— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XII

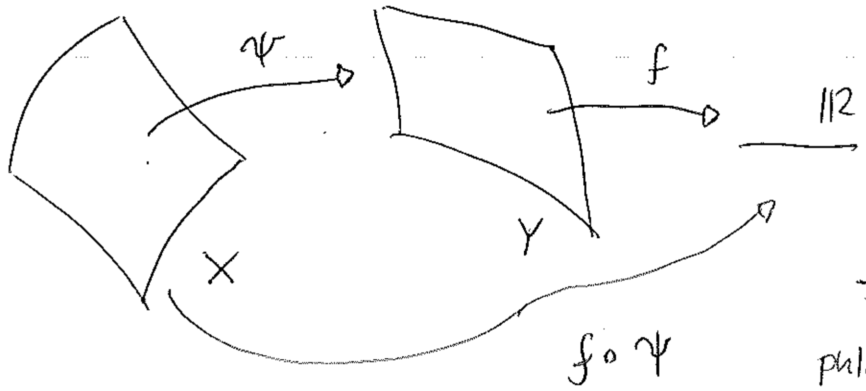
★ Smooth maps between manifolds

Let X, Y be differentiable manifolds of dimension n and m respectively

★ A map $\psi : X \longrightarrow Y$ is said to be smooth

if, whenever $\underbrace{f \in C^\infty(Y, \mathbb{R})}_{\text{smooth function on } Y}$, then $f \circ \psi \in C^\infty(X, \mathbb{R})$

" smooth is who smooth does "



$f \circ \psi =: \psi^* f$

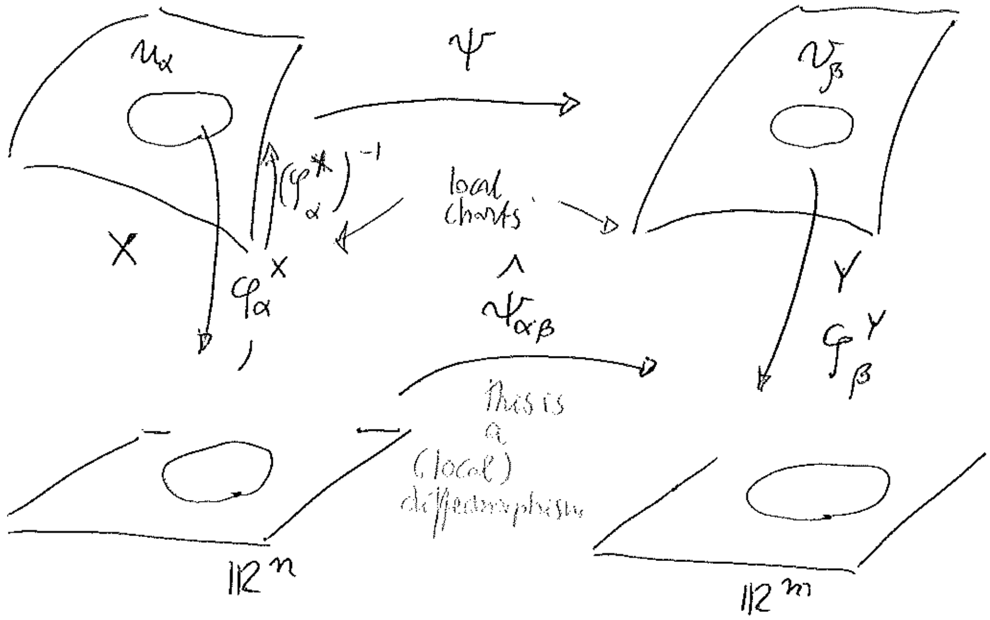
pull-back of f via ψ

" knowing functions on X one knows X "

Def. $\psi : X \longrightarrow Y$ is said to be a (smooth) diffeomorphism

if
- ψ is bijective (and smooth)
- ψ^{-1} is smooth

Locally, $\psi : X \to Y$ induces smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ (upon applying the definition to local coordinate functions)



$$\hat{\Psi}_{\alpha\beta} = \varphi_\beta^Y \circ \Psi \circ \left(\varphi_\alpha^X\right)^{-1}$$

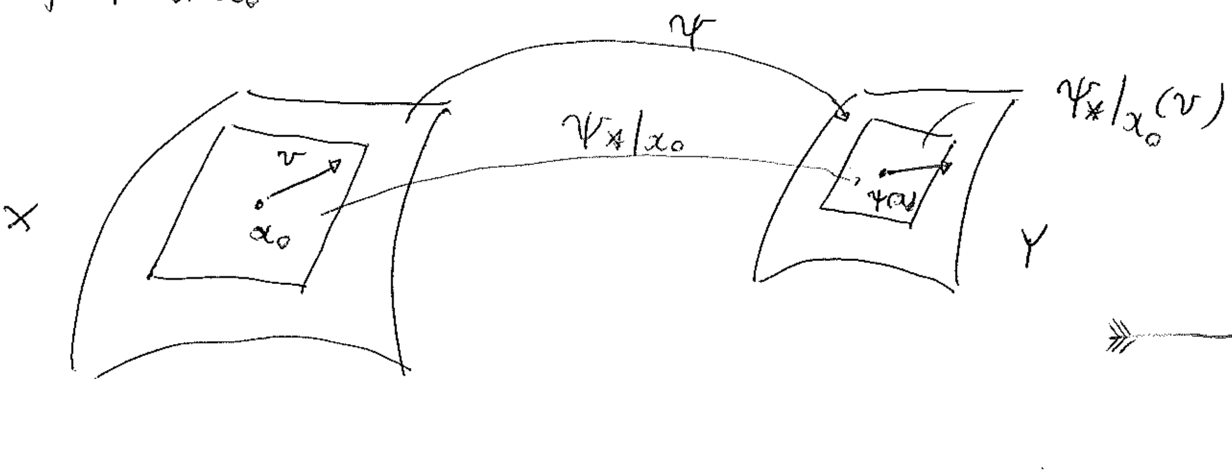
if Ψ is smooth, and $f \in C^\infty(Y, \mathbb{R})$, then $f \circ \Psi = \Psi^* f \in C^\infty(X, \mathbb{R})$ by definition

✦ Differential of a smooth map.

Let $\Psi : X \to Y$ smooth and $x_0 \in X$. One can define

$$d\Psi|_{x_0} \equiv \Psi_*|_{x_0} : T_{x_0}X \longrightarrow T_{\Psi(x_0)}Y$$
$$\qquad\qquad\qquad\qquad \cup\!\!\!| \qquad\qquad\qquad$$
$$v \longmapsto \Psi_*|_{x_0}(v)$$

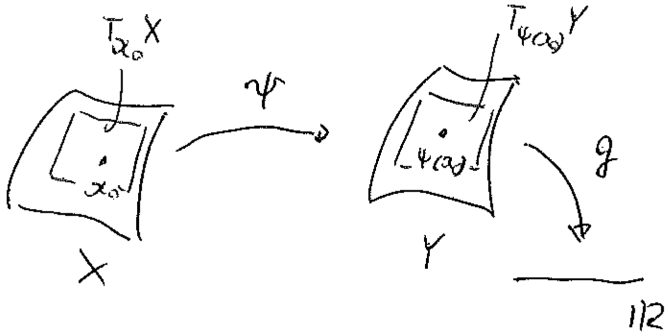
differential (push-forward) of Ψ at x_0



via the position

$$\boxed{\Big(\psi_*\big|_{x_0}(\underset{\substack{\cap \\ T_{x_0}X}}{v})\Big)(\underset{\substack{\cap \\ C^\infty(Y,\psi(x_0),\mathbb{R})}}{g}) := v(\underset{\substack{\cap \\ C^\infty(X,x_0,\mathbb{R})}}{\overset{\psi^* g}{g\circ\psi}})}$$

This is a derivation, acting on functions defined on a neighbourhood of $\psi(x_0)$



One finds that

$$\boxed{\Big(\psi_*\big|_{x_0}(v)\Big)(g) = \sum_{j=1}^{m}\left[v(y^j\circ\psi)\frac{\partial}{\partial y^j}\right](g)}$$

In particular $$\psi_*\big|_{x_0}\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{m}\frac{\partial}{\partial x^i}(y^j\circ\psi)\frac{\partial}{\partial y^j}$$

$$\frac{\partial}{\partial x^i} \longmapsto \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

$$\frac{\partial}{\partial x} \longmapsto J^T\frac{\partial}{\partial y}$$

$$\equiv \ \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} \ \equiv \ J^T$$

Let us provide some details $\qquad \psi : X \to Y$

φ : coordinate system $(x^1 \dots x^n)$ around $x_0 \in X$
local chart

τ : $\quad$ = $\quad (y^1 \dots y^m)$ around $y_0 \in Y$

Compute:

$$\left(\psi_*\big|_{x_0}(v) \right)(g) = v(g \circ \psi) = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}(g \circ \psi)$$

$$= \sum_{i=1}^{n} a^i \frac{\partial}{\partial r^i} \left(\underbrace{g \circ \tau^{-1}} \circ \underbrace{\tau \circ \psi \circ \varphi^{-1}} \right)\Big|_{\varphi(x_0)}$$

chain rule

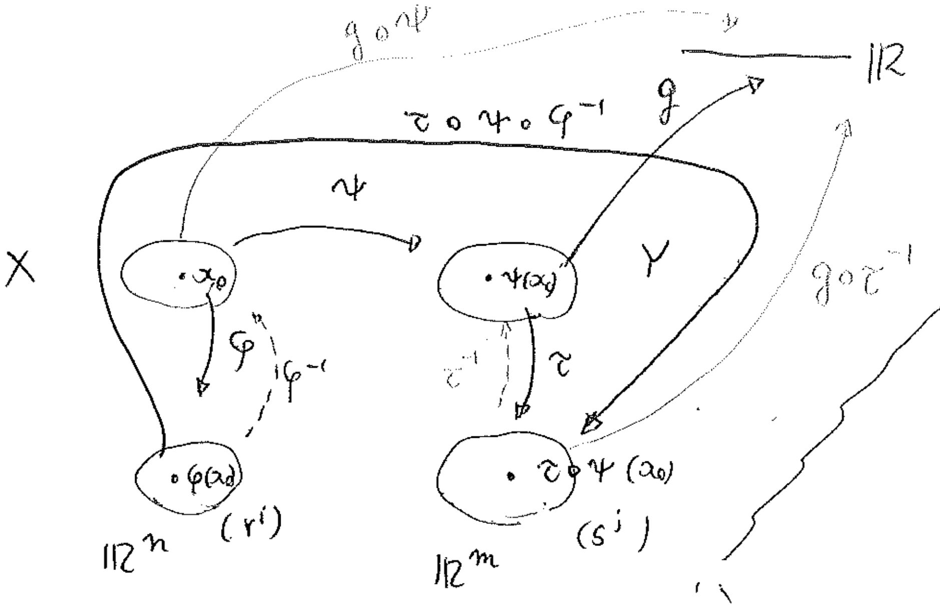
$$= \sum_{i=1}^{n} a^i \sum_{j=1}^{m} \frac{\partial}{\partial s^j}\left(g \circ \tau^{-1}\right)\Big|_{\tau \circ \psi(x_0)} \cdot \frac{\partial}{\partial r^i}\left(s_j \circ \tau \circ \psi \circ \varphi^{-1}\right)\Big|_{\varphi(x_0)}$$

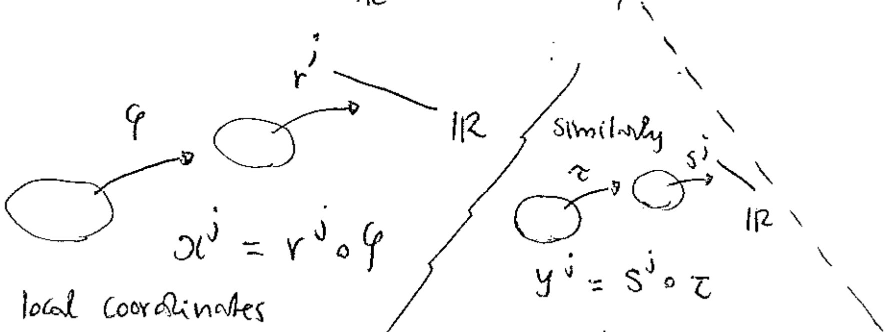
coordinate maps on $\mathbb{R}^m$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a^i \frac{\partial g}{\partial y^j} \cdot \frac{\partial}{\partial x^i}(y_j \circ \psi)$$

$$= \underbrace{\left[\sum_{j=1}^{m} v(y_j \circ \psi) \frac{\partial}{\partial y^j} \right]}(g)$$

this is indeed a tangent vector at $\psi(x_0)$





We find, in particular

$$\psi_*\big|_{x_0} : \quad T_{x_0}X \longrightarrow T_{\psi(x_0)}Y$$

$$\frac{\partial}{\partial x^i} \longmapsto \sum_{j=1}^{m} \underbrace{\frac{\partial}{\partial x^i}(y_j \circ \psi)}_{(\psi_*|_x)_{ji} \,\sim\, J^T} \frac{\partial}{\partial y_j}$$

Let us also check that if $X \xrightarrow{\psi} Y \xrightarrow{\Phi} Z \overset{h}{\dashrightarrow} \mathbb{R}$

then $d(\Phi \circ \psi) = d\Phi \circ d\psi$ generalized chain rule

as homomomorphisms
no matrix product.

$$\boxed{(\Phi \circ \psi)_* = \Phi_* \circ \psi_*}$$

associativity of $\circ$

$$\left[(\Phi \circ \psi)_*(v)\right](h) \underset{\text{def}}{=} v\left(h \circ (\Phi \circ \psi)\right) = v\left((h \circ \Phi) \circ \psi\right)$$

$$= \psi_*(v)\left(h \circ \Phi\right) = \Phi_*\left(\psi_*(v)\right)(h)$$

$$= \left[(\Phi_* \circ \psi_*)(v)\right](h) \qquad \square$$

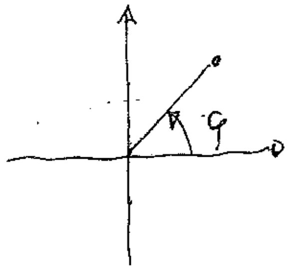
★ Example. Let $M = \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$

$$\Psi : \underset{\substack{\Psi \\ M}}{(\rho, \varphi)} \longmapsto (x,y) \in \mathbb{R}^2 \qquad \Psi : \begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases}$$

$\rho > 0$

$\varphi \in (0, 2\pi)$

$$\underset{\substack{\parallel \\ \Psi_*}}{d\Psi} : \begin{cases} dx = d\rho \cos\varphi - \rho \sin\varphi \, d\varphi \\ dy = d\rho \sin\varphi + \rho\cos\varphi \, d\varphi \end{cases}$$



$$\begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}$$

$$\frac{\partial}{\partial \rho} \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{\partial}{\partial \varphi} \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Psi_*\left(\frac{\partial}{\partial \rho}\right) = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} = \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}$$

$$\Psi_*\left(\frac{\partial}{\partial \varphi}\right) = -\rho\sin\varphi \frac{\partial}{\partial x} + \rho\cos\varphi \frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

This is of course in agreement with the general definition

$$(\Psi_* X)(f) = X(f \circ \Psi)$$

$$\begin{tikzcd} X \arrow[r, "\Psi"] \arrow[rr, bend right, "f \circ \Psi"'] & Y \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Take $X = \frac{\partial}{\partial \rho}$. Compute:

$$\Psi_*\left(\frac{\partial}{\partial \rho}\right)(f) = \frac{\partial}{\partial \rho}(f \circ \Psi) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho}$$

"$f = f(x,y)$" "$f = f(\rho, \varphi)$"

$$= \frac{\partial f}{\partial x} \underset{\substack{\parallel \\ \frac{x}{\sqrt{x^2+y^2}}}}{\cos\varphi} + \frac{\partial f}{\partial y} \underset{\substack{\parallel \\ \frac{y}{\sqrt{x^2+y^2}}}}{\sin\varphi} \underset{\text{"remove } f\text{"}}{\Longrightarrow} \boxed{\Psi_*\left(\frac{\partial}{\partial \rho}\right) = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}} \quad \checkmark$$

Similarly:

$$\Psi_*\left(\frac{\partial}{\partial\varphi}\right)(f) = \frac{\partial}{\partial\varphi}(f\circ\Psi) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi}$$

$$= \frac{\partial f}{\partial x}\underbrace{(-\rho\sin\varphi)}_{-y} + \frac{\partial f}{\partial y}\underbrace{(\rho\cos\varphi)}_{x} \Rightarrow \boxed{\Psi_*\left(\frac{\partial}{\partial\varphi}\right) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}}$$

Let us examine this "classical" computation as well (useful in general)

$$\frac{\partial f}{\partial\rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\rho} \qquad \begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \end{pmatrix} = J^t$$

$$\frac{\partial f}{\partial\varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi}$$

that is:
(remove f)

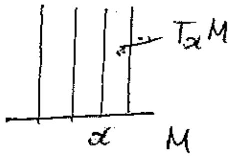
$$\begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\rho\sin\varphi & \rho\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{cases} \frac{\partial}{\partial\rho} = \overbrace{\cos\varphi}^{\frac{x}{r}}\frac{\partial}{\partial x} + \overbrace{\sin\varphi}^{\frac{y}{r}}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial\varphi} = \underbrace{-\rho\sin\varphi}_{-y}\frac{\partial}{\partial x} + \underbrace{\rho\cos\varphi}_{x}\frac{\partial}{\partial y} \end{cases}$$

$\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\varphi}$ are viewed, concretely as linear combinations of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ but

★ The tangent bundle TM

Let M be a (smooth) manifold



disjoint union

Let $\quad TM = \bigsqcup_{x \in M} T_x M$ tangent space at x

★ tangent bundle of M

★ cotangent bundle of M $\quad T^*M = \bigsqcup_{x \in M} T_x^* M$ $= (T_x M)^*$ dual of $T_x M$

$\equiv$ cotangent space at x

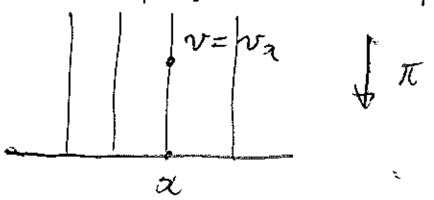
★ TM and T^*M are naturally endowed with a manifold structure. Let us focus on TM (T^*M is treated similarly)

Let $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in \Omega}$ be an atlas for M

some index set

Let $\pi : TM \longrightarrow M$ be the natural projection map.

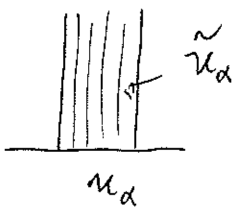
$v \equiv v_x \longmapsto x$



any $v \in TM$ belongs to one and only one $T_x M$ for $x \in M$

Let $\tilde{\mathcal{U}}_\alpha = \pi^{-1}(\mathcal{U}_\alpha) \approx \mathcal{U}_\alpha \times \mathbb{R}^n$

they give rise to



Set

$$\tilde{\varphi}_\alpha : \quad \Big(x, \underbrace{\sum_{i=1}^n b^i \frac{\partial}{\partial x^i}}_{\in\, T_x M}\Big) \longmapsto (x^j, b^j)$$

$(x^j) = r_j \circ \varphi$ local coordinates

$\tilde{\mathcal{A}} = \{(\tilde{\mathcal{U}}_\alpha, \tilde{\varphi}_\alpha)\}_{\alpha \in \Omega}$ becomes an atlas for TM

★ TM can be topologized according to the second definition; it is clear that it has a countable basis and is Hausdorff if M is such.

The transition functions are readily obtained via the following computation (abridged notation, together with Einstein's convention)

$y = y(x)$
coordinate change
$y = \varphi(x)$

insert argument

$$b^i \frac{\partial f}{\partial x^i} = b^i \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}$$

$$= \underbrace{\left(b^i \frac{\partial y^j}{\partial x^i} \right)}_{(b'^j)} \frac{\partial f}{\partial y^j} \qquad (b^i) \mapsto (b'^i)$$

$$b'^i = b^k \frac{\partial y^i}{\partial x^k} = \frac{\partial y^i}{\partial x^k} b^k$$

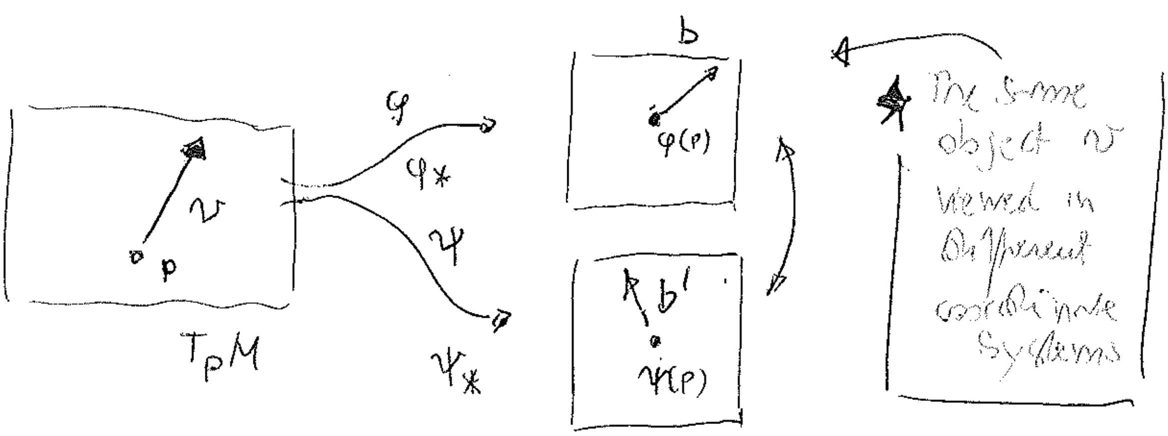
just a relabeling

$$b' = \boxed{\varphi_*}\, b$$

this is fully consistent with the interpretation of the b's as velocity vectors of curves

transition maps:

$$(x, b) \longmapsto (y, b')$$

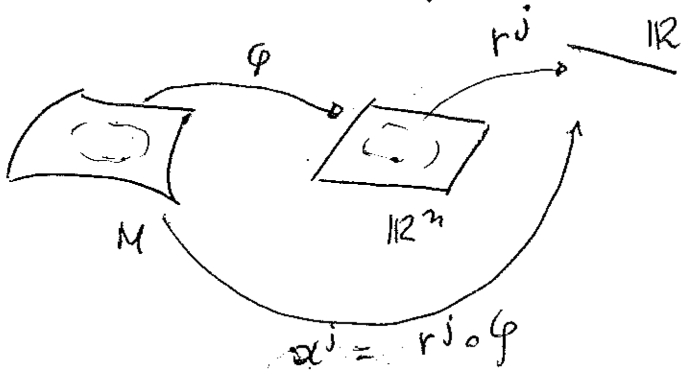


★ The cotangent bundle

A similar treatment can be devised for the cotangent bundle T^*M. Charts (given an atlas for M):

$$\tilde{\varphi}_\alpha : (x, \underbrace{\sum_{i=1}^{n} b_i \, dx^i}_{\in T_x^* M}) \longmapsto (x^i, b^i)$$

notice: dx^i is the differential of the jth-coordinate function x^j



in accordance with the general definition

$\psi : X \to Y$

$\psi_*|_x : T_x X \to T_{\psi(x)} Y$

Remark: Cotangent spaces are fundamental in mechanics, being examples of phase spaces (symplectic manifolds), i.e. receptacles of positions and momenta of point particles.

momenta ≡ duals of velocities

remember that identification with dual space is not canonical

Again work out the transition functions

$$b_i \, dx^i = \overbrace{b_i \frac{\partial x^i}{\partial y^j}}^{b'_j} dy^j$$

$$b \longmapsto b'$$

$$(b_i) \longmapsto \left(b_k \frac{\partial x^k}{\partial y^i}\right) = \left(\frac{\partial x^k}{\partial y^i} b_k\right) \equiv (b'_i)$$

relabeling

* again notice the different behaviours (contravariance vs covariance)

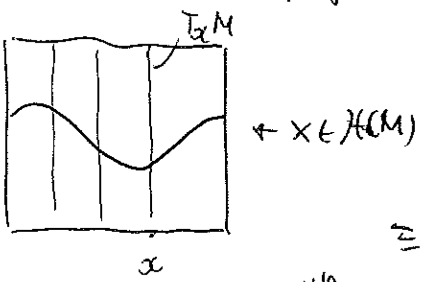
transition maps

$$(x, b) \longmapsto (y, b')$$

covector

— · — · —

* vector fields on M (notation: $\mathfrak{X}(M)$) are the smooth sections of TM, i.e. given $\pi : TM \to M$ (canonical projection; it is a smooth map), $X : M \to TM$ (smooth) such that $\pi \circ X = \mathrm{id}_M$, i.e. $X(x) \in T_xM$ $\forall x \in M$



* Differential 1-forms (notation: $\Lambda^1(M)$) = smooth sections of T^*M; $\omega : M \to T^*M$ with $\pi \circ \omega = \mathrm{id}_M$ ($\pi : T^*M \to M$ canonical projection)

* Tensor bundles

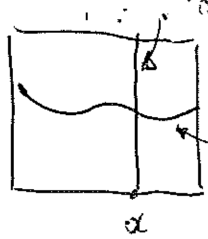
One can similarly define tensor bundles, whose sections are tensor fields $\quad \pi: T^{(p,q)}(M) \to M$

$$T^{(p,q)}(M) \ni \Big(x,\ t^{I}_{J}\,\frac{\partial}{\partial x^{I}} \otimes dx^{J}\Big)$$

$I \leftarrow p$, $J \leftarrow q$

$$\underbrace{T^*_xM \otimes \dots T^*_xM}_{q} \otimes \underbrace{T_xM \otimes \dots T_xM}_{p}$$

$\downarrow$ local chart



notation: $\mathcal{T}^{(p,q)}(M)$ $\qquad (x,\ t^{I}_{J})$

$\nwarrow$ components

* transition maps: $\quad y = y(x)$

$$t^{i_1 \dots i_p}_{j_1 - j_q}\ \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_p}}\ dx^{j_1} \otimes . \, dx^{j_q}$$

insert f fictitiously / Einstein

$$\frac{\partial f}{\partial x^{i_1}} = \frac{\partial f}{\partial y^{l_1}}\,\frac{\partial y^{l_1}}{\partial x^{i_1}} \quad \text{etc...}$$

$$\|$$

$$dx^{j_1} = \frac{\partial x^{j_1}}{\partial y^{h_1}}\,dy^{h_1} \quad \text{et cetera}$$

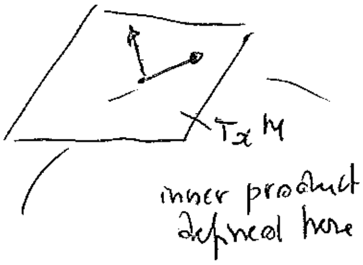
$$\underbrace{t^{i_1 - i_p}_{j_1 - j_q}\ \frac{\partial y^{l_1}}{\partial x^{i_1}} \cdot \frac{\partial y^{l_p}}{\partial x^{i_p}}\ \frac{\partial x^{j_1}}{\partial y^{h_1}} \cdots \frac{\partial x^{j_q}}{\partial y^{h_q}}}_{= \, t'^{\,l_1 - l_p}_{\ h_1 - h_q}}\ \frac{\partial}{\partial y^{l_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{l_p}}\ dy^{h_1} \otimes \dots \otimes dy^{h_q}$$

Concisely: $\quad t^{I}_{J}\,\frac{\partial}{\partial x^{I}} \otimes dx^{J} = \underbrace{t^{I}_{J}\,\frac{\partial y^{L}}{\partial x^{I}}\,\frac{\partial x^{J}}{\partial y^{H}}}_{t'^{\,L}_{\ H}}\,\frac{\partial}{\partial y^{L}} \otimes dy^{H}$

$$\boxed{t'^{\,L}_{\ H} = t^{I}_{J}\,\frac{\partial y^{L}}{\partial x^{I}}\,\frac{\partial x^{J}}{\partial y^{H}}}$$

$\nwarrow$ sum over I and J

★ Example : A Riemannian metric on M is a smoothly varying family of inner products on $T_x M$, $x \in M$; it is a symmetric, positive definite (at each point) element of $\mathcal{T}^{(0,2)}(M)$



locally :

$$x \longmapsto g_{ij}\, dx^i dx^j$$

$dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + \frac{1}{2} dx^j \otimes dx^i)$:

★ symmetric tensor product

Let us check its transformation law :

$$g_{ij}\, dx^i dx^j = g_{ij} \frac{\partial x^i}{\partial y^k} dy^k \frac{\partial x^j}{\partial y^l} dy^l$$

as a function of x

$$= g_{ij} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} dy^k dy^l$$

as a function of y

$$\equiv g'_{kl}\, dy^k dy^l$$

$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$

★ antisymmetric tensor product

$\boxed{dx \otimes dy = dx dy + dx \wedge dy}$

$$\boxed{g'_{kl} = g_{ij} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l}}$$

(this is a function of y)

Once one gets used to this kind of computations they are easily performed automatically.

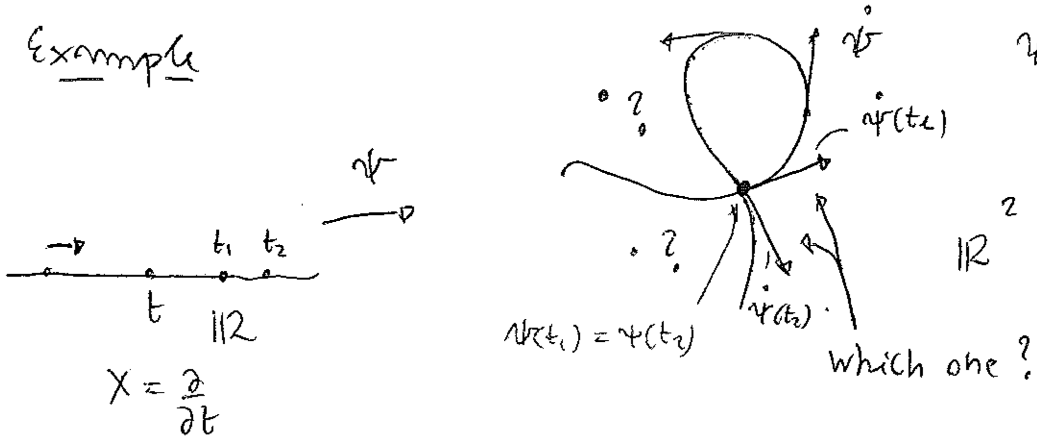
★ Behaviour of vector fields and differential forms under smooth maps

★ $\mathcal{X}(M)$ | Notice that in general, the push-forward

$$\Psi_* : TM \to TN \qquad \text{of} \qquad \Psi : M \to N$$

does NOT map vector fields on M to vector fields on N, in general: (although, of course, Ψ_* sends tangent vectors to M to tangent vectors to N) ⚠

Example



Ψ is neither surjective but, what is worse, is not injective

No problem arises if Ψ is a diffeomorphism:

$$(\Psi_* X)(f)(y) = X(\Psi^* f)(x) = X(\Psi^* f)(\Psi^{-1}(y))$$

$f \in \mathcal{C}^\infty(N)$, $y \in N$, $\Psi^* f \in \mathcal{C}^\infty(M)$, $x \in M$, $x = \Psi^{-1}(y)$

★ $\Lambda^k(M)$ | On the other hand, given $\omega \in \Lambda^k(N)$, the pulled-back form $\Psi^*\omega \in \Lambda^k(M)$ is always defined (for any Ψ smooth). One has:

$$(\Psi^*\omega)(x)(X_1 \dots X_k) := \omega(\Psi(x))(\Psi_* X_1 \dots \Psi_* X_k)$$

$X_1, \dots, X_k \in T_x M$, $\quad \Psi_* X_1, \dots, \Psi_* X_k \in T_{\Psi(x)} N$

(This definition extends the one given in the $\mathbb{R}^n$-case)