COURSE OF OPTIMIZATION
Class exercises
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1. Exercises on convex analysis

EXERCISE 1.1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Prove that the following facts are equivalent:

(1) for every $\alpha \in \mathbb{R}$ the set $C_\alpha := \{x : f(x) \le \alpha\}$ is either empty or convex.

(2) $f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$, for all $x, y \in \mathbb{R}^n$, $\lambda \in]0,1[$.

Can we conclude that f is convex? Give a proof or a counterexample.

EXERCISE 1.2. Find all the functions $f : H \to]-\infty, +\infty]$, defined on an Hilbert space H, satisfying $f^* = f$.

EXERCISE 1.3. Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ be a matrix, $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Assume that $f(y) = +\infty$ for every $y \notin \mathrm{Im}(A) := \{z \in \mathbb{R}^m : z = Ax \text{ for a certain } x \in \mathbb{R}^n\}$. Compute $(f \circ A)^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

EXERCISE 1.4. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper function such that $f(x+y) \le f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$. Prove that there exists a unique closed convex set $C \subseteq \mathbb{R}^d$ such that $\overline{f} = \sigma_C$, where σ_C denotes the support function to C. Extend the result when $f : X \to \mathbb{R} \cup \{+\infty\}$, where X is a normed space.

EXERCISE 1.5. Let X be a normed space. Give an example showing that, in general, the supremum appearing in

$$\|p\|_{X'} = \sigma_{\overline{B_X(0,1)}}(p) = \sup_{\|q\|_X \le 1} \langle p, q\rangle_{X',X}.$$

may not be attained.

EXERCISE 1.6. Let X be a vector space, $K \subseteq X$ be a convex set such that $0 \in K$, $\alpha K \subseteq K$ for all $|\alpha| \le 1$, and for every $x \in X$ there exists $r > 0$ such that $x \in rK$. Define

$$p_K(x) = \inf\{r > 0 : x \in rK\},$$

and prove that p_K is a seminorm on X. Is p_K a norm? Provide a proof or give a counterexample.

EXERCISE 1.7. Let H be an Hilbert space, $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be not identically $+\infty$, $\ell : H \to \mathbb{R}$ a linear and continuous map, $\ell \neq 0$, $\alpha \in \mathbb{R}$. Define $g : H \to \mathbb{R} \cup \{+\infty\}$ by $g(x) := f(\ell(x) + \alpha)$ and compute $g^* : H \to \mathbb{R} \cup \{+\infty\}$.

EXERCISE 1.8. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x_1, x_2) = e^{x_1 + 2x_2}$. Compute g^*.

2. Simulation of first partial test

EXERCISE 2.1. Let Ω be a bounded open subset of $\mathbb{R}^2$. Consider the problem:

$$\inf_{u \in H_0^1(\Omega)} \int_\Omega \Bigg(5\,|\nabla u(x_1,x_2)|^2 - 2\partial_{x_2} u(x_1,x_2)\,\partial_{x_1} u(x_1,x_2) + \left(\left(x_1^4 + 3x_2^2\right) u(x_1,x_2) - 2 \right)^2 +$$

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$$+4[\partial_{x_1} u(x_1,x_2)]^2 + [\partial_{x_2} u(x_1,x_2)]^2\Big)\, dx_1\, dx_2.$$

(1) Prove that the problem admits a unique solution.
(2) State the problem in the form $\mathscr{F}(u) = F(u) + G \circ \Lambda(u)$, where $F : X \to]-\infty,+\infty]$, $G : Y \to]-\infty,+\infty]$ and $\Lambda : X \to Y$, carefully precising the function spaces X, Y and discussing the regularity properties of F, G, Λ.
(3) Write the dual problem and the extremality conditions, establish whether the dual problems admits a unique solution.
(4) Use the previous results to write a partial differential equations satisfied by the minimum.

EXERCISE 2.2. Let Ω be an open bounded subset of $\mathbb{R}^d$, $q \in H_0^1(\Omega;\mathbb{R})$ be fixed. Set:

$$\mathscr{C} := \{v \in H_0^1(\Omega;\mathbb{R}) : \|\nabla v - \nabla q\|_{L^2(\Omega;\mathbb{R}^d)} \le 1\}.$$

Consider the problem

$$\inf_{u \in \mathscr{C}} \int_\Omega \frac{|u(x)|^2}{2}\, dx.$$

(1) Prove that the problem admits a unique solution.
(2) Formulate the problem in the whole space in the form $\mathscr{F}(u) = F(u) + G \circ \Lambda(u)$, where $F : X \to]-\infty,+\infty]$, $G : Y \to]-\infty,+\infty]$, and $\Lambda : X \to Y$, carefully precising the functional spaces X, Y and discuting the regularity of F, G, Λ.
(3) Write the dual problem and the extremality relations. Establish if the dual problem admits an unique solution.

EXERCISE 2.3.

(1) Prove that the two marginals of a convex functions $\Phi : X \times Y \to \mathbb{R} \cup]-\infty,+\infty]$ are convex.
(2) Let Ω_1, Ω_2 be nonempty convex subsets of a Banach space X. We say that Ω_1, Ω_2 are an *extremal system* if for every $\varepsilon > 0$ there exists $a \in X$, $\|a\| \le \varepsilon$ such that $(\Omega_1 + a) \cap \Omega_2 = \emptyset$. Prove that Ω_1, Ω_2 are an extremal system if and only if $0 \notin \operatorname{int}(\Omega_1 - \Omega_2)$ where $\Omega_1 - \Omega_2 := \{x_1 - x_2 : x_i \in \Omega_i = 1, 2\}$.
(3) Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be defined as $f(x_1,x_2) = (3x_1 + 4x_2)^3$ if $3x_1 + 4x_2 > 0$ and $f(x_1,x_2) = +\infty$ if $3x_1 + 4x_2 \le 0$. Prove that f is convex and compute f^* and f^{**}.
(4) Let C be a closed nonempty convex subset of $\mathbb{R}^d$ with $\operatorname{int} C \neq \emptyset$. Prove that $C = \overline{\operatorname{int} C}$.
(5) Discuss the continuity properties of convex functions defined on a Banach space, proving some relevant results.

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