6.5 Exercises - Part 3

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Exercise 9. (a) Let $M = N_1 \oplus N_2$ be a module and let P_1 and P_2 be projective covers of N_1 and N_2, respectively. Show that $P_1 \oplus P_2$ is a projective cover of M.

(b) Let M be a module of finite length with $M/\operatorname{Rad}(M) = S_1 \oplus \dots S_r$. Show that there exists a superfluous epimorphism $P(S_1) \oplus \cdots \oplus P(S_r) \to M$ and conclude that $P(M) = P(M/\operatorname{Rad}(M)) = P(S_1) \oplus \cdots \oplus P(S_r)$.

(Hint: $\operatorname{Rad}(M)$ is superfluous in M, so...)

(c) Prove that the injective envelope $E(S)$ of any simple module S is indecomposable.

(d) Show that any indecomposable injective module E is the injective envelope of its socle. Deduce that $\operatorname{Soc} E$ is a simple module.

Exercise 10. (a) Let M be an indecomposable left R-module of finite length, and let $f \in \operatorname{End}_R(M)$. Show that the following statements are equivalent.
(i) f is a monomorphism,
(ii) f is an epimorphism,
(iii) f is an isomorphism,
(iv) f is not nilpotent.

In particular, if f is not invertible, then gf is not invertible for any $g \in \operatorname{End}_R(M)$.

(b) Prove Schur's Lemma: If S is a simple module, then $\operatorname{End}_R S$ is a skew field.
Is the converse true?

Exercise 11. Let $p \in \mathbb{N}$ a prime and $M = \{\frac{a}{p^n} \in \mathbb{Q} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$.

(a) Verify that $\mathbb{Z} \leq M \leq \mathbb{Q}$ in $\mathbb{Z}\operatorname{Mod}$.

(b) Let $\mathbb{Z}_{p^\infty} = M/\mathbb{Z}$. Show that $\mathbb{Z}_{p^\infty}$ is a divisible group.

(c) show that any $L \leq \mathbb{Z}_{p^\infty}$ is cyclic, generated by an element $\frac{1}{p^l}$, $l \in \mathbb{N}$.

Conclude the the lattice of the subgroups of $\mathbb{Z}_{p^\infty}$ is a well-ordered chain, and $\mathbb{Z}_{p^\infty}$ does not have any maximal subgroup.

Exercise 12. (a) Let $F : \mathcal{B} \longrightarrow \mathcal{C}$ be a functor and let B and B' be two objects in $\mathcal{B}$. Show that:
- if B and B' are isomorphic in $\mathcal{B}$, then the objects $F(B)$ and $F(B')$ are isomorphic in $\mathcal{C}$;
- if F is full and faithful, then the converse is also true.

(b) Let R and S be two rings and let $G : Mod(R) \longrightarrow Mod(S)$ be an equivalence of categories. Show that G is an exact functor.