Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2013/14 - N.5

Pb 1. Prove that the subset of $L^2(0,1)$ defined by

$$M = \Big\{ u \in L^2(0,1) : \int_0^1 u(x)dx = 1 \Big\}$$

is closed, convex and nonempty. Then, prove that if $u \in M$, then $\|u\|_{L^2} \geq 1$ and determine the orthogonal projection of 0 on M. Finally, prove that the set $S = \{\lambda u : \lambda \in \mathbb{R},\, u \in M\}$ is dense in $L^2(0,1)$ and deduce from this that $M^\perp = \{0\}$.

Pb 2. Let (a_j) be a real positive sequence and let

$$C = \{x \in \ell^2(\mathbb{N}) : |x_j| \leq a_j, \text{ for all } j \in \mathbb{N}\}.$$

Prove that C is nonempty, convex and closed in $\ell^2(\mathbb{N})$. Moreover, determine for every $x \in \ell^2(\mathbb{N})$ the projection $P_C x$ and prove that $C^\perp = \{0\}$.

Pb 3. Let X be a Hilbert space, let (C_h) be a nonincreasing sequence of closed and convex subspaces of X with nonempty intersection and let C denote the intersection. Prove that C is convex and closed in X and that for all $x \in X$ and $h \in \mathbb{N}$

$$\|x - P_{C_h}x\| \leq \|x - P_{C_{h+1}}x\| \leq \|x - P_C x\|.$$

Finally, for every $x \in X$ determine the limit of the sequence $(P_{C_h}x)$ in X.

Pb 4. Let $\alpha \in \mathbb{R}$ and $f_\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_\alpha(x,y) = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+1)^\alpha}.$$

For all $k \in \mathbb{N} \cup \{\infty\}$, let us set $A_k = \{\alpha \in \mathbb{R} : f_\alpha \in L^k(\mathbb{R}^2)\}$. Prove that (A_k) is strictly increasing, namely $A_k \subsetneq A_{k+1}$, and that

$$\bigcup_{k=1}^{\infty} A_k \subsetneq A_\infty.$$

Pb 5. Let $\psi : (0,1) \to \mathbb{R}$ me a measurable function and let

$$C = \big\{ u \in L^2(0,1) : u(x) \geq \psi(x) \text{ for a.e. } x \in (0,1) \big\}.$$

Prove that C is closed, convex, nonempty if and only if $\psi^+ \in L^2(0,1)$. In the latter case determine the function $P_C : L^2(0,1) \to C$.

Pb 6. Let Y be the set

$$Y = \big\{ u \in L^2(\mathbb{R}^2) : \; u(x,y) = u(y,x) \text{ for a.e. } (x,y) \in \mathbb{R}^2 \big\}.$$

Prove that Y is a close subspace of $L^2(\mathbb{R}^2)$ and that, for every $u \in L^2(\mathbb{R}^2)$ the function

$$\frac{u(x,y)+u(y,x)}{2},$$

belongs to Y. Furthermore, determine $Y^\perp$.

Pb 7. Let X be a real Hilbert space and $\{e_h : h \in \mathbb{N}\}$ be a complete orthonormal system in X. Define, for every $k \geq 1$, the operator $L_k : X \to X$ by setting

$$L_k x := \sum_{h=0}^{k} (x, e_h) e_h, \quad \text{for all } x \in X.$$

Prove that each L_k is linear and continuous and that

$$\lim_k \|L_k x - x\| = 0, \quad \text{for all } x \in X.$$

Furthermore, prove that each $L_k : X \to X$ is a compact operator and that the sequence (L_k) cannot be convergent in the operator norm $\mathcal{L}(X,X)$.

Pb 8. Let $E \subset \mathbb{R}^n$ be a measurable set and $\varphi : E \to [0,+\infty)$ me a measurable function and let

$$C = \big\{ u \in L^2(E) : |u(x)| \leq \varphi(x) \text{ for a.e. } x \in E \big\}.$$

Prove that $C \subset L^2(E)$ is closed, convex, nonempty. Determine the projection $P_C : L^2(E) \to C$.

Pb 9. Let $E \subset \mathbb{R}^n$ be a measurable set, $1 \leq p < +\infty$ and

$$Y = \Big\{ u \in L^p(E) : \int_{E \cap B(0,1)} u(x)dx = 0 \Big\}.$$

Say when $Y \neq \emptyset$. Prove that Y is a vectorial subspace of $L^p(E)$. When $Y \neq \emptyset$, determine an element $v \in L^p(E) \setminus Y$ and $\varphi \in (L^p(E))'$ such that $\varphi v = 1$ and $\varphi u = 0$ for any $u \in Y$.

Pb 10. Consider the subspace Y of even functions of $L^2(-\pi,\pi)$. Prove that Y is closed and describe the orthogonal decomposition $L^2(-\pi,\pi) = Y \oplus Y^\perp$.

Pb 11. In $L^2(0,\infty)$ consider the elements $e_h = \chi_{]h,h+1[}$ with $h \in \mathbb{N}$. Prove that $\{e_h\}_{h\in\mathbb{N}}$ is an orthogonal system but that it cannot be a complete system in $L^2(0,\infty)$.

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