Background: Sine and Cosine Functions

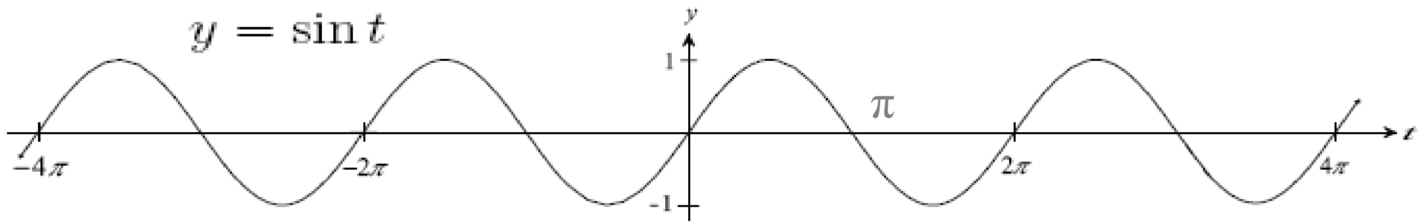
- Periodic functions
- General form of sine and cosine functions:

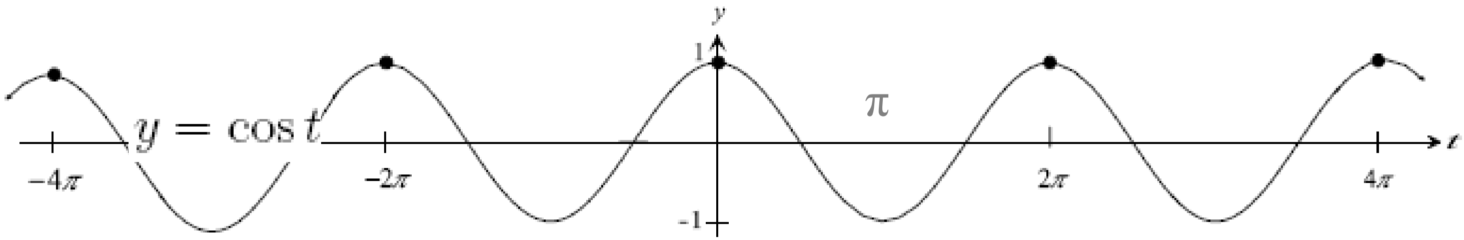
$$y(t) = A\sin[a(t+b)] \qquad y(t) = A\cos[a(t+b)]$$

$\lvert A\rvert$	amplitude
$\frac{2\pi}{\lvert a\rvert}$	period
b	phase shift

Mathematical Background: Sine and Cosine Functions

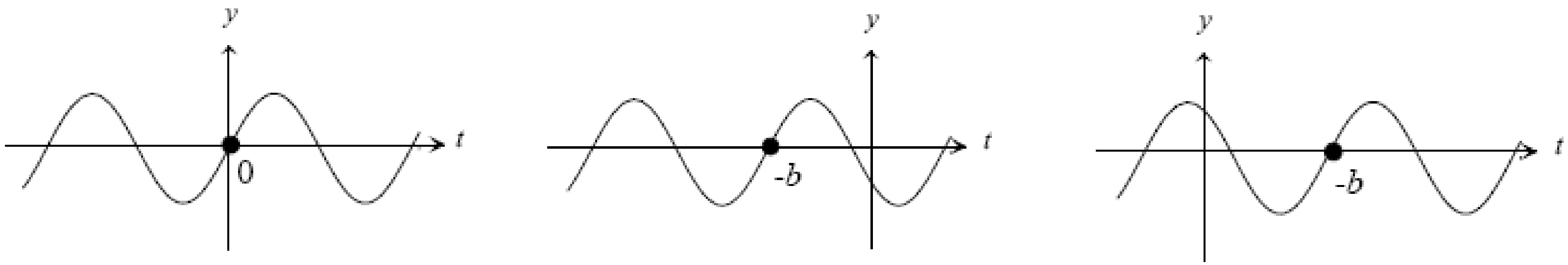
Special case: A=1, b=0, a=1





Mathematical Background: Sine and Cosine Functions (cont’d)

- Shifting or translating the sine function by a const b



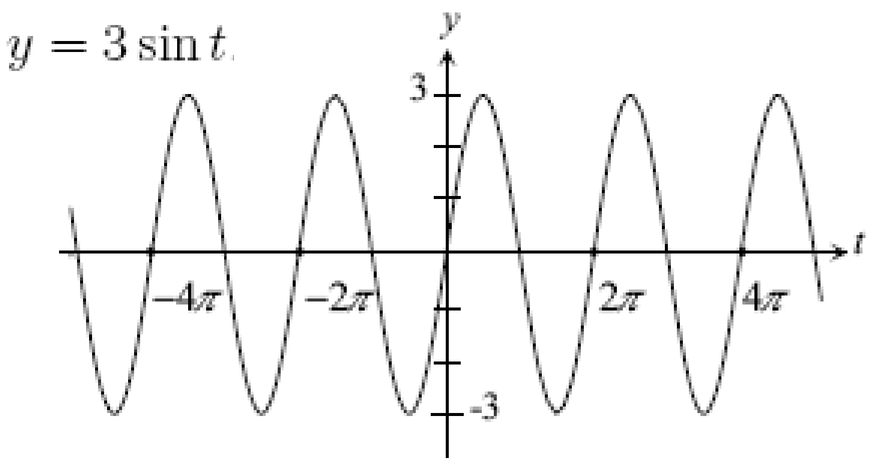
(a) $y = \sin t$ (b) $y = \sin(t+b)$, $b > 0$ (c) $y = \sin(t+b)$, $b < 0$

<u>Note:</u> cosine is a shifted sine function:

$$\cos(t) = \sin(t + \frac{\pi}{2})$$

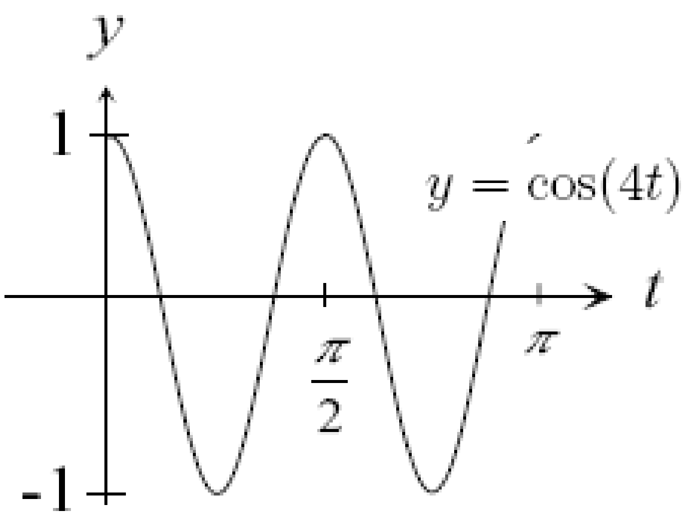
Mathematical Background: Sine and Cosine Functions (cont' d)

- Changing the amplitude A



Mathematical Background: Sine and Cosine Functions (cont' d)

- Changing the period $T=2\pi/|\alpha|$
  consider A=1, b=0: $y=\cos(\alpha t)$



$\alpha = 4$
period $2\pi/4=\pi/2$

shorter period
higher frequency
(i.e., oscillates faster)

Frequency is defined as $f=1/T$

Alternative notation: $\sin(\alpha t)=\sin(2\pi t/T)=\sin(2\pi f t)$

Fourier Series Theorem

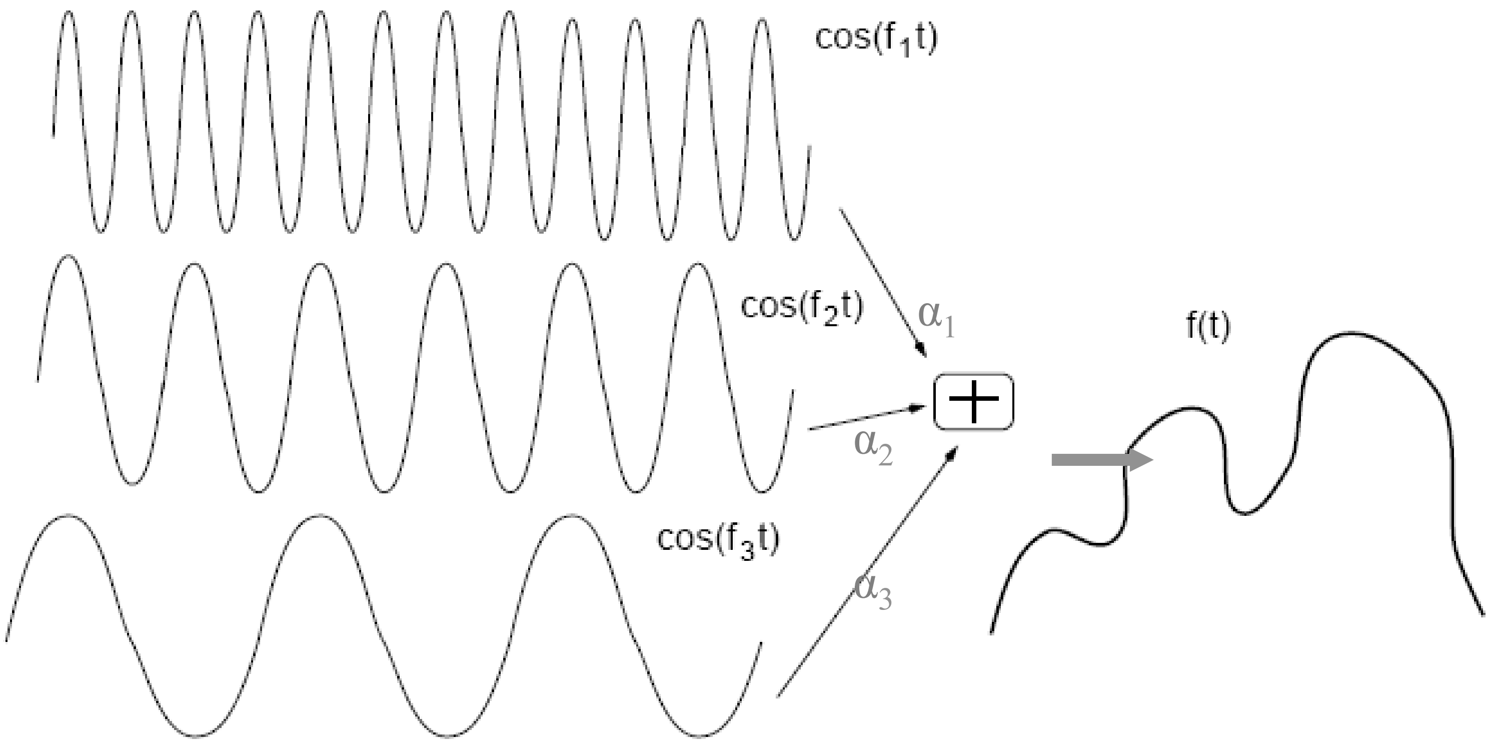
- Any periodic function can be expressed as a weighted sum (infinite) of sine and cosine functions of varying frequency:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(nf_0 t) + \sum_{n=1}^{\infty} b_n sin(nf_0 t)$$

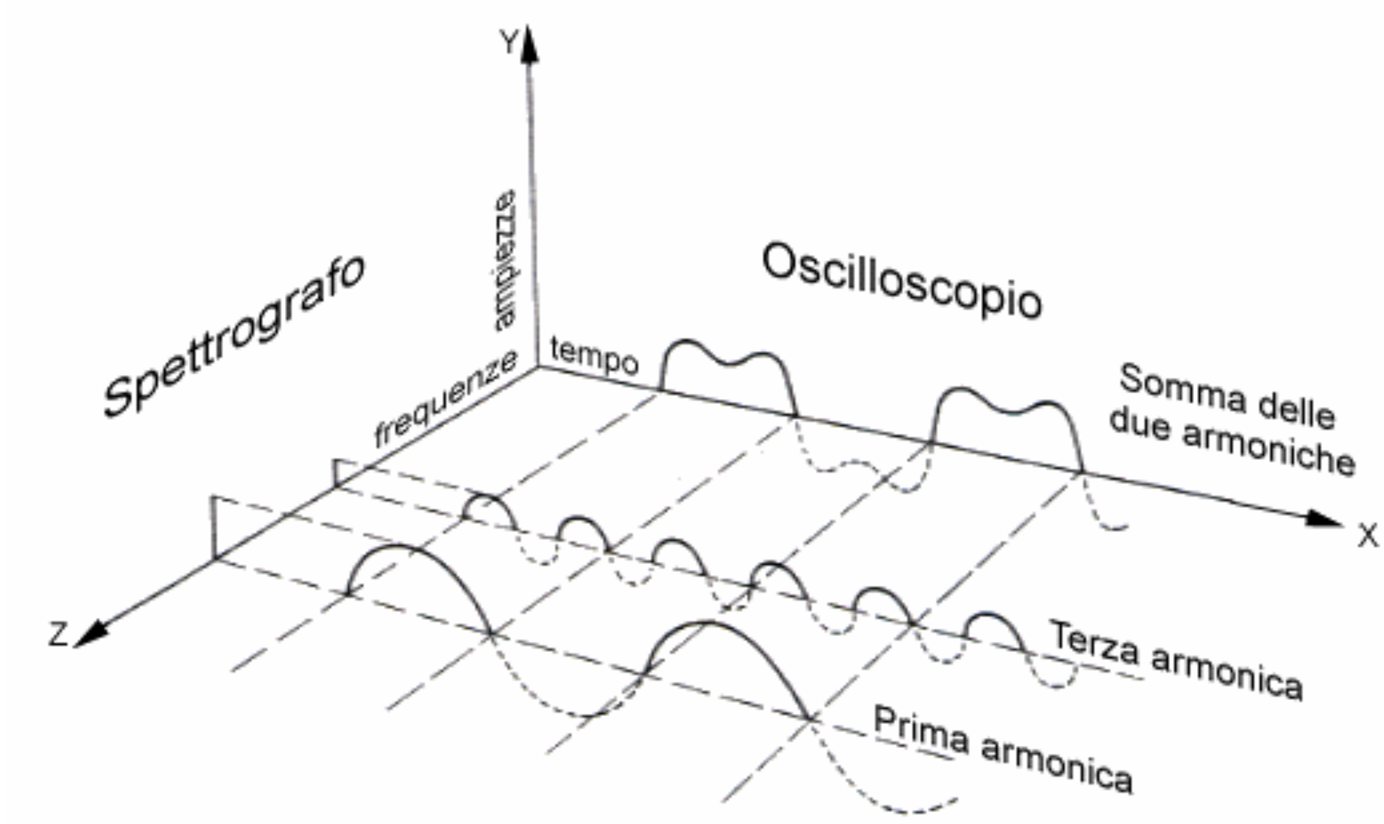
f_0 is called the "fundamental frequency"

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos(nf_0 t)\, dt \qquad b_n = 1/\pi \int_{-\pi}^{\pi} f(x) \sin(nf_0 t)\, dt$$

Fourier Series (cont' d)



Concept


Y
ampiezze
Spettrografo
Oscilloscopio
tempo
frequenze
Somma delle
due armoniche
X
Z
Terza armonica
Prima armonica

Continuous Time Fourier Transform (FT)

- Transforms a signal (i.e., function) from the **spatial** domain to the **frequency** domain.

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

Time domain

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

Forward FT: $\mathcal{F}(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$

Spatial domain

Inverse FT: $\mathcal{F}^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$

where $e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$

CTFT

- Change of variables for simplified notations: ω=2πu

$$F(2\pi u) = F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx =$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{j2\pi ux}d\left(2\pi u\right) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

- More compact notations (same as in GW)

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$$

$$f\left(x\right) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

Definitions

- **F(u)** is a complex function:

$$F(u) = R(u) + jI(u)$$

- Magnitude of FT (spectrum):

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Phase of FT: $\phi(F(u)) = \tan^{-1}(\frac{I(u)}{R(u)})$

- Magnitude-Phase representation:

$$F(u) = |F(u)|e^{j\phi(u)}$$

$$R^2(u) + I^2(u)$$

- Energy of f(x): **P(u)**=|F(u)|^2

Continuous Time Fourier Transform (CTFT)

Time is a real variable (t)

Frequency is a real variable (ω)

Signals : 1D

CTFT: Concept

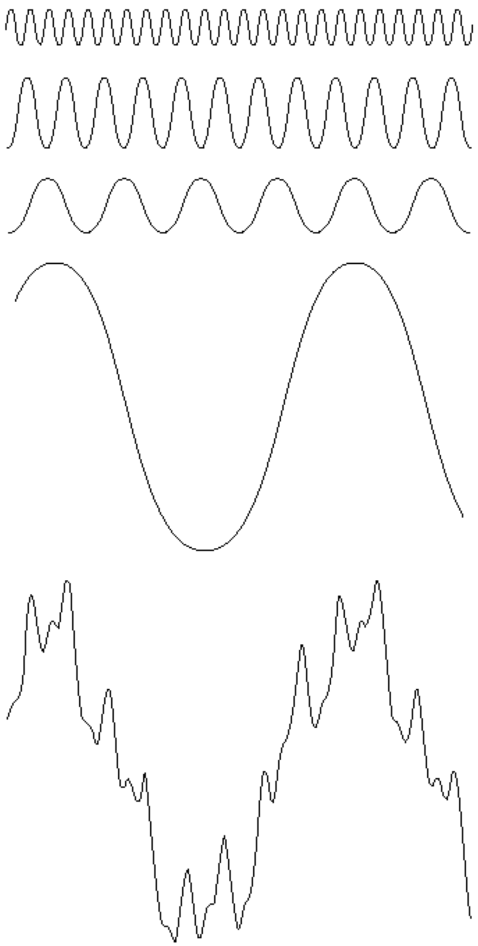


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- A signal can be represented as a weighted sum of sinusoids.

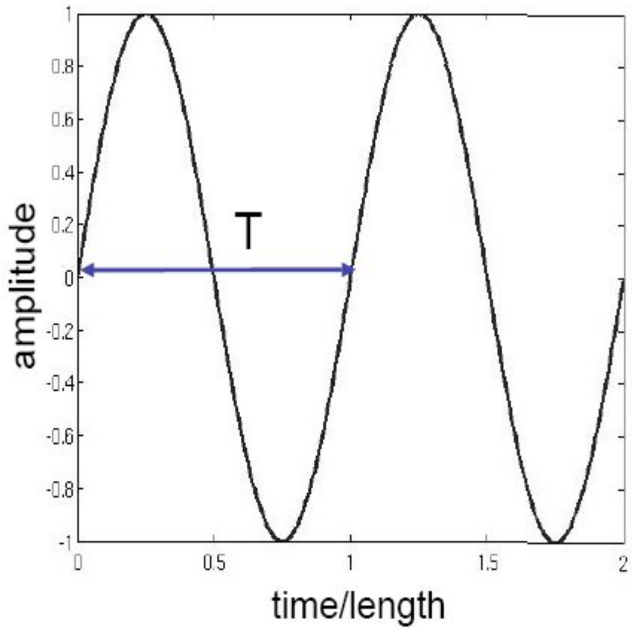
- Fourier Transform is a change of basis, where the basis functions consist of sins and cosines (complex exponentials).

[Gonzalez Chapter 4]

Continuous Time Fourier Transform (CTFT)

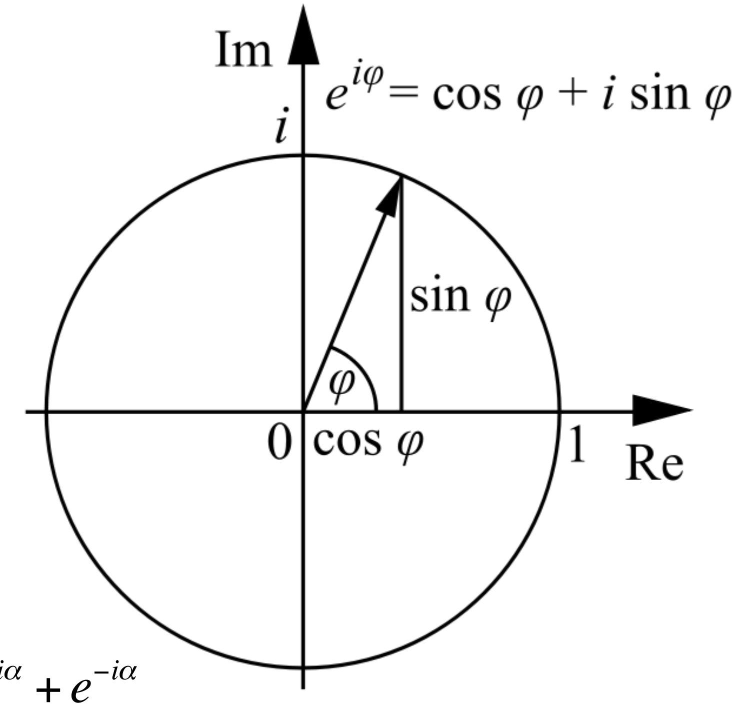
- Define frequency
  = 1/T
  cycles per unit time
  cycles per unit distance

- Here f = 1   T=1



Fourier Transform

- Cosine/sine signals are easy to define and interpret.
- Analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.
- A complex number has real and imaginary parts: $z = x+j\,y$
- *The Eulero formula* links complex exponential signals and trigonometric functions



$$r\,e^{j\alpha} = r(\cos\alpha + j\sin\alpha) \quad \Longleftrightarrow \quad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}, \qquad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

CTFT

- Continuous Time Fourier Transform
- Continuous time *a-periodic* signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency ω is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

Fourier *integral*

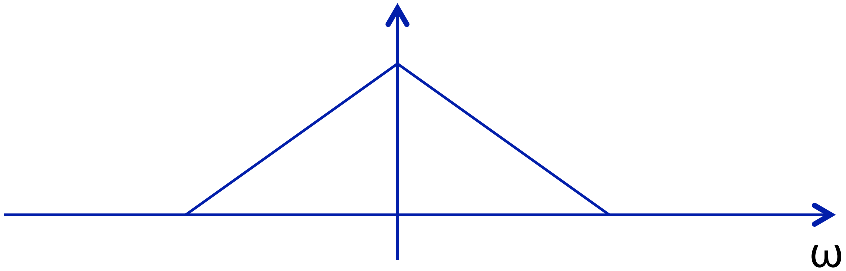
$$F(\omega) = \int_t f(t) e^{-j\omega t} dt$$ analysis

$$f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$$ synthesis

CTFT of real signals

- Real signals: each signal sample is a real number
- Property: the CTFT is Hermttian-symmetric -> the spectrum is symmetric

$$\hat{f}(-\omega) = \hat{f}^*(\omega)$$



$$f(t) \to \hat{f}(\omega)$$

$$f(-t) \to \hat{f}(-\omega) = \hat{f}^*(\omega)$$

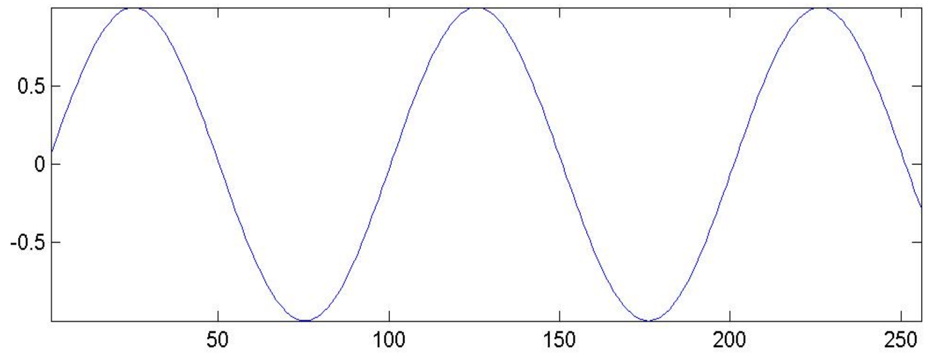
$$\mathrm{Pr}\,oof$$

$$\Im\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t) e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} f(t') e^{j\omega t'}\, dt' = \hat{f}^*(-\omega)$$

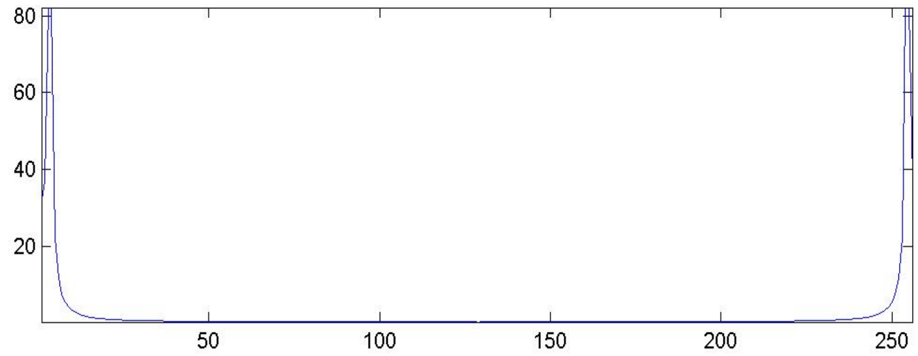
Sinusoids

- Frequency domain characterization of signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$



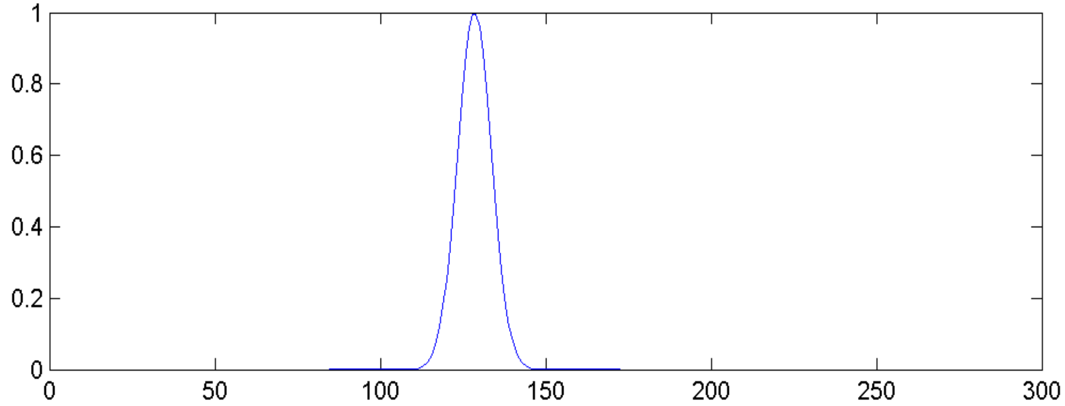
Signal domain



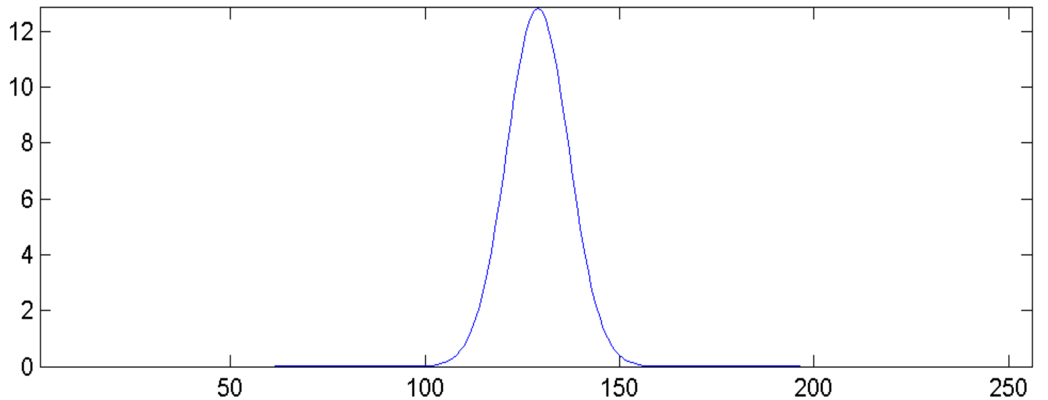
Frequency domain (spectrum, absolute value of the transform)

Gaussian

Time domain

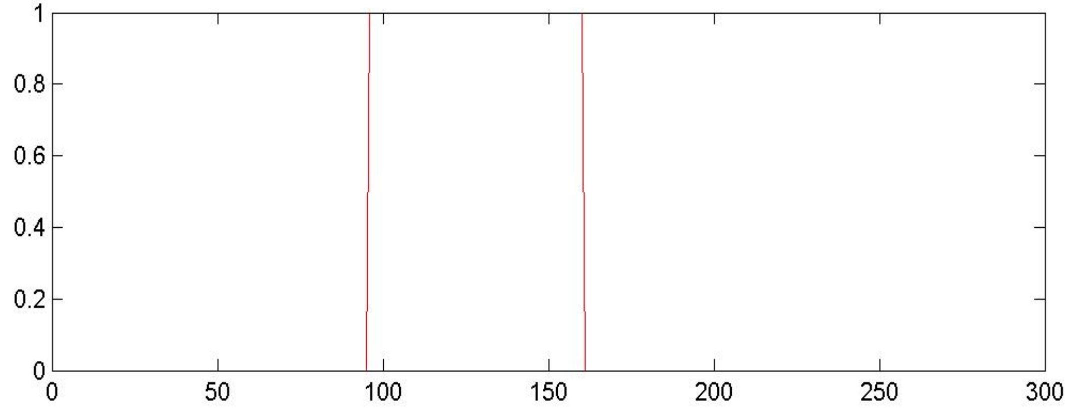


Frequency domain

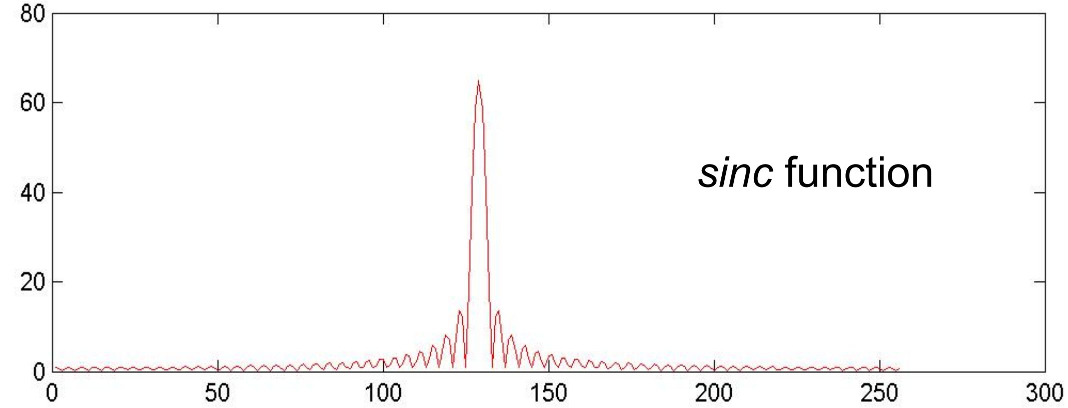


rect

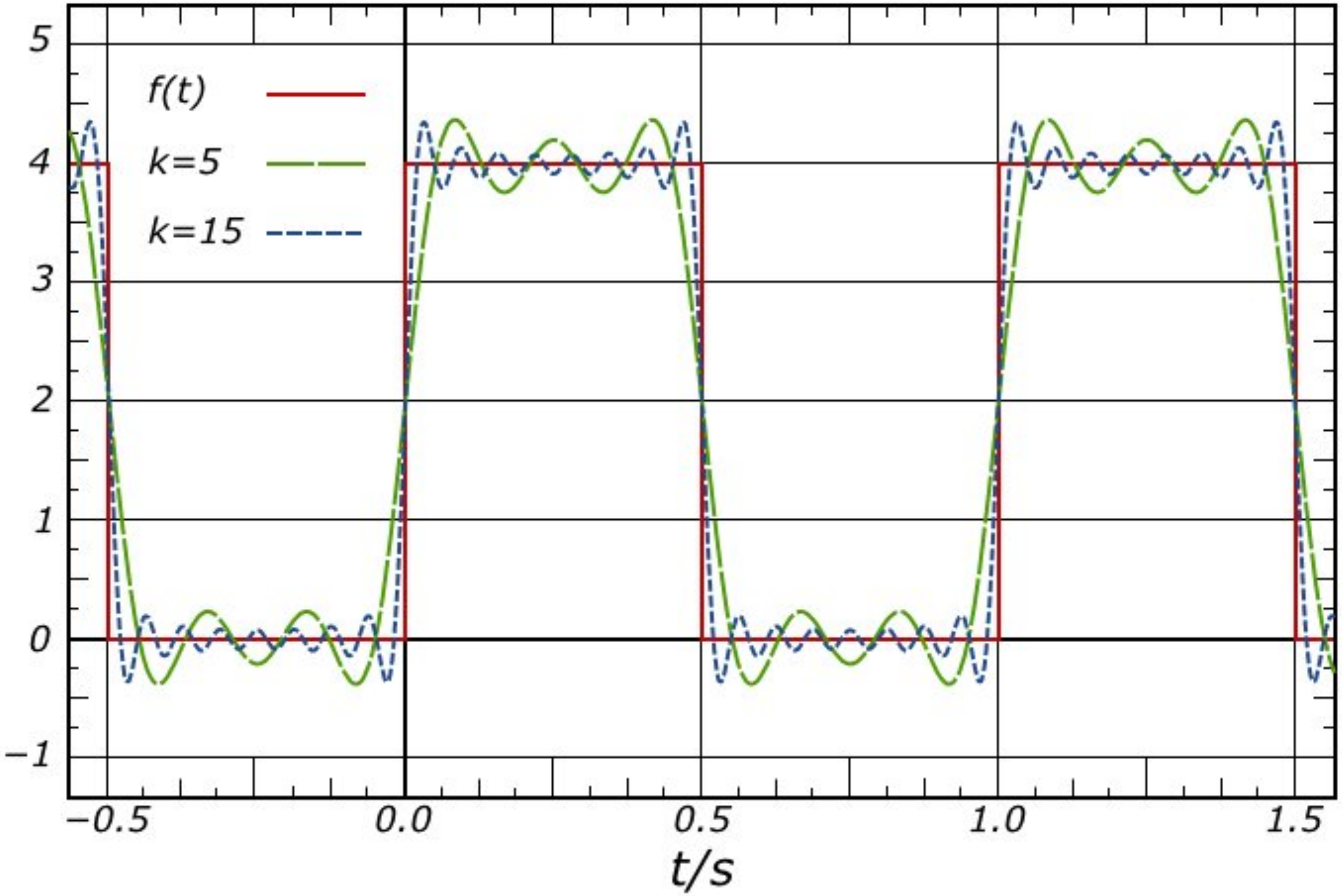
Time domain



Frequency domain



Example



Properties

Table 2.1 Fourier Transform Properties

Property	Function	Fourier Transform	
	$f(t)$	$\hat{f}(\omega)$	
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$	**(2.15)**
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$	**(2.16)**
Multiplication	$f_1(t)\, f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$	**(2.17)**
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$	**(2.18)**
Modulation	$e^{i\xi t} f(t)$	$\hat{f}(\omega-\xi)$	**(2.19)**
Scaling	$f(t/s)$	$\|s\|\,\hat{f}(s\,\omega)$	**(2.20)**
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p \hat{f}(\omega)$	**(2.21)**
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$	**(2.22)**
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$	**(2.23)**
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$	**(2.24)**

Discrete Time FT (DTFT)

- If f[n] is a function of the discrete variable n then the DTFT is given by

$$\hat{f}_{2\pi}(\omega) = \sum_{n=-\infty}^{+\infty} f[n] \mathrm{e}^{-j\omega n}$$

$$f[n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) \mathrm{e}^{j\omega n} \, d\omega$$

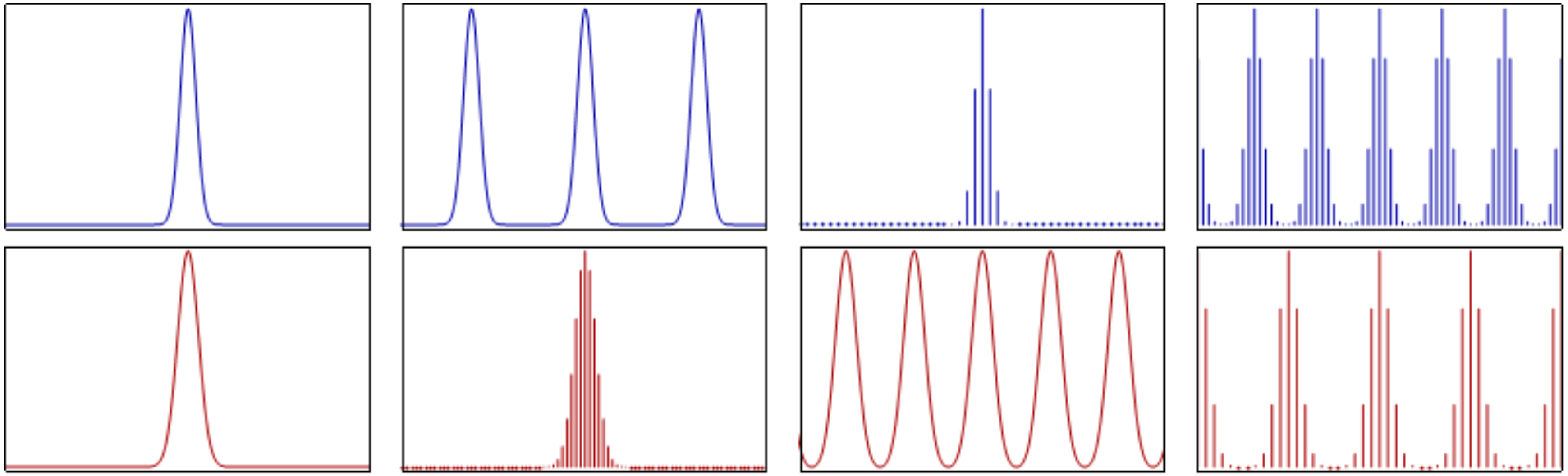
DTFT

$$\hat{f}_{1/T}(\omega) = \sum_{n=-\infty}^{+\infty} T \cdot f[nT] \mathrm{e}^{-j\omega nT} =$$

$$= \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{T}\right) = F\left\{\sum_{n=-\infty}^{+\infty} f[nT] \cdot \delta(t - nT)\right\}$$

$$f[nT] = \frac{1}{T}\int_{-\infty}^{+\infty} \hat{f}_{1/T}(\omega) e^{j\omega n} d\omega$$

CT versus DT FT



Relationship between the (continuous) Fourier transform and the discrete Fourier transform.
Left column: A continuous function (top) and its Fourier transform (bottom).
Center-left column: Periodic summation of the original function (top). Fourier transform (bottom) is zero except at discrete points. The inverse transform is a sum of sinusoids called Fourier series.
Center-right column: Original function is discretized (multiplied by a Dirac comb) (top). Its Fourier transform (bottom) is a periodic summation (DTFT) of the original transform.
Right column: The DFT (bottom) computes discrete samples of the continuous DTFT. The inverse DFT (top) is a periodic summation of the original samples. The FFT algorithm computes one cycle of the DFT and its inverse is one cycle of the DFT inverse.

Discrete Fourier Transform (DFT)

Applies to **finite length** discrete time (sampled) signals and time series

The easiest way to get to it

Time is a discrete variable (t=n)

Frequency is a discrete variable (f=k)

DFT

- The DFT can be considered as a generalization of the CTFT to discrete series of a finite number of samples
- It is the FT of a discrete (sampled) function of one variable

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/N}$$

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j2\pi kn/N}$$

  - The 1/N factor is put either in the analysis formula or in the synthesis one, or the 1/sqrt(N) is put in front of both.
- Calculating the DFT takes about N^2 calculations

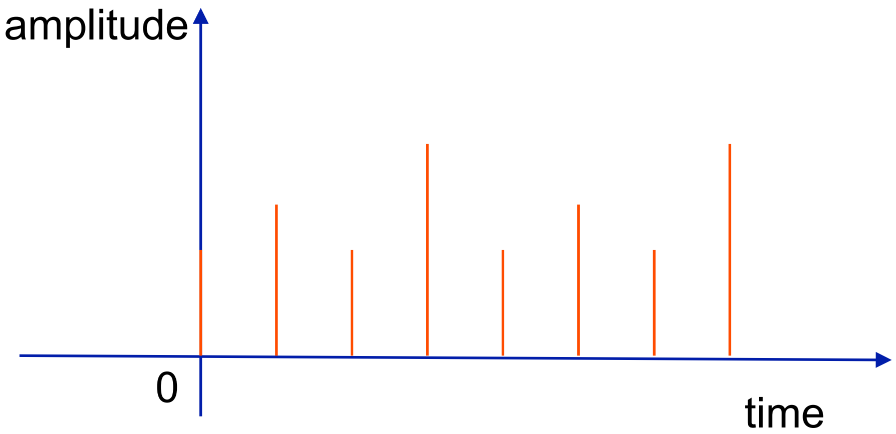
In practice..

- In order to calculate the DFT we start with k=0, calculate F(0) as in the formula below, then we change to u=1 etc

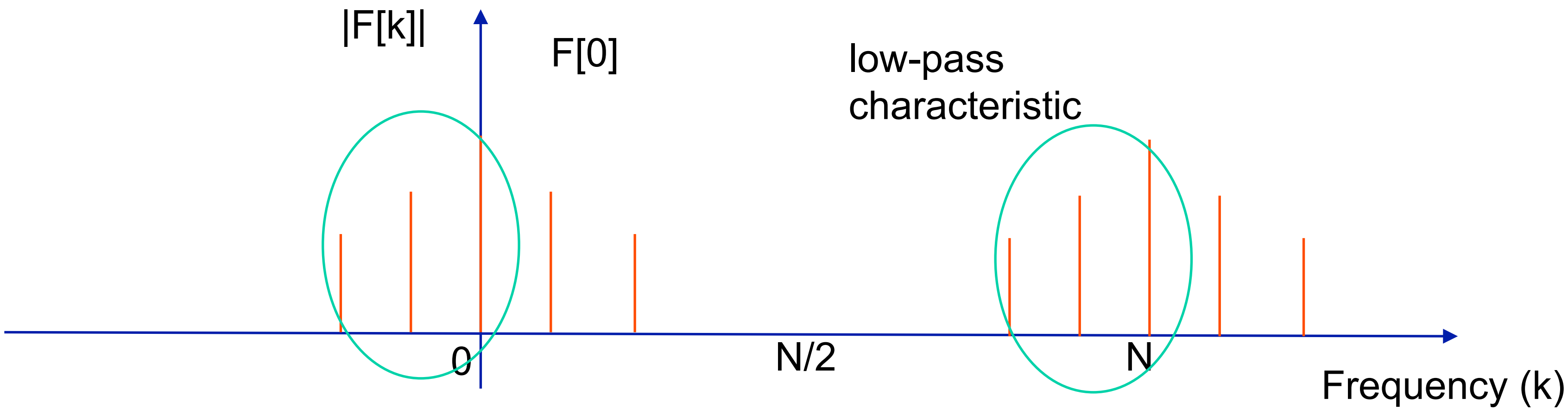
$$F[0] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] e^{-j2\pi 0n/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n] = \overline{f}$$

- F[0] is the mean value of the function f[n]
  - This is also the case for the CTFT
- The transformed function F[k] has the same number of terms as f[n] and always exists
- The transform is always reversible by construction so that we can always recover f given F

Highlights on DFT properties



The DFT of a **real** signal is **symmetric (Hermitian symmetry)**
The DFT of a **real symmetric** signal (even like the cosine) is **real and symmetric**
The DFT is **N-periodic**
Hence
The DFT of a real symmetric signal only needs to be specified in [0, N/2]



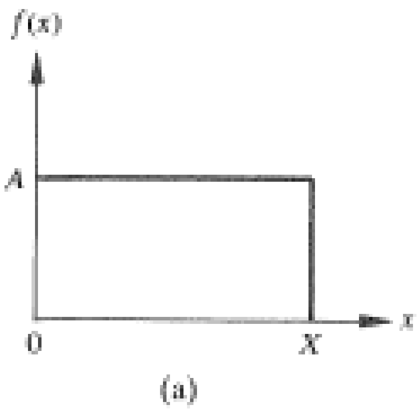
Visualization of the basic repetition

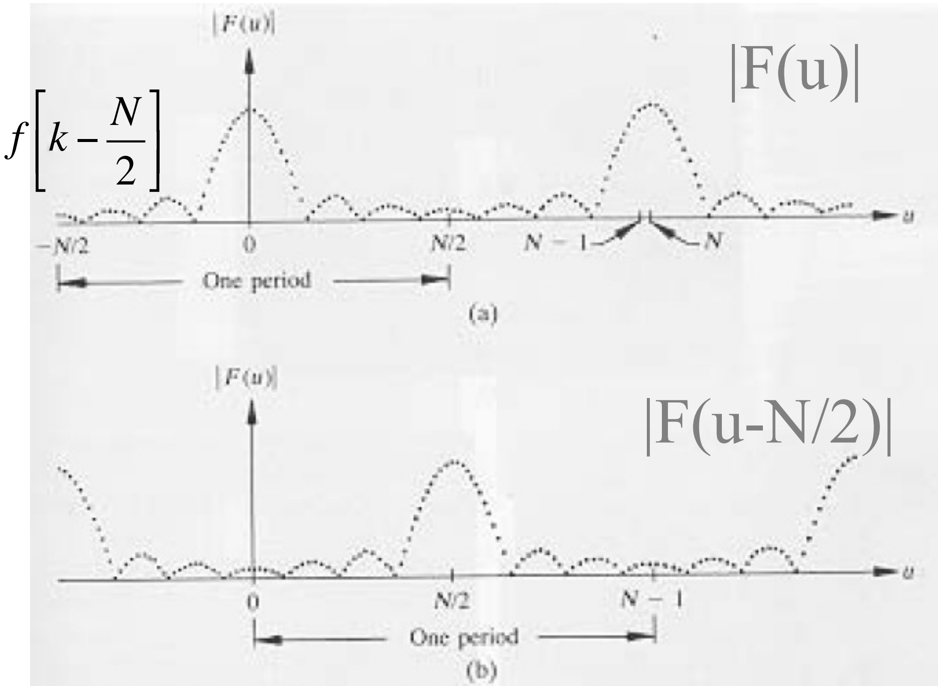
- To show a full period, we need to translate the origin of the transform at **u=N/2** (or at **(N/2,N/2)** in 2D)

$$f[n]e^{2\pi u_0 n} \rightarrow f[k-u_0]$$

$$u_0 = \frac{N}{2}$$

$$f[n]e^{2\pi\frac{N}{2}n} = f[n]e^{\pi N n} = (-1)^n f[n] \rightarrow f\left[k-\frac{N}{2}\right]$$





DFT

- About N^2 multiplications are needed to calculate the DFT
- The transform F[k] has the same number of components of f[n], that is N
- The DFT always exists for signals that do not go to infinity at any point
- Using the Eulero's formula

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

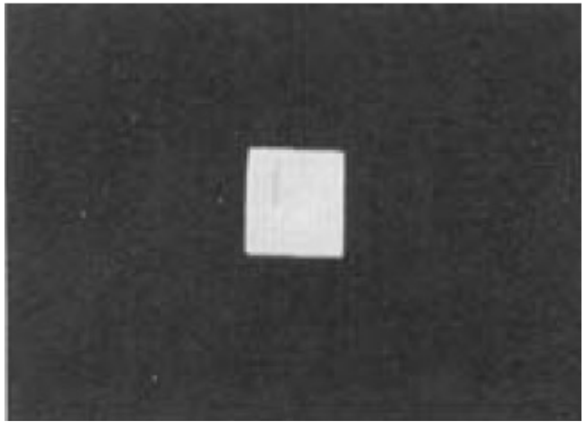
$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\big(\cos(j2\pi kn/N) - j\sin(j2\pi kn/N)\big)$$

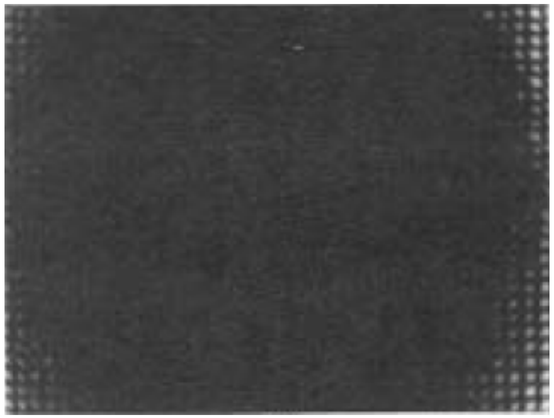
frequency component *k*

discrete trigonometric functions

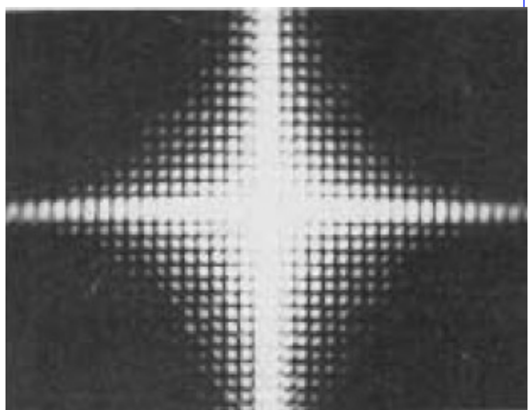
2D example

$$f(x,y)(-1)^{x+y} \Longleftrightarrow F(u-N/2,\ v-N/2)$$





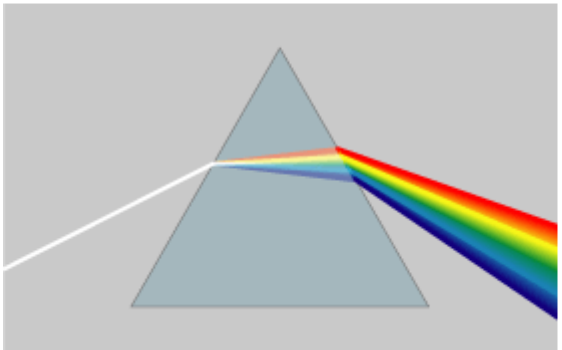
no translation



after translation

Going back to the intuition

- The FT decomposed the signal over its harmonic components and thus represents it as a sum of linearly independent complex exponential functions
- Thus, it can be interpreted as a "mathematical prism"



DFT

- Each term of the DFT, namely each value of F[k], results of the contributions of all the samples in the signal (f[n] for n=1,..,N)
- The samples of f[n] are multiplied by trigonometric functions of different frequencies
- The domain over which F[k] lives is called *frequency domain*
- Each term of the summation which gives F[k] is called *frequency component* of *harmonic component*

DFT is a complex number

- F[k] in general are complex numbers

$$F[k] = \mathrm{Re}\{F[k]\} + j\,\mathrm{Im}\{F[k]\}$$

$$F[k] = |F[k]| \exp\{j \mathrm{R} F[k]\}$$

$$\left\{\begin{array}{l} |F[k]| = \sqrt{\mathrm{Re}\{F[k]\}^2 + \mathrm{Im}\{F[k]\}^2} \\ \mathrm{R} F[k] = \tan^{-1}\left\{-\dfrac{\mathrm{Im}\{F[k]\}}{\mathrm{Re}\{F[k]\}}\right\} \end{array}\right\}$$

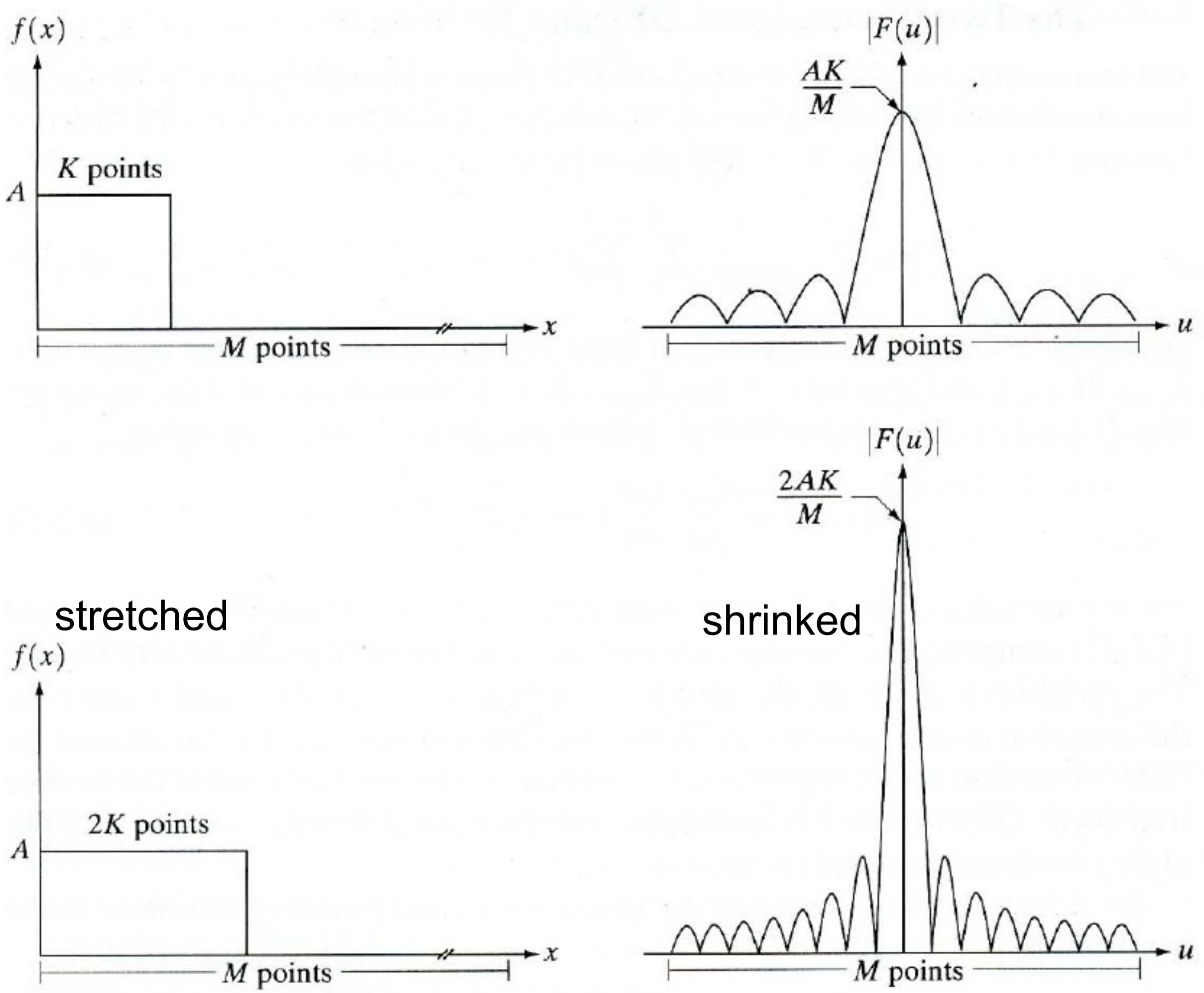
magnitude or spectrum

phase or angle

$$P[k] = |F[k]|^2$$

power spectrum

Stretching vs shrinking

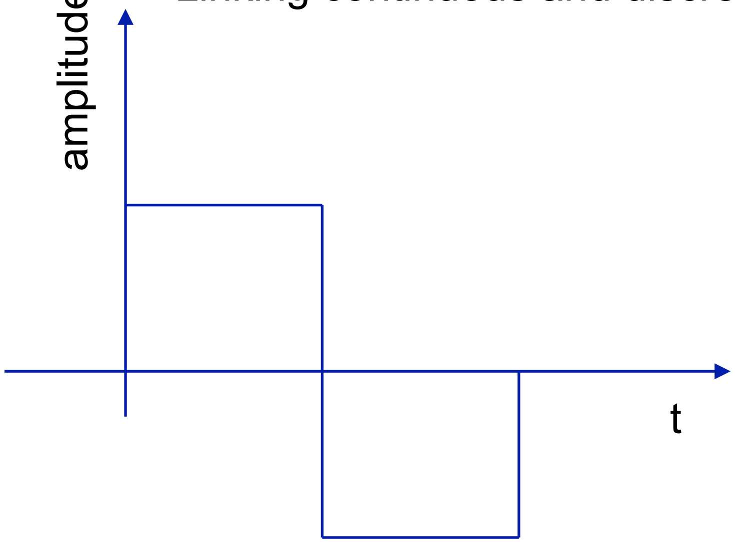


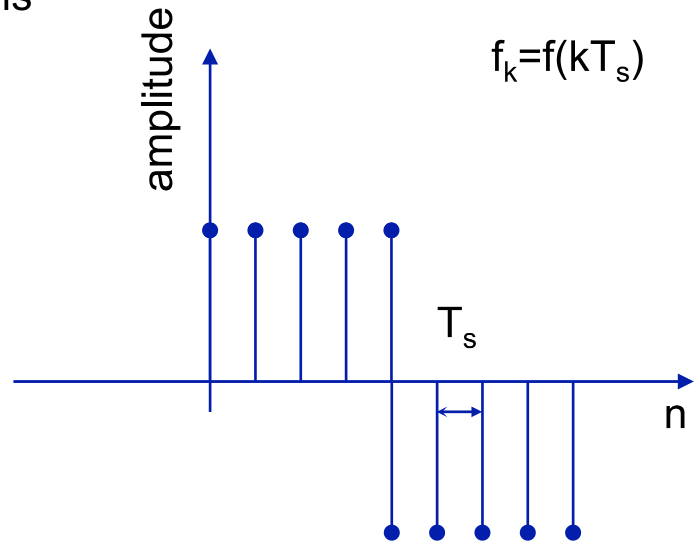
a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Periodization vs discretization

Linking continuous and discrete domains





- DT (discrete time) signals can be seen as *sampled* versions of CT (continuous time) signals
- Both CT and DT signals can be of finite duration or periodic
- There is a duality between *periodicity* and *discretization*
  - Periodic signals have discrete frequency (sampled) transform
  - Discrete time signals have periodic transform
  - DT periodic signals have discrete (sampled) periodic transforms

Increasing the resolution by Zero Padding

- Consider the analysis formula

$$F[k]=\frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-\frac{2\pi jkn}{N}}$$

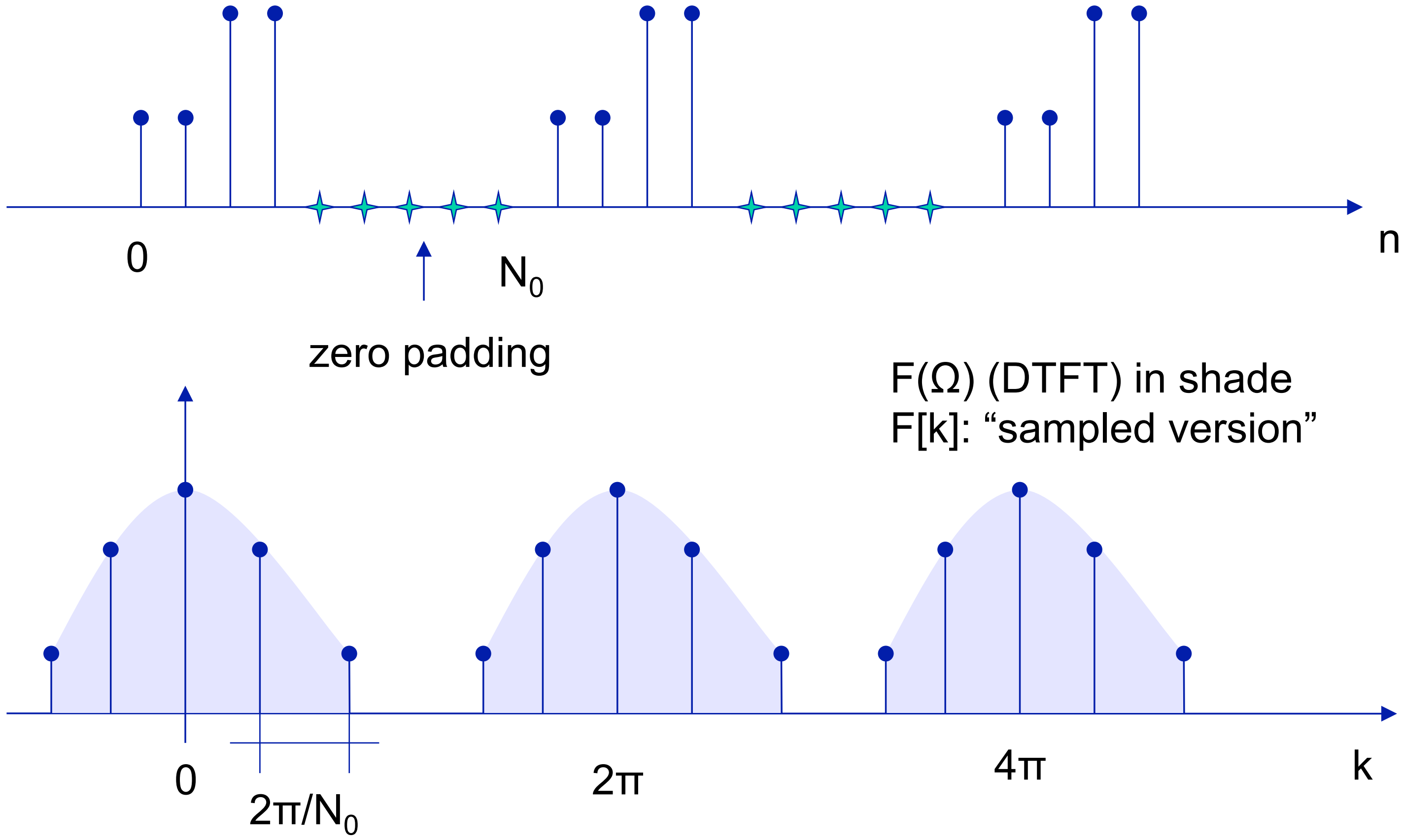
- If f[n] consists of N samples than F[k] consists of N samples as well, it is discrete (k is an integer) and it is periodic (because the signal f[n] is discrete time, namely n is an integer)
- The value of each F[k], for all k, is given by a weighted sum of the values of f[n], for n=1,.., N-1
- Key point: if we artificially increase the length of the signal adding M zeros on the right, we get a signal $f_1[m]$ for which m=1,…,N+M-1. Since

$$f_1[m]=\begin{cases} f[m] \quad for \quad 0 \le m < N \\ 0 \quad for \quad N \le m < N+M \end{cases}$$

Increasing the resolution through ZP

- Then the value of each F[k] is obtained by a weighted sum of the "real" values of f[n] for $0 \leq k \leq N-1$, which are the only ones different from zero, but they happen at different "normalized frequencies" since the frequency axis has been rescaled. In consequence, F[k] is more "densely sampled" and thus features a higher resolution.

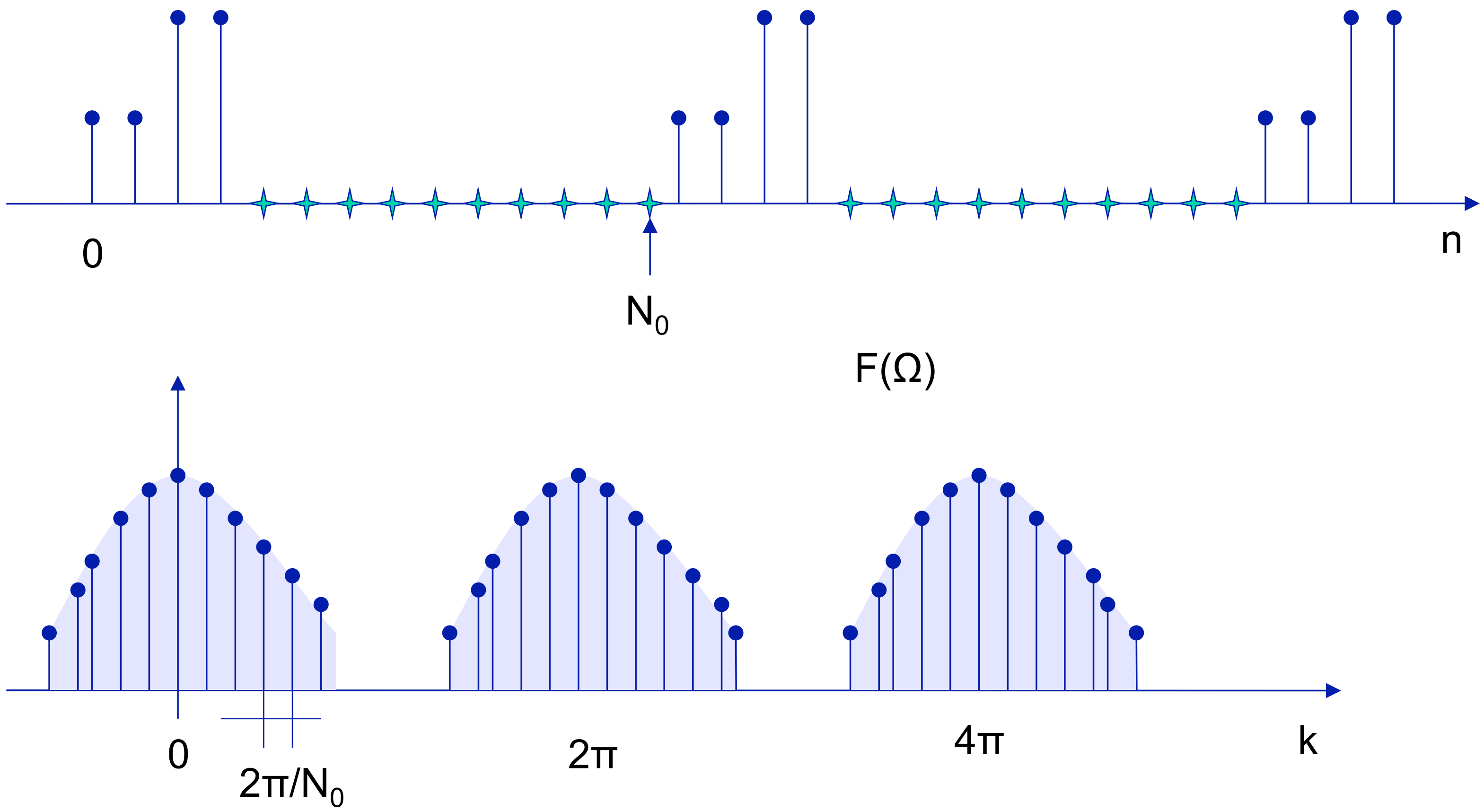
Increasing the resolution by Zero Padding



Zero padding

Increasing the number of zeros augments the "resolution" of the transform since the samples of the DFT get "closer"

zero padding



Summary of dualities

SIGNAL DOMAIN		FOURIER DOMAIN
Sampling	DTFT	Periodicity
Periodicity	CTFS	Sampling
Sampling+Periodicity	DTFS/DFT	Sampling +Periodicity

Discrete Cosine Transform (DCT)

Applies to digital (sampled) finite length signals AND uses only **cosines**.

The DCT coefficients are all **real numbers**

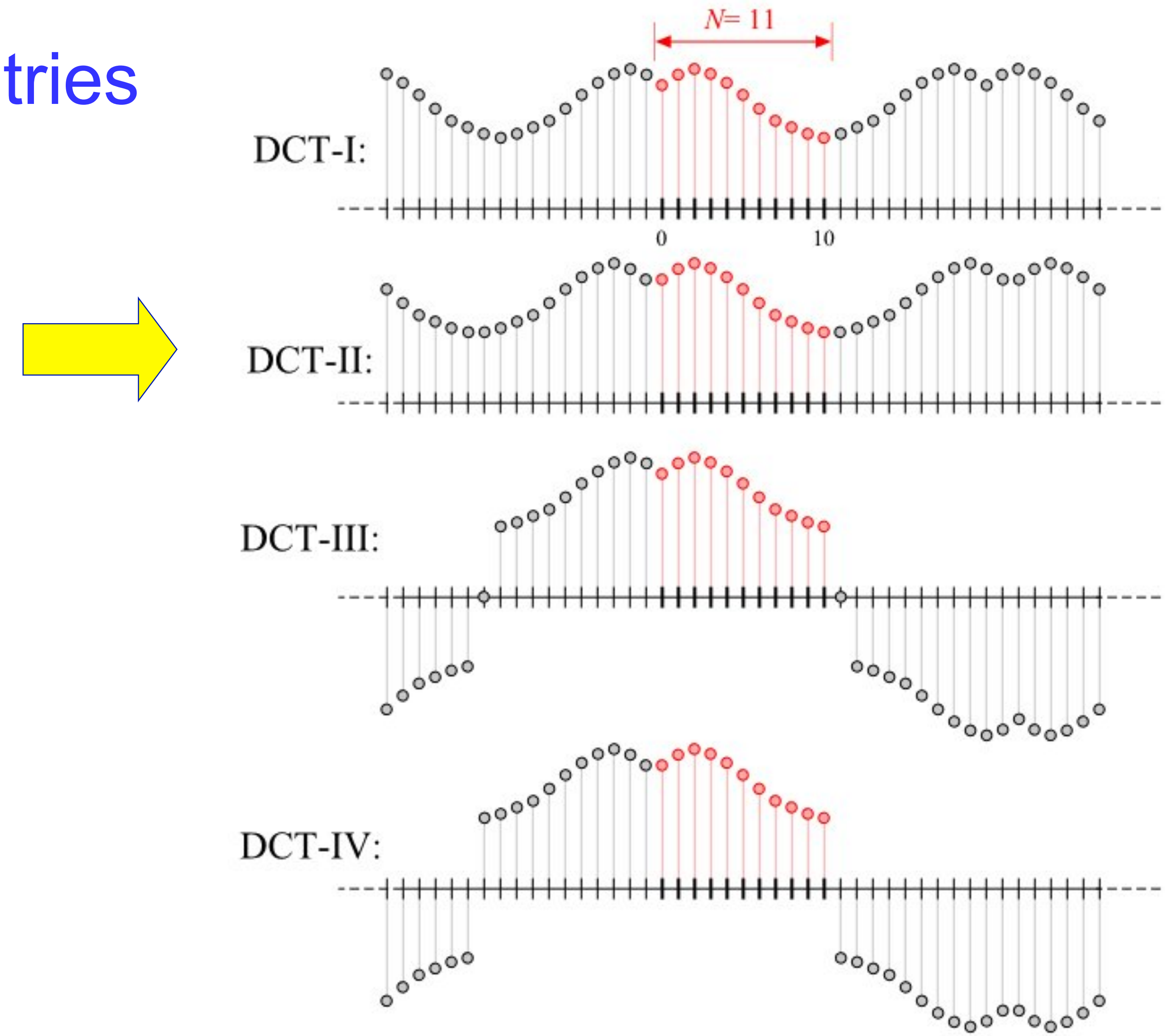
Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, out of which four are common
- Strong connection with the Karunen-Loeven transform
  - VERY important for signal compression

DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is even about the sample *a*, in which case the even extension is ***dcbabcd***, or the data is even about the point *halfway* between *a* and the previous point, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

Symmetries



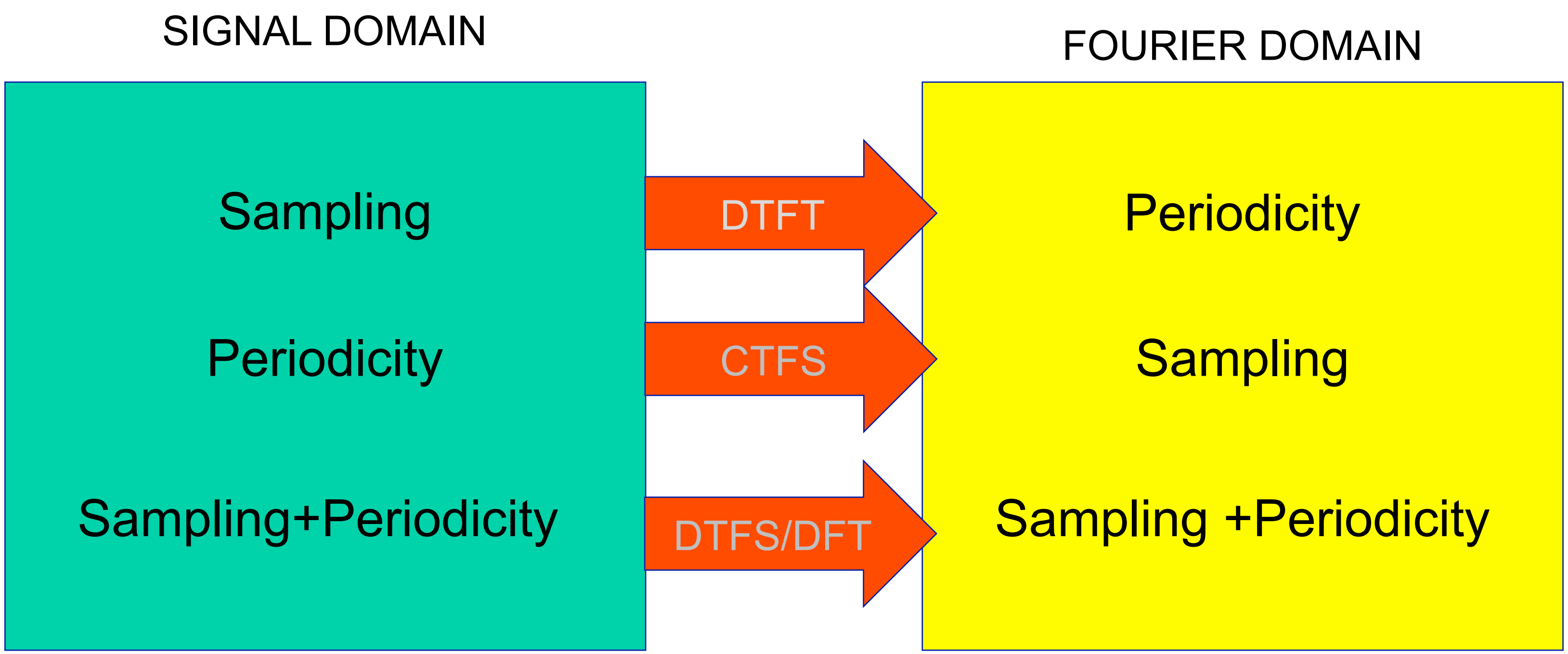
DCT

$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots.,N_0-1$$

$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of $\sqrt{2}$ (see above), this can be used to make the transform matrix orthogonal.

Summary of dualities



Sampling

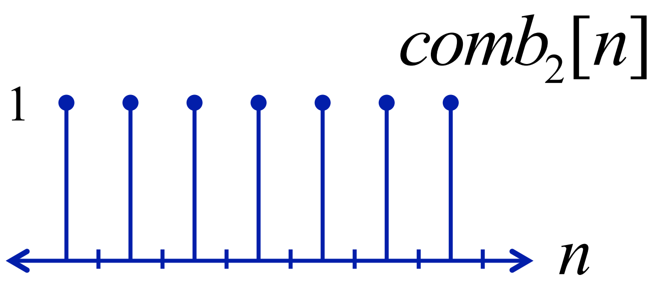
From continuous to discrete time

Impulse Train

- Define a *comb* function (impulse train) as follows

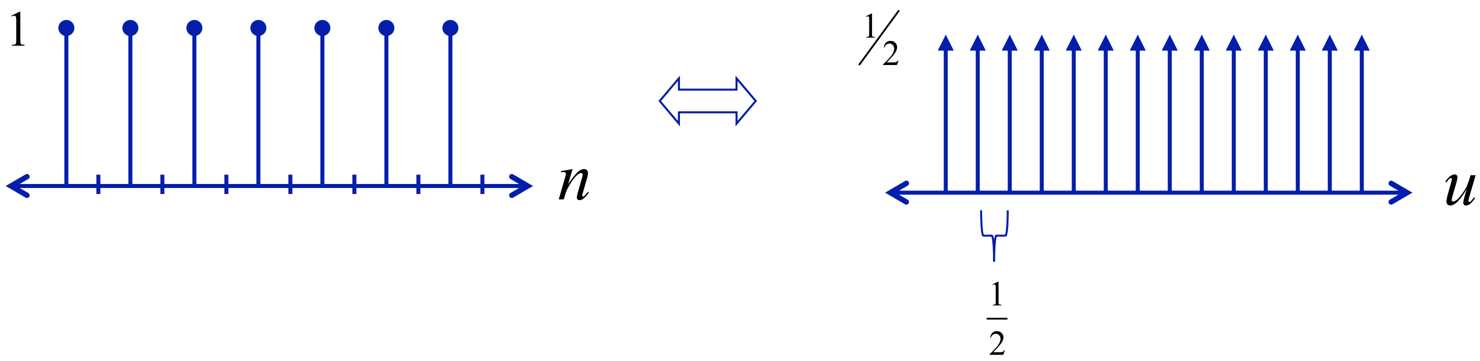
$$comb_N[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN]$$

where *M* and *N* are integers



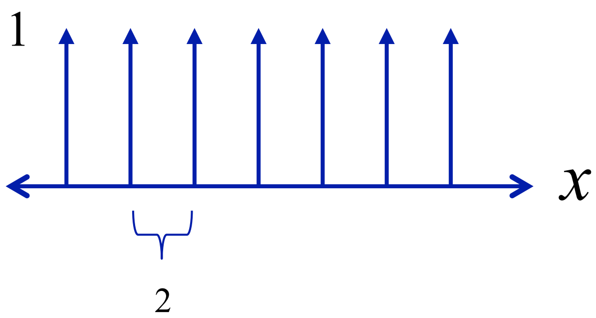
Impulse Train

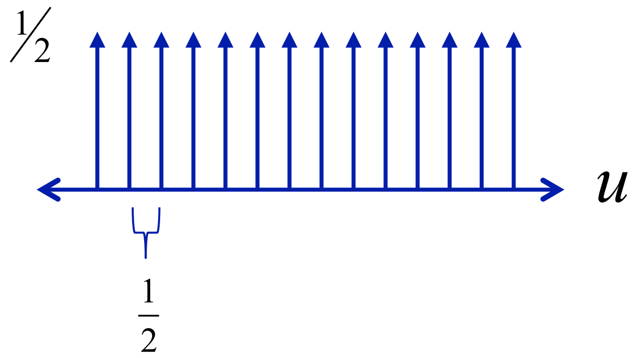
$$comb_2[n] \qquad \Longleftrightarrow \qquad \frac{1}{2}comb_{\frac{1}{2}}(u)$$



Impulse Train

$$comb_2(x) \quad \Leftrightarrow \quad \frac{1}{2}comb_{\frac{1}{2}}(u)$$

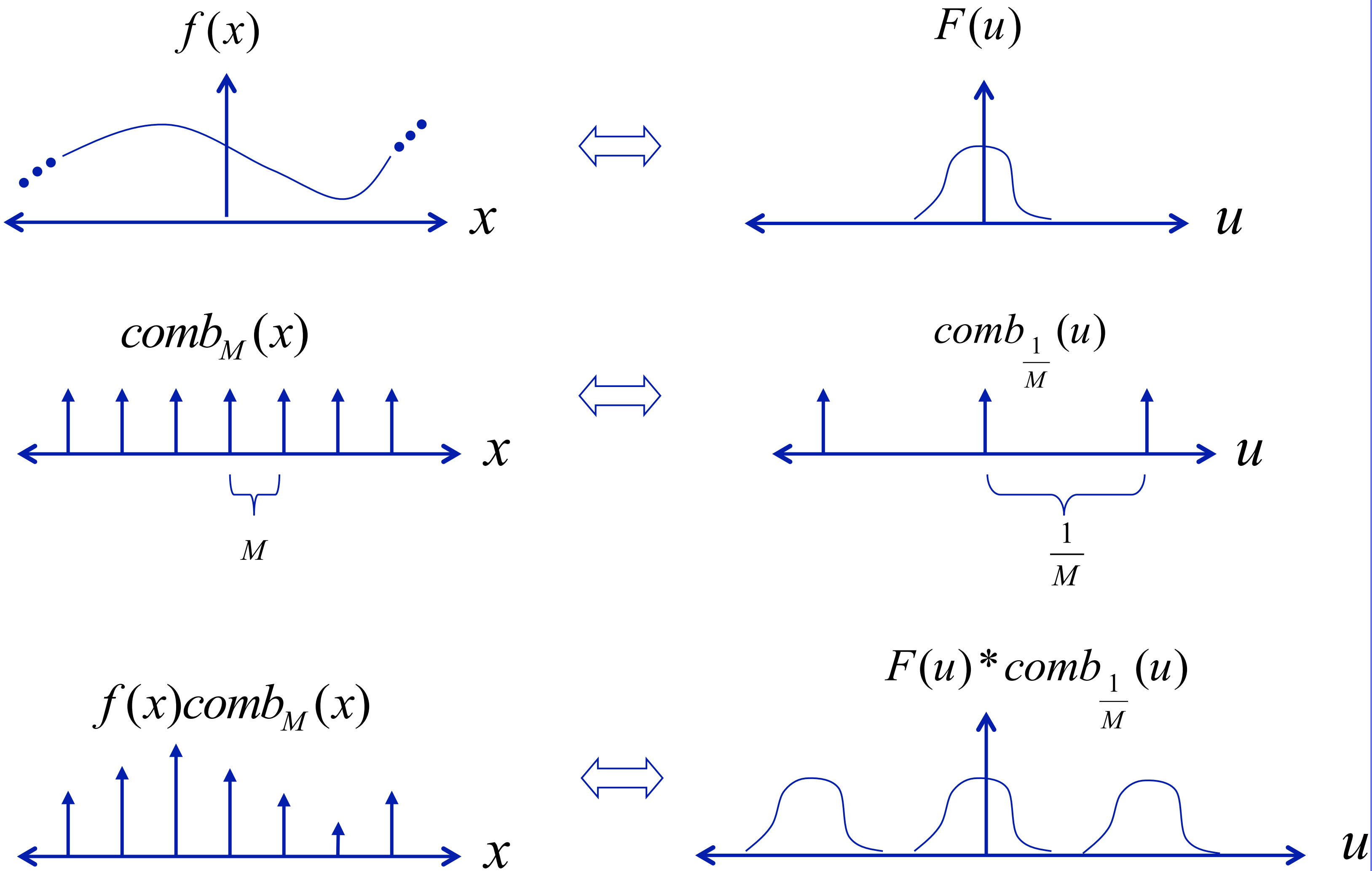




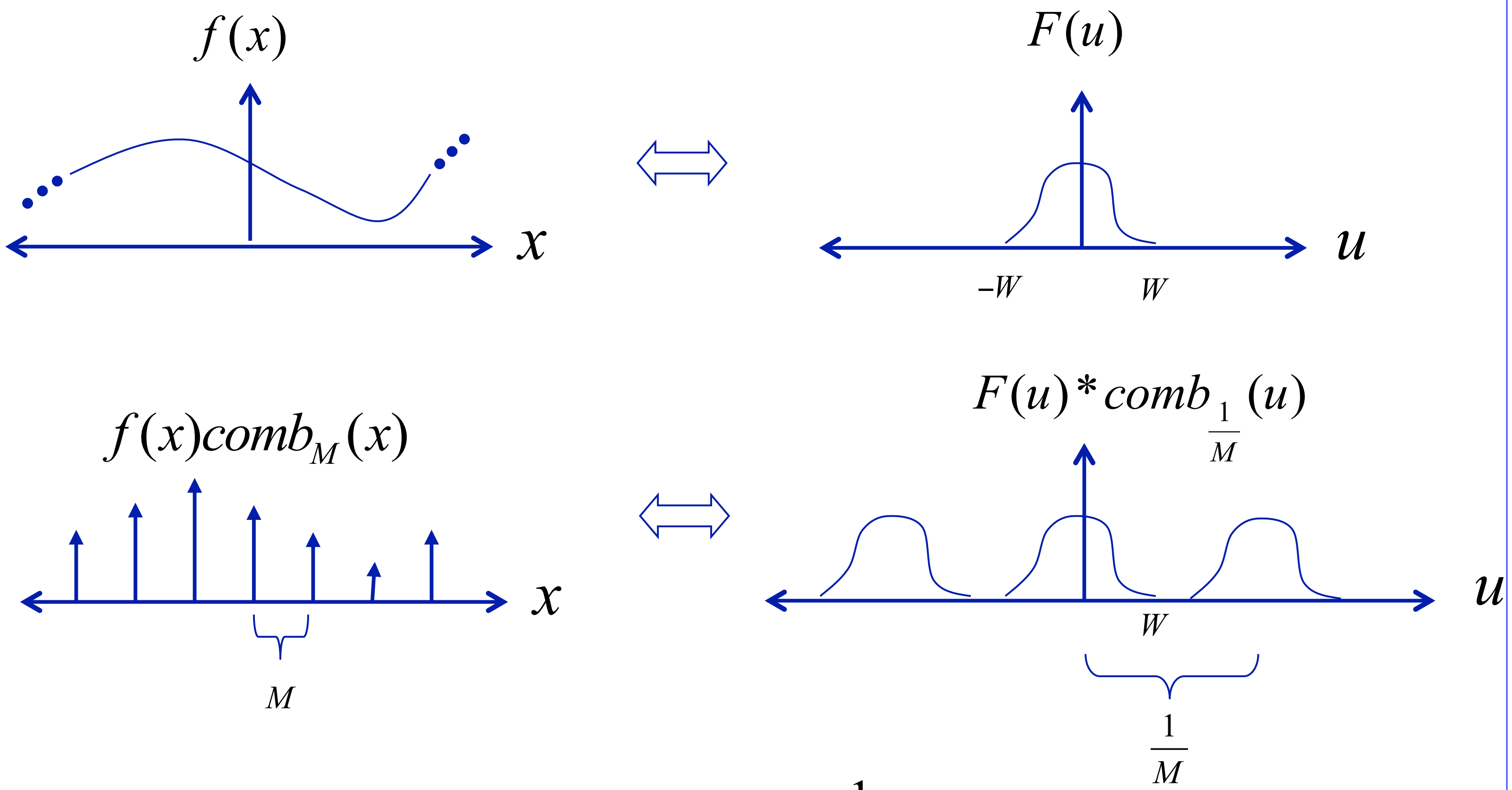
Consequences

Sampling (Nyquist) theorem

Sampling

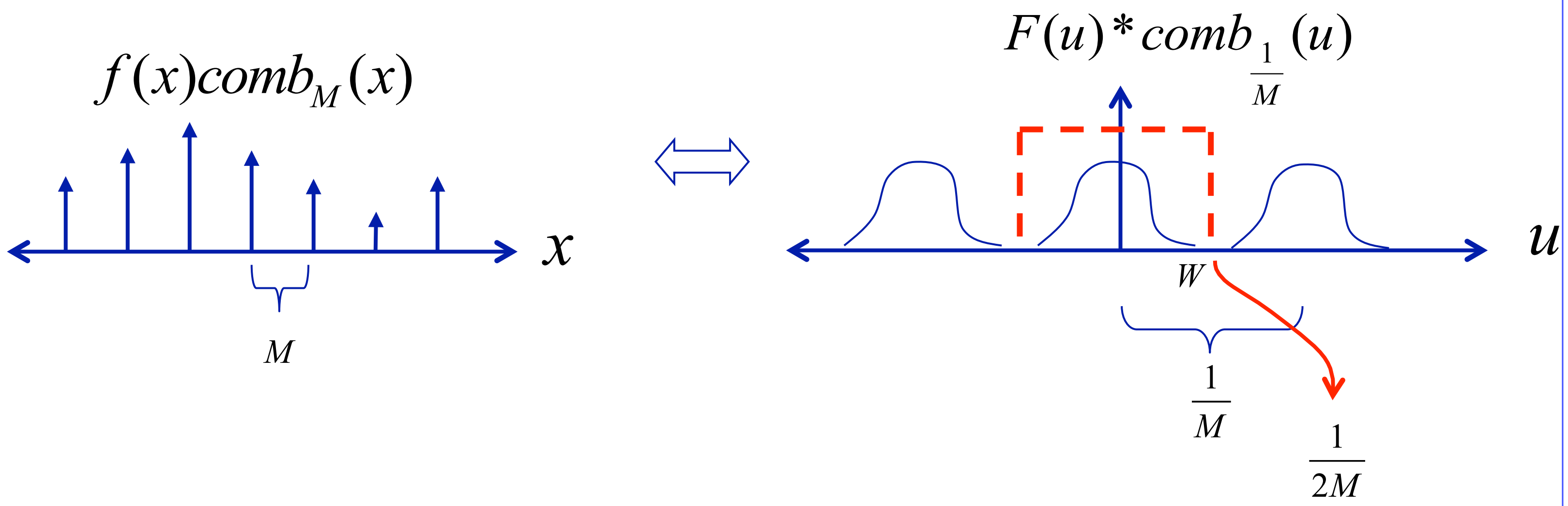

$f(x)$
x
$F(u)$
u
$comb_M(x)$
x
M
$comb_{\frac{1}{M}}(u)$
u
$\frac{1}{M}$
$f(x)comb_M(x)$
x
$F(u) * comb_{\frac{1}{M}}(u)$
u

Sampling



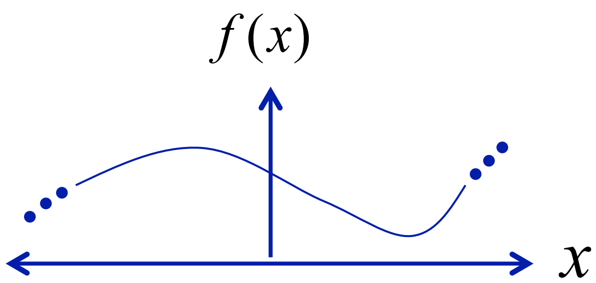
Nyquist theorem: No aliasing if $\frac{1}{M} > 2W$

Sampling

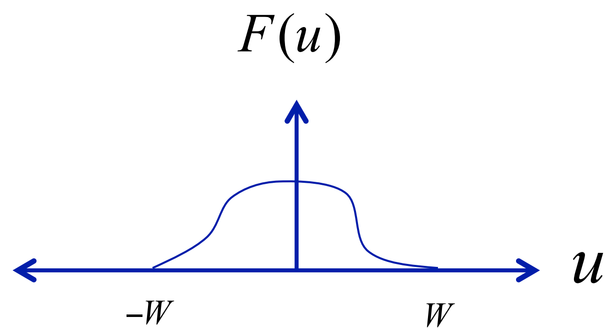


If there is no aliasing, the original signal can be recovered from its samples by low-pass filtering.

Sampling

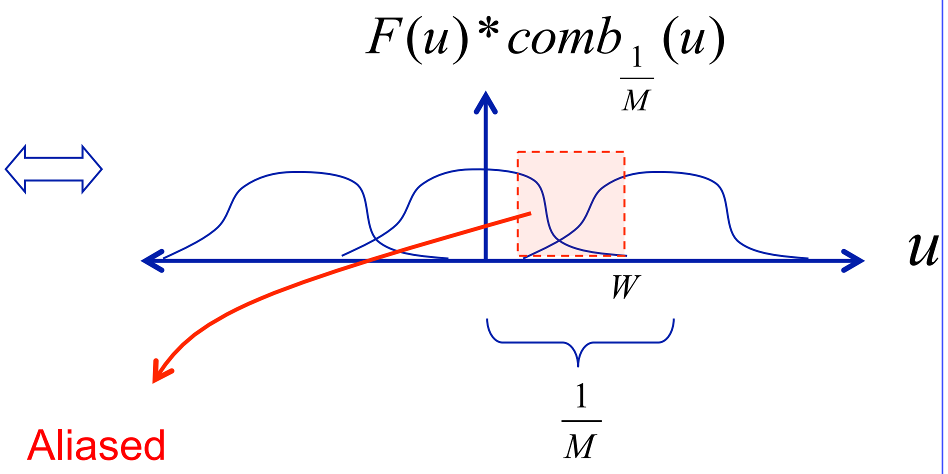


$\Longleftrightarrow$

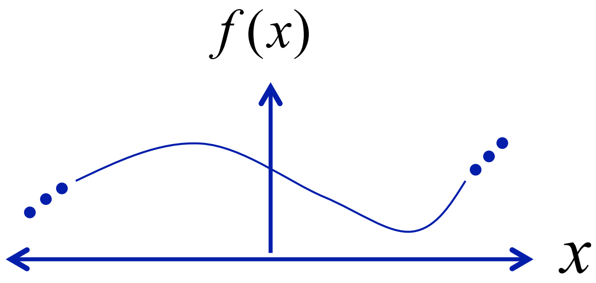


$f(x)comb_M(x)$

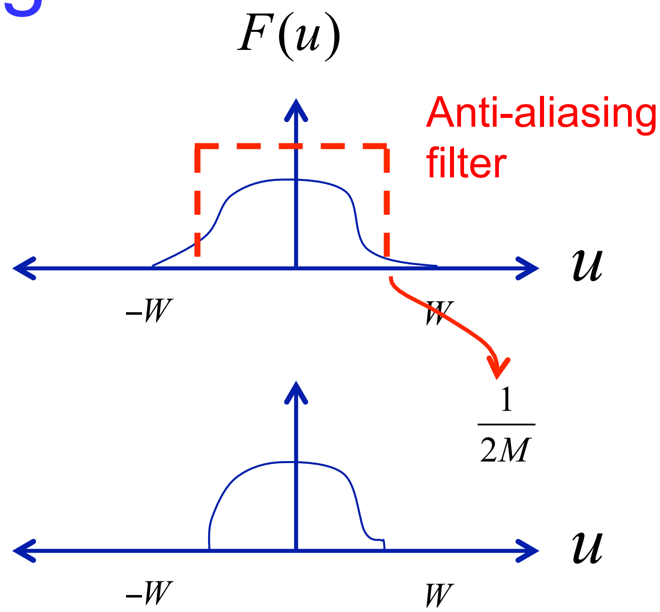
$\Longleftrightarrow$



Sampling



$\Longleftrightarrow$

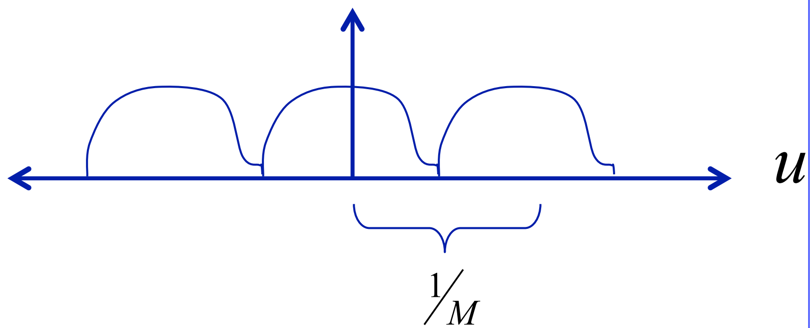


$$f(x) * h(x)$$

$\Longleftrightarrow$

$$\left[f(x) * h(x)\right] comb_M(x)$$

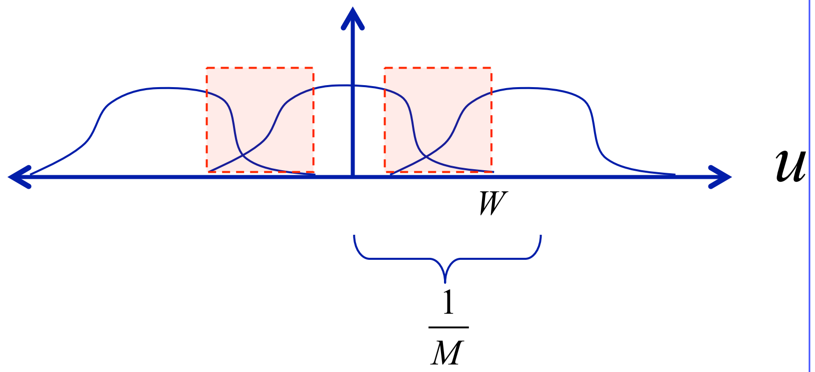
$\Longleftrightarrow$



Sampling

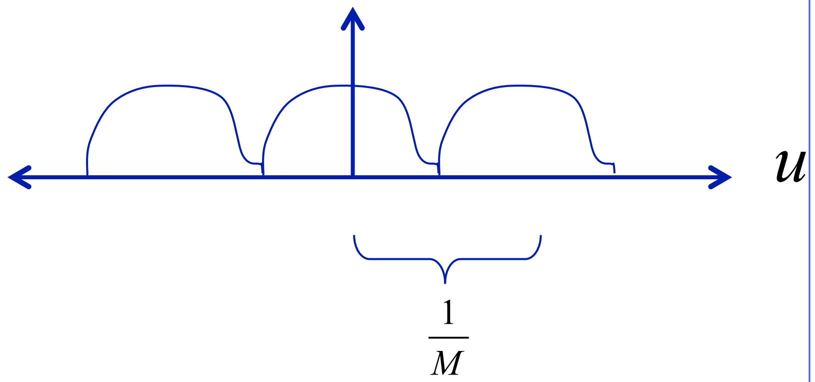
- Without anti-aliasing filter:

$$f(x)comb_M(x) \Longleftrightarrow$$



- With anti-aliasing filter:

$$[f(x)*h(x)]comb_M(x) \Longleftrightarrow$$



Images vs Signals

1D

- Signals
- Frequency
  - Temporal
  - Spatial
- Time (space) frequency characterization of signals
- Reference space for
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ….

2D

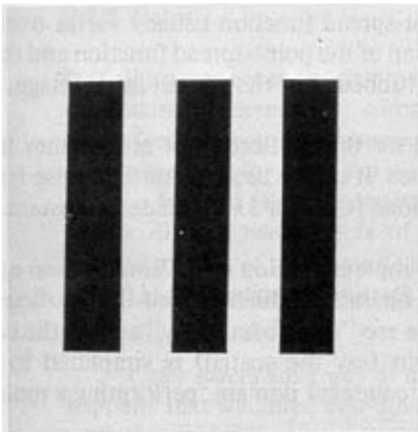
- Images
- Frequency
  - Spatial
- Space/frequency characterization of 2D signals
- Reference space for
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ….

2D Continuous FT

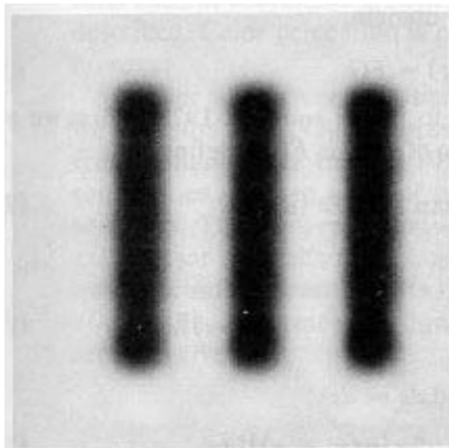
How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)

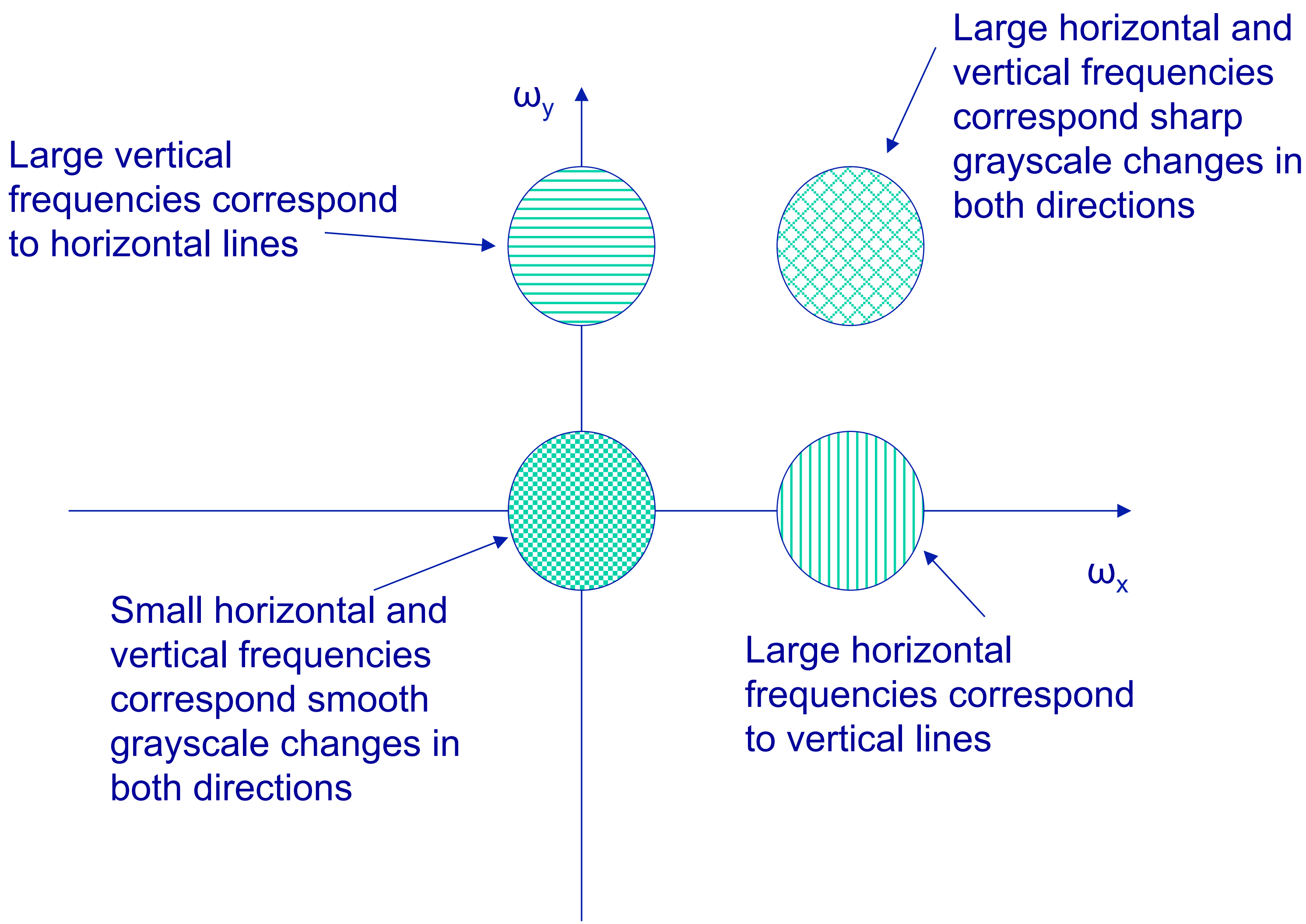
Original Image



Low-passed

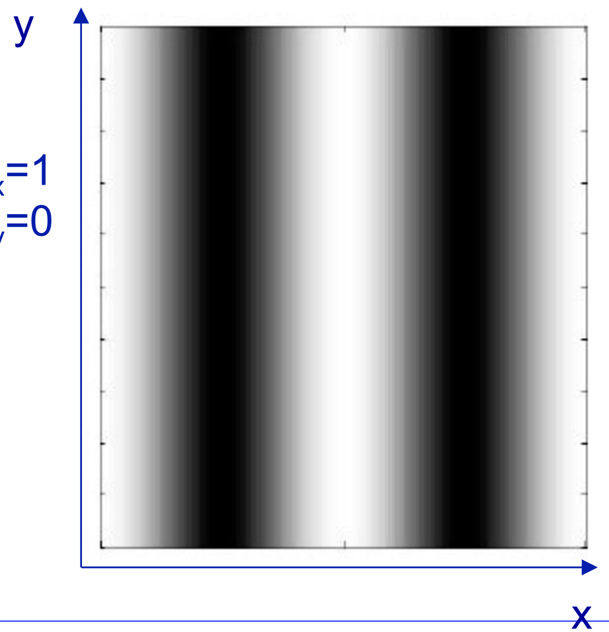


2D Frequency domain

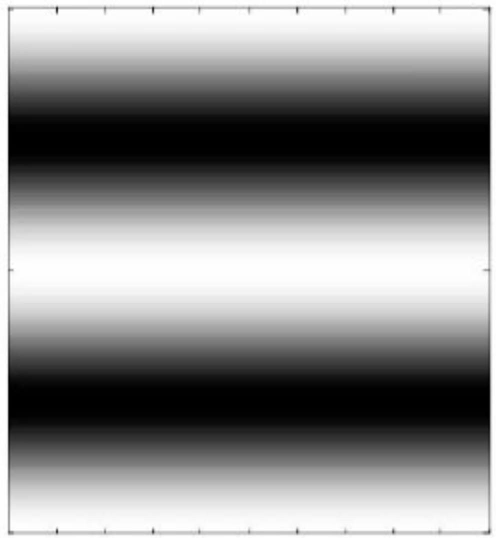


2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies



$\omega_x=1$
$\omega_y=0$



$\omega_x=0$
$\omega_y=1$

2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dxdy$$

$$f(x,y) = \int_{-\infty}^{+\infty} \hat{f}(u,v)\, e^{j2\pi(ux+vy)}\, dudv =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|f(x,y)\right|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\hat{f}(u,v)\right|^2 dudv$$

Plancherel’ s equality

Delta

- Sampling property of the 2D-delta function (Dirac's delta)

$$\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) f(x, y) dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\left\{\delta(x, y)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) e^{-j2\pi(ux+vy)} dxdy = 1$$

$$F\left\{\delta(x - x_0, y - y_0)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) e^{-j2\pi(ux+vy)} dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

Constant functions

- Inverse transform of the impulse function

$$F^{-1}\left\{\delta(u,v)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u,v) e^{j2\pi(ux+vy)} du dv = e^{j2\pi(0x+v0)} = 1$$

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y) = 1 \qquad \forall x, y$$

$$F\left(u,v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)} dx dy = \delta(u,v)$$

Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

$$s(x,y) = \cos(2\pi fx)$$

$$F\left\{\cos(2\pi fx)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2\pi fx)e^{-j2\pi(ux+vy)}dxdy =$$

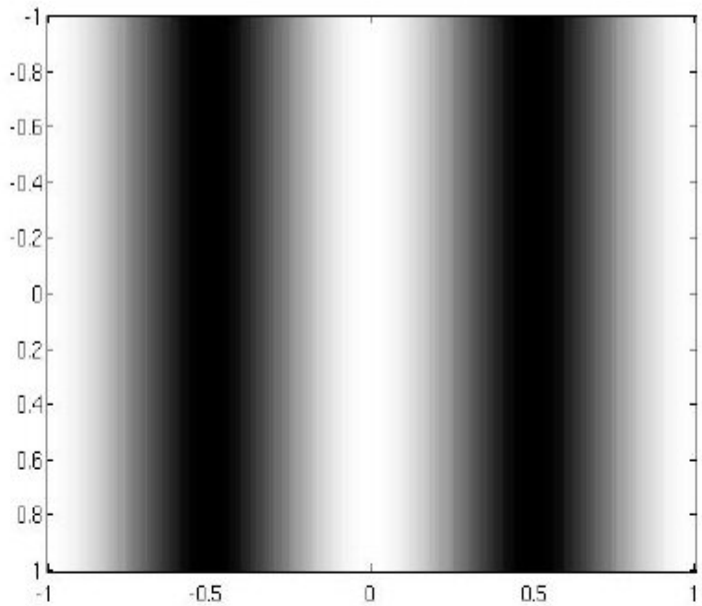
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{e^{j2\pi(fx)}+e^{-j2\pi(fx)}}{2}\right]e^{-j2\pi(ux+vy)}dxdy$$

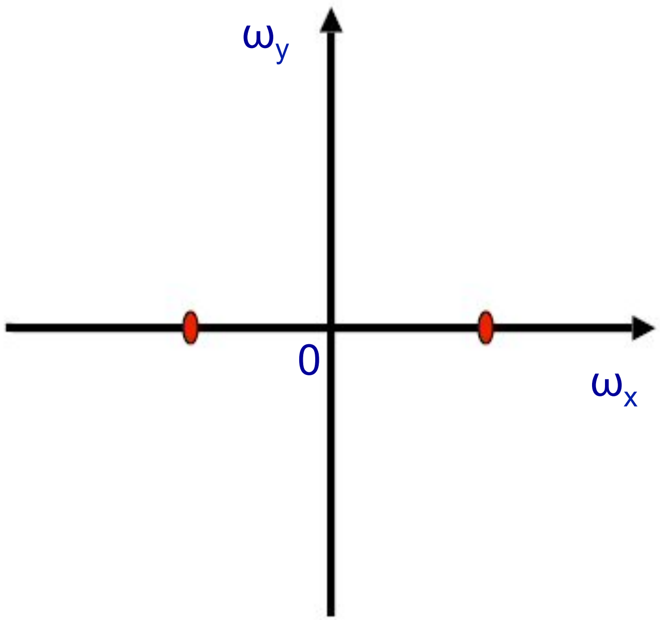
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]e^{-j2\pi vy}dxdy =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}e^{-j2\pi vy}dy\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx = \frac{1}{2}1\int_{-\infty}^{\infty}\left[e^{-j2\pi(u-f)x}+e^{-j2\pi(u+f)x}\right]dx =$$

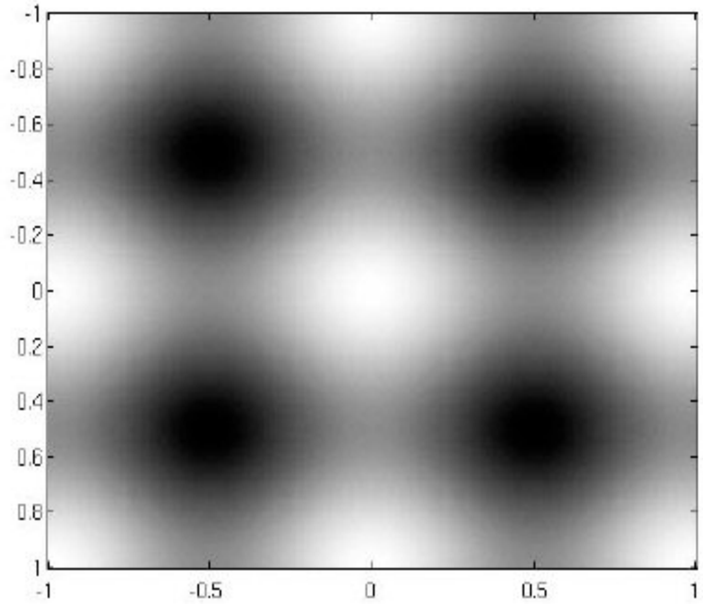
$$\frac{1}{2}\left[\delta(u-f)+\delta(u+f)\right]$$

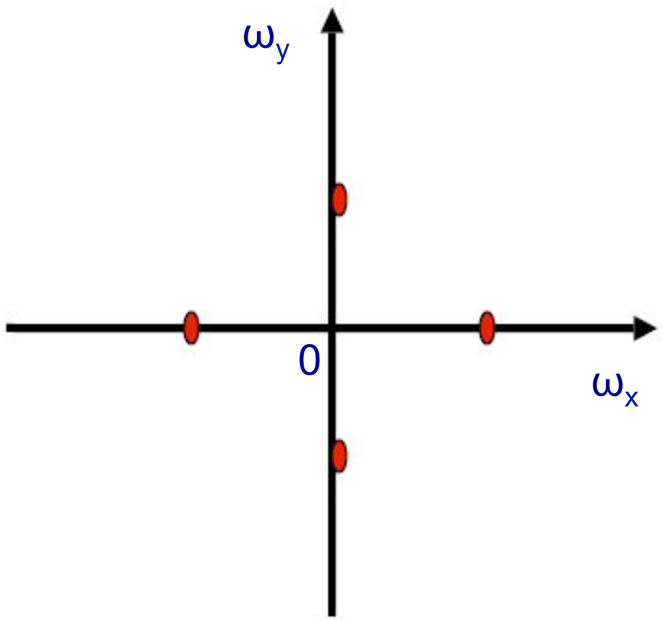
Vertical grating



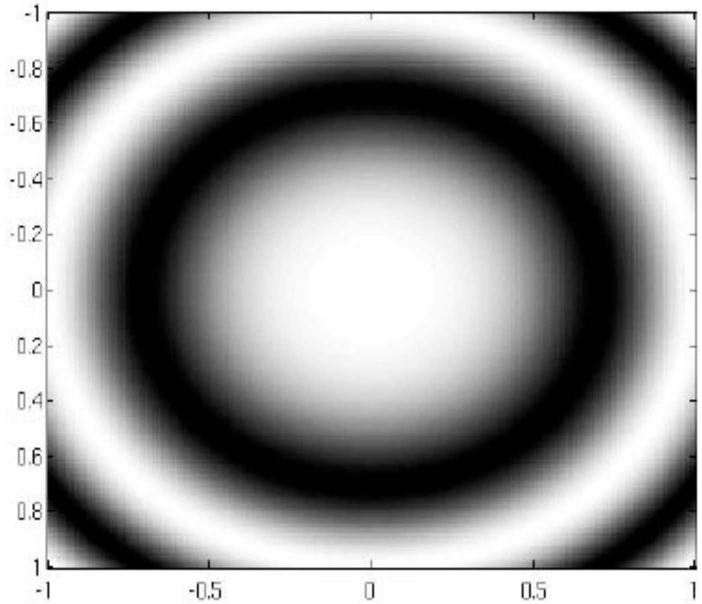


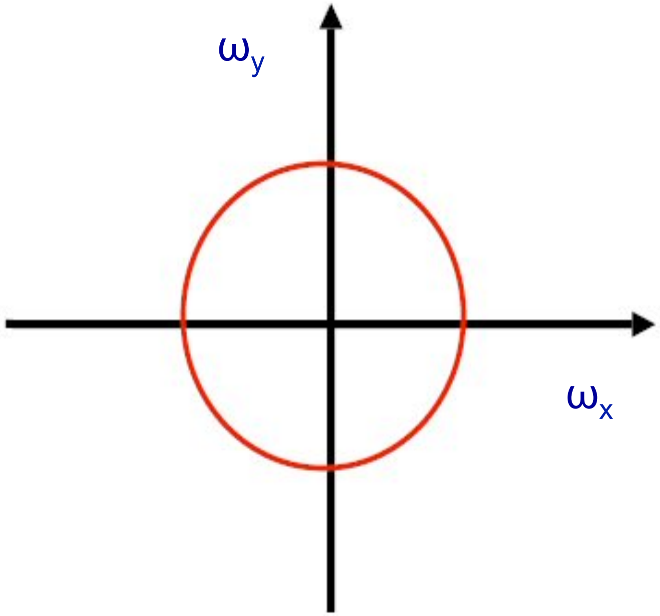
Double grating



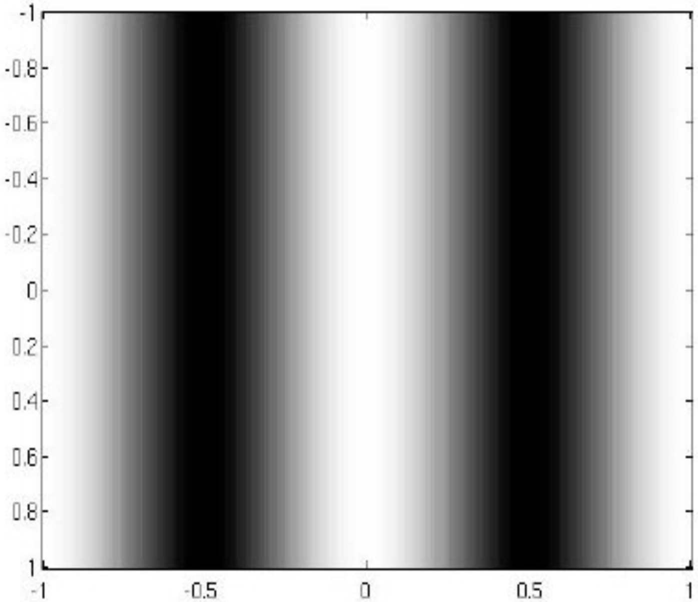


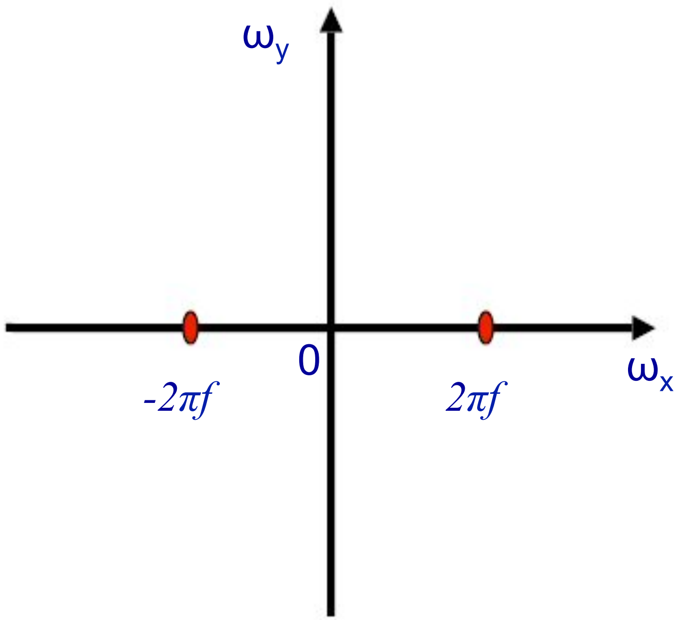
Smooth rings





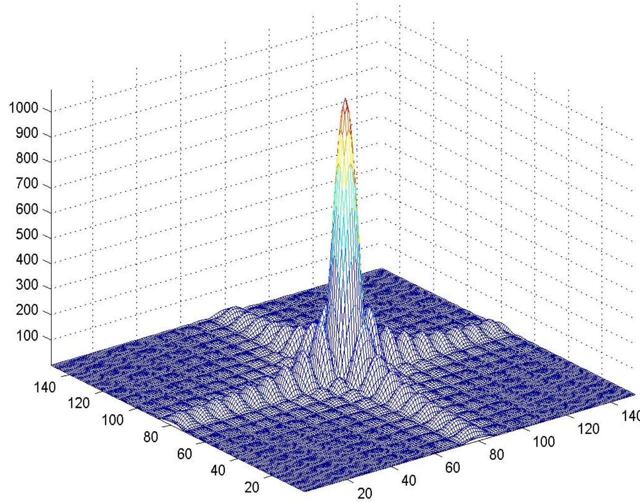
Vertical grating

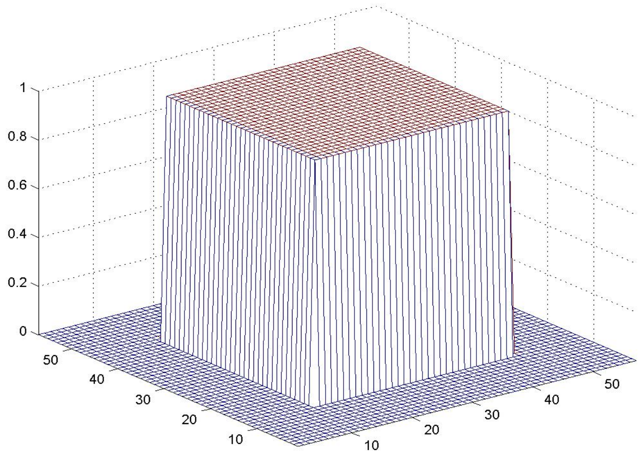


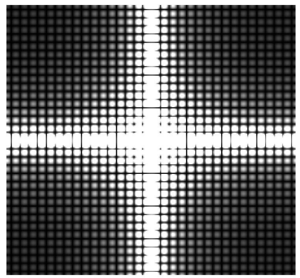


2D box

2D sinc







CTFT properties

- Linearity $$af(x,y)+bg(x,y) \Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f(x-x_0, y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0x+v_0y)}f(x,y) \Leftrightarrow F(u-u_0, v-v_0)$$
- Convolution $$f(x,y)*g(x,y) \Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f(x,y)g(x,y) \Leftrightarrow F(u,v)*G(u,v)$$
- Separability $$f(x,y)=f(x)f(y) \Leftrightarrow F(u,v)=F(u)F(v)$$

Separability

1. Separability of the 2D Fourier transform
   – 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} e^{-j2\pi vy}dy\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx =$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy = F\left(u,v\right)$$

2D FT $\Rightarrow$ 1D FT along the rows $\rightarrow$ 1D FT along the cols

Separability

- Separable functions can be written as $f(x,y) = f(x)g(y)$

2. The FT of a separable function is the product of the FTs of the two functions

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy\int_{-\infty}^{\infty} h(x)e^{-j2\pi ux}dx =$$

$$= H(u)G(v)$$

$$f(x,y) = h(x)g(y) \Rightarrow F(u,v) = H(u)G(v)$$

Discrete Time Fourier Transform (DTFT)

Applies to Discrete domain signals and time series - 2D

Fourier Transform: 2D Discrete Signals

- Fourier Transform of a 2D discrete signal is defined as

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

where $\frac{-1}{2} \leq u,v < \frac{1}{2}$

- Inverse Fourier Transform

$$f[m,n] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{j2\pi(um+vn)} du dv$$

Properties

- Periodicity: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - *The period is 1 for the unitary frequency notations and 2π for normalized frequency notations*.
  - Proof (referring to the firsts case)

$$F(u+k, v+l) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

1 1

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)} e^{-j2\pi km} e^{-j2\pi ln}$$

Arbitrary integers

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

$$= F(u,v)$$

Fourier Transform: Properties

- Linearity, shifting, modulation, convolution, multiplication, separability, energy conservation properties also exist for the 2D Fourier Transform of discrete signals.

DTFT Properties

- Linearity $$af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0m+v_0n)}f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$$
- Convolution $$f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$$
- Separable functions $$f[m,n]=f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$$
- Energy conservation $$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|f[m,n]\right|^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$$

Fourier Transform: Properties

- Define *Kronecker delta function*

$$\delta[m,n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Fourier Transform of the Kronecker delta function

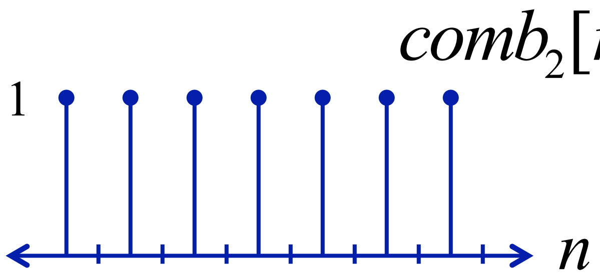
$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\delta[m,n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1$$

Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

where *M* and *N* are integers



Impulse Train

$$comb_{M,N}[m,n] \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

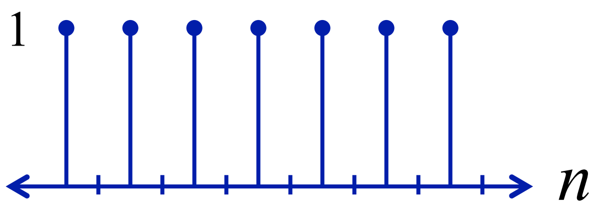
$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta\left(x-kM, y-lN\right)$$

- Fourier Transform of an impulse train is also an impulse train:

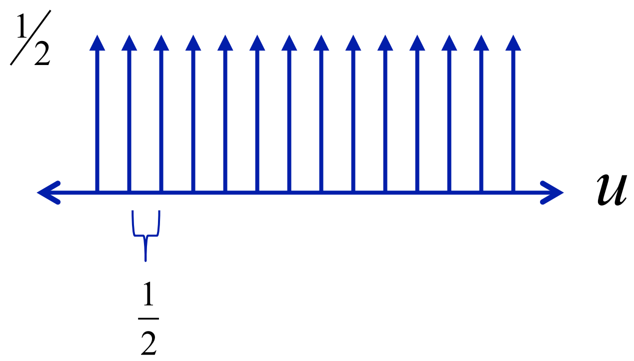
$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta\left[m-kM, n-lN\right]}_{comb_{M,N}[m,n]} \Leftrightarrow \underbrace{\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \delta\left(u-\frac{k}{M}, v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

Impulse Train

$$comb_2[n] \qquad\qquad \frac{1}{2}comb_{\frac{1}{2}}(u)$$



$\Leftrightarrow$



Impulse Train

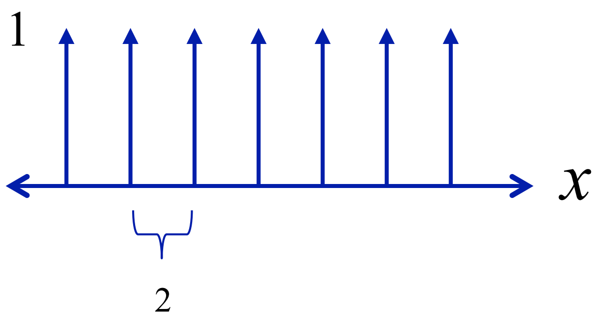
$$comb_{M,N}(x,y)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)$$

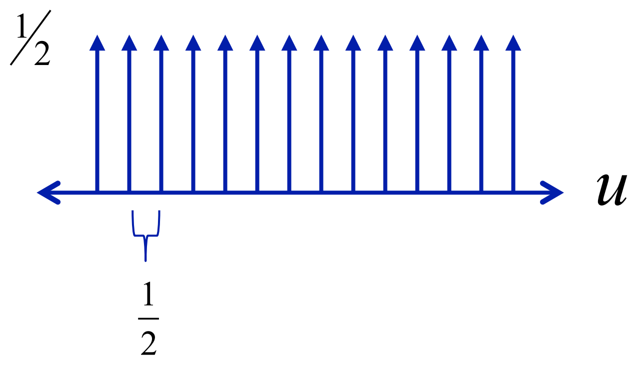
- In the case of continuous signals:

$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)}_{comb_{M,N}(x,y)}\Leftrightarrow\frac{1}{MN}\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M},v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

Impulse Train

$$comb_2(x) \quad \Leftrightarrow \quad \frac{1}{2}comb_{\frac{1}{2}}(u)$$





2D DTFT: constant

- Fourier Transform of 1

$$f[k,l] = 1, \forall k,l$$

$$F[u,v] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[1 \times e^{-j2\pi(uk+vl)} \right] =$$

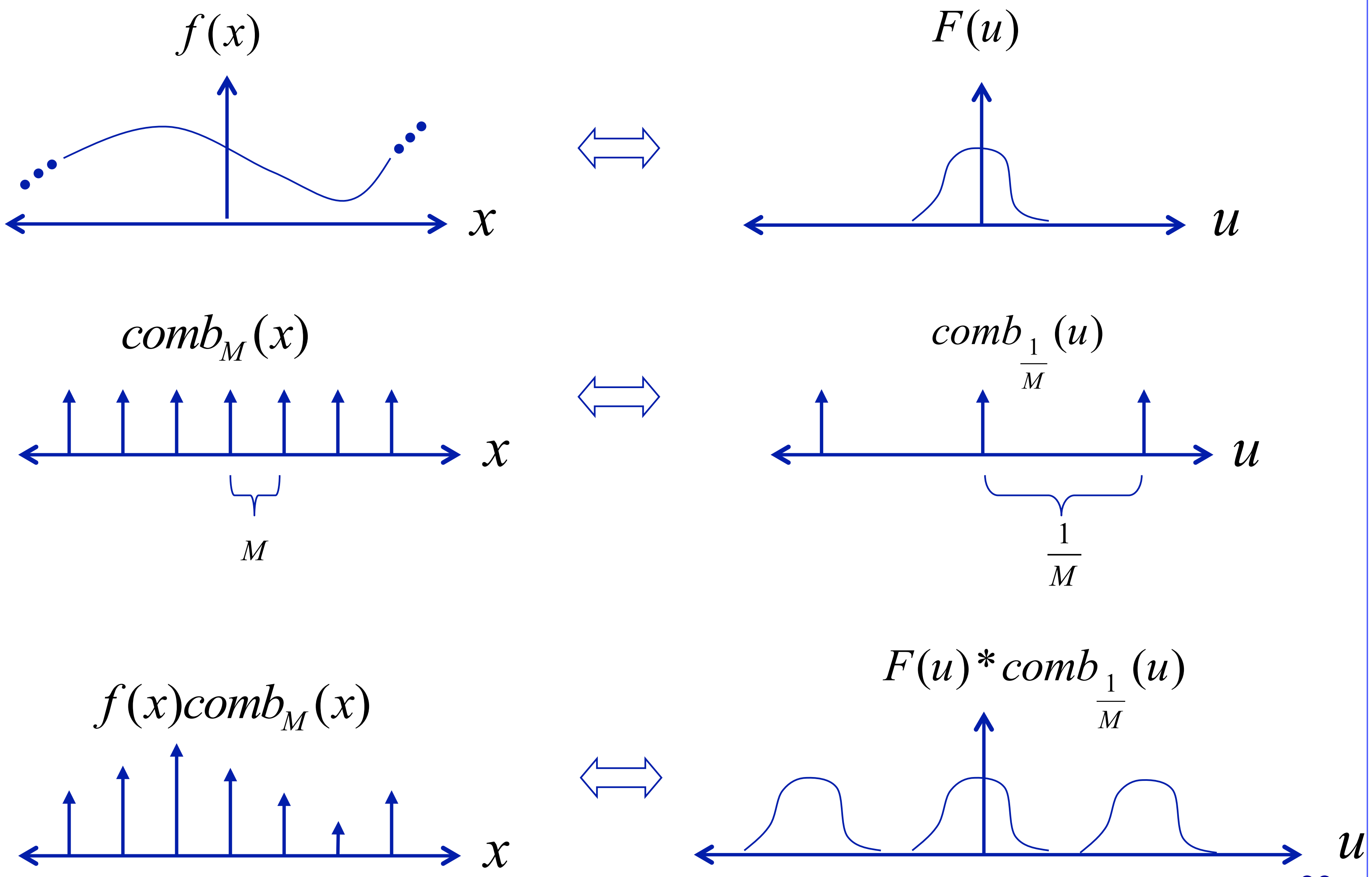
$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(u-k, v-l)$$

periodic with period 1
along u and v

To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.

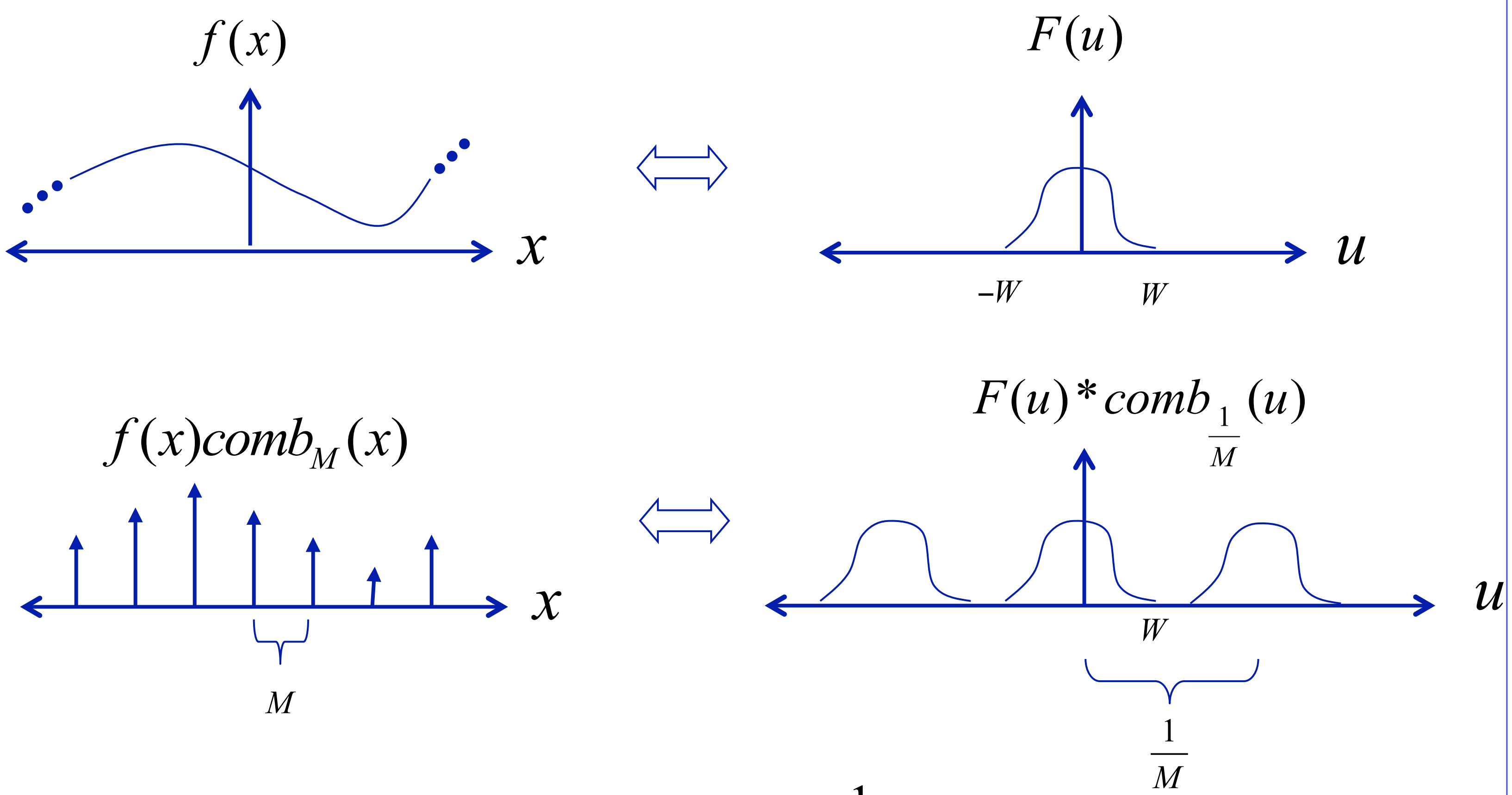
Sampling theorem revisited

Sampling



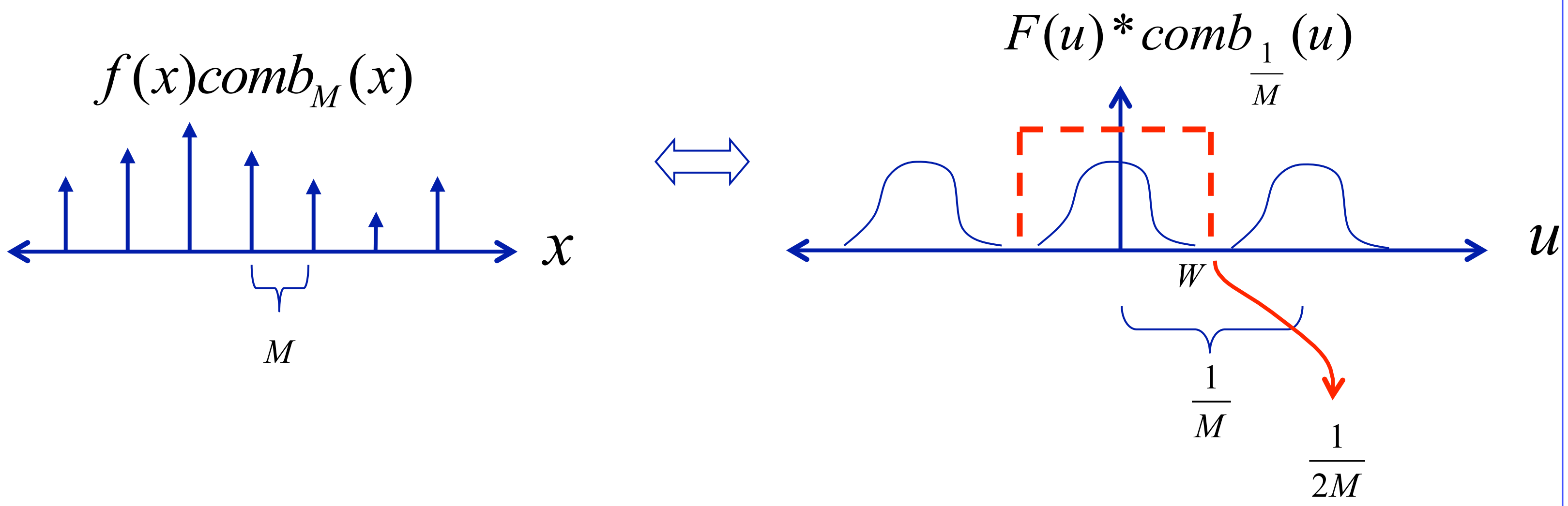
$f(x)$
x
$F(u)$
u
$comb_M(x)$
x
M
$comb_{\frac{1}{M}}(u)$
u
$\frac{1}{M}$
$f(x)comb_M(x)$
x
$F(u) * comb_{\frac{1}{M}}(u)$
u

Sampling



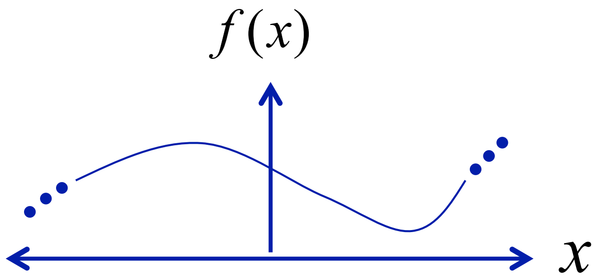
Nyquist theorem: No aliasing if $\frac{1}{M} > 2W$

Sampling

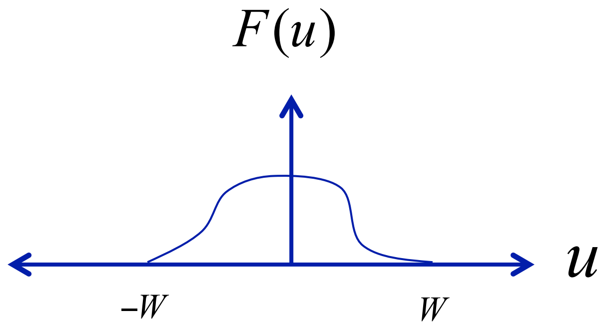


If there is no aliasing, the original signal can be recovered from its samples by low-pass filtering.

Sampling

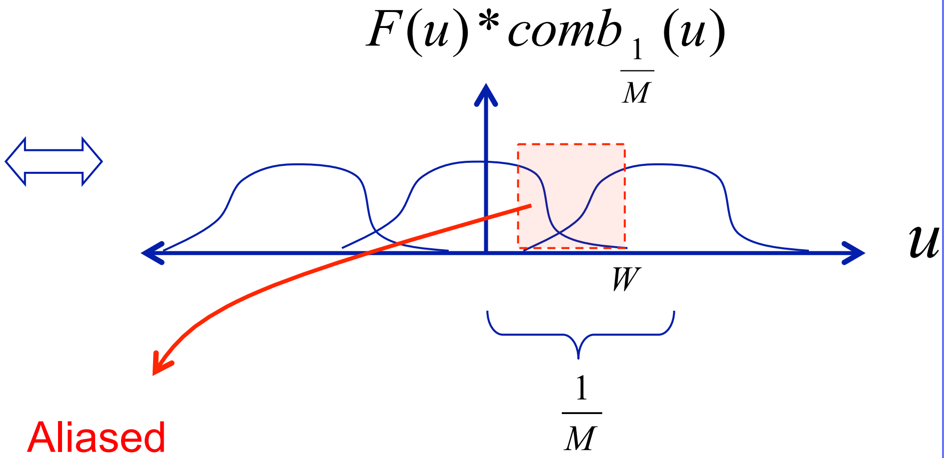


$\Leftrightarrow$

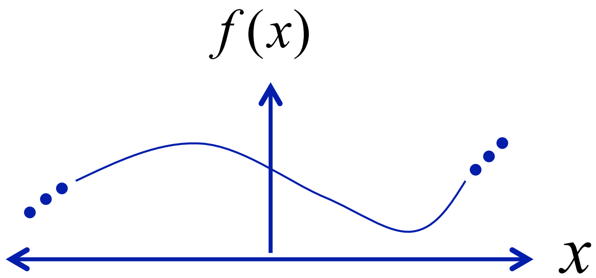


$$f(x)comb_M(x)$$

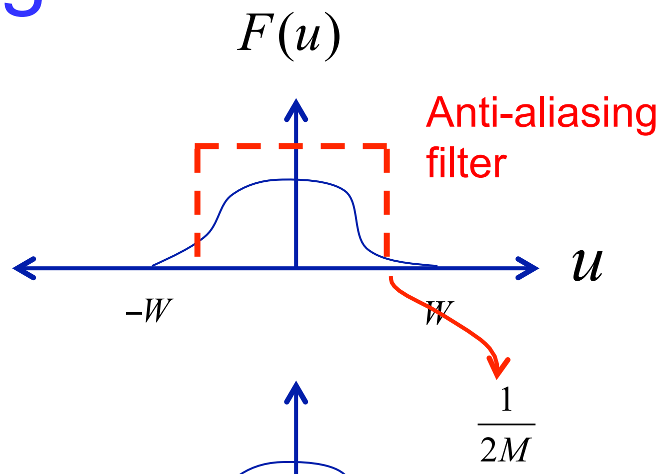
$\Leftrightarrow$



Sampling

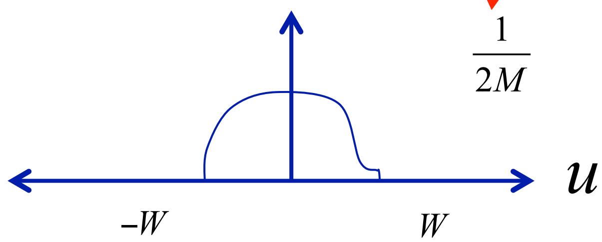


$\Longleftrightarrow$



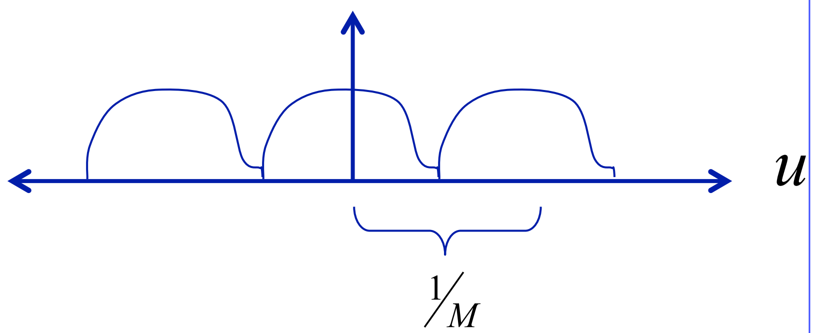
$$f(x)*h(x)$$

$\Longleftrightarrow$



$$\left[f(x)*h(x)\right]comb_M(x)$$

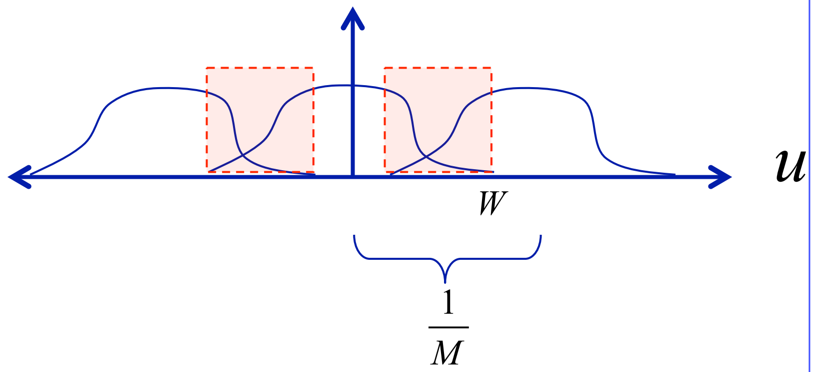
$\Longleftrightarrow$



Sampling

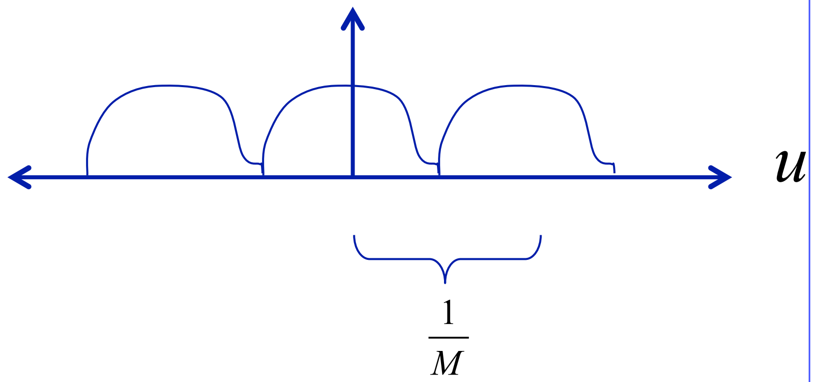
- Without anti-aliasing filter:

$$f(x)comb_M(x) \Leftrightarrow$$

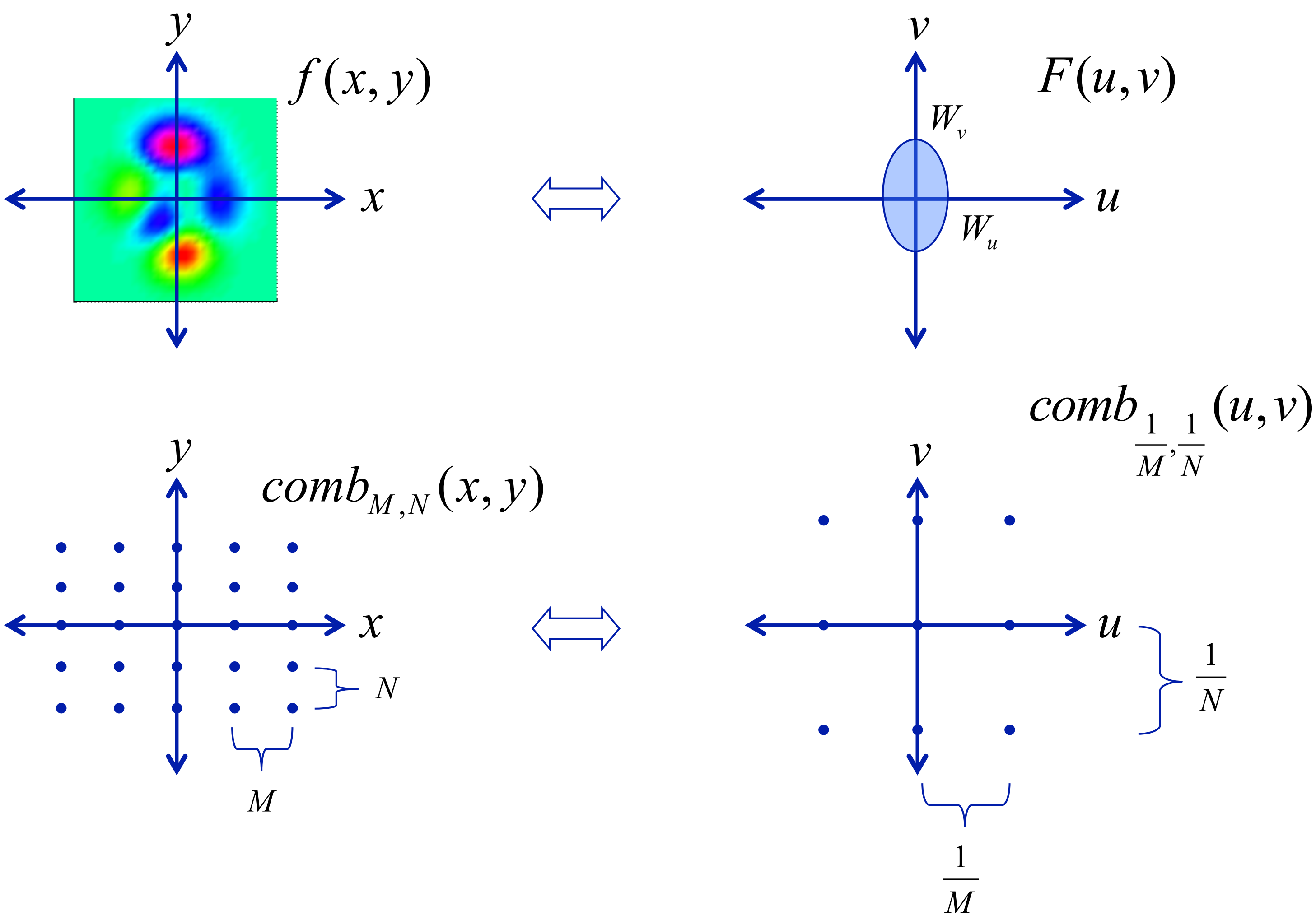


- With anti-aliasing filter:

$$\left[f(x)*h(x)\right]comb_M(x) \Leftrightarrow$$

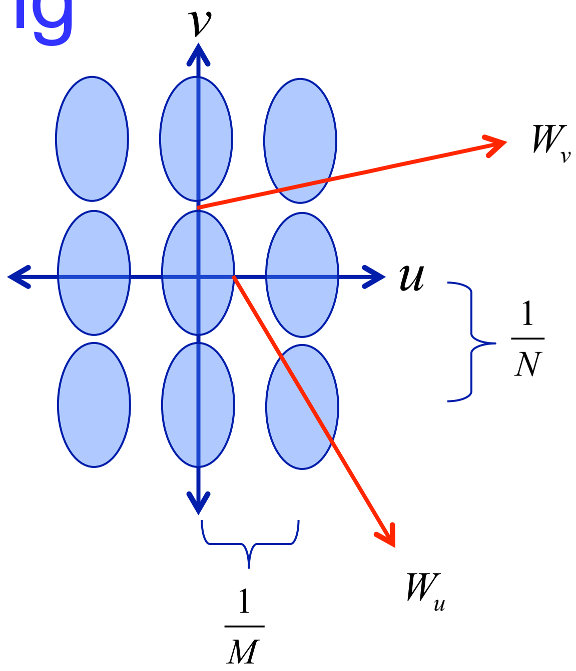


Sampling in 2D (images)



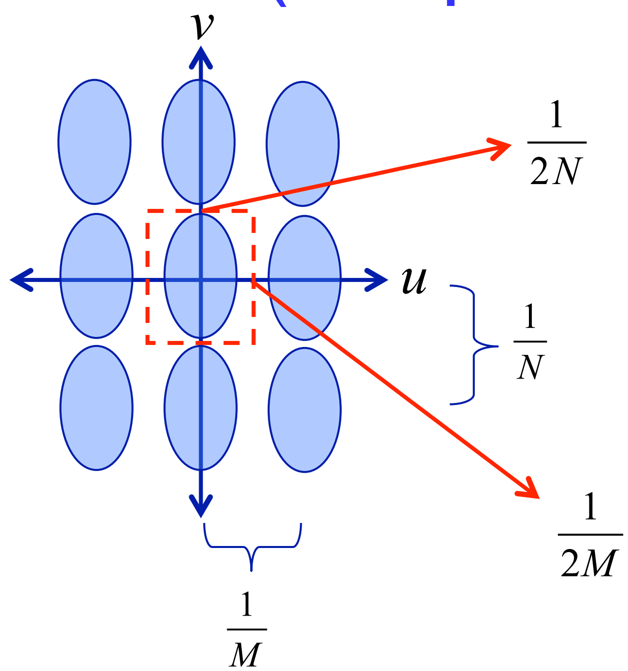
Sampling

$$f(x,y)comb_{M,N}(x,y) \quad \Longleftrightarrow$$



No aliasing if $\frac{1}{M} > 2W_u$ *and* $\frac{1}{N} > 2W_v$

Interpolation (low pass filtering)



Ideal reconstruction filter:

$$H(u,v) = \begin{cases} MN, \text{ for } u \le \dfrac{1}{2M} \text{ and v} \le \dfrac{1}{2N} \\ 0, \text{ otherwise} \end{cases}$$

Anti-Aliasing

```
a=imread('barbara.tif');
```

Anti-Aliasing

```
a=imread('barbara.tif');
b=imresize(a,0.25);
c=imresize(b,4);
```

Anti-Aliasing

```
a=imread('barbara.tif');
b=imresize(a,0.25);
c=imresize(b,4);

H=zeros(512,512);
H(256-64:256+64, 256-64:256+64)=1;

Da=fft2(a);
Da=fftshift(Da);
Dd=Da.*H;
Dd=fftshift(Dd);
d=real(ifft2(Dd));
```