Nov. 17, 2012

III. Exercises Representation Theory

7. Exercise: Let R be a ring. Show that every submodule N of a semisimple R-module M is a direct summand.
(Hint: Write $M = \bigoplus_{i\in I} S_i$ with simple modules S_i and apply Zorn's Lemma on the set of all subsets $J \subset I$ such that $N \cap \bigoplus_{i\in J} S_i = 0$).

8. Exercise: Let Λ be a finite dimensional algebra with Jacobson radical J, and M a finitely generated Λ-module. Recall from 17.14 that $JM \subset \mathrm{Rad}(M)$.

1. Show that $(\mathrm{Rad}M)/JM \subset \mathrm{Rad}(M/JM)$.
2. Construct an epimorphism $(\Lambda/J)^n \to M/JM$ for a suitable n, and use Exercise 7 to deduce that M/JM is semisimple.
3. Conclude that $JM = \mathrm{Rad}(M)$.

9. Exercise: **a)** Let $\Phi\colon \mathrm{Mod}(KQ) \longrightarrow \mathrm{Rep}_K(Q)$ and $\Psi\colon \mathrm{Rep}_K(Q) \longrightarrow \mathrm{Mod}(KQ)$ be as constructed in the lecture. Complete the proof of the lecture by showing that

1. Φ and Ψ actually are K-linear covariant functors;
2. $\Phi \circ \Psi \simeq 1_{\mathrm{Rep}_K(Q)}$.

b) Let Q_1 be the one-loop quiver and Q_2 be the quiver with underlying graph $\mathbb{A}_2$ (two vertices, one arrow between them). Show that as algebras $KQ_1 \simeq K[T]$, the polynomial algebra in one variable, and $KQ_2 \simeq \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, considered as subalgebra of $\mathrm{M}_2(K)$.

Submission of solutions: In the lecture on Nov. 28, 2012.