Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2013/14 - N.4

Pb 1. Prove that $C = \{f \in L^2(0,1) : \|f\|_{L^2(0,1)} \leq 1\}$ is *not* a compact set of $L^2(0,1)$.

Pb 2. Let $p > 1$, E a subset of $\mathbb{R}^N$ of finite lebesgue measure, $f_n : E \to \mathbb{R}$ a sequence of measurable functions bounded in $L^p(E)$ which converges pointwise to a function $f : E \to \mathbb{R}$. Prove that (f_n) converges to f in $L^q(E)$ for every $q \in [1,p)$.

Pb 3. Prove that

$$C = \big\{f \in C([0,1]) : f(0) = 0 \text{ e } |f(x) - f(y)| \leq |x - y| \text{ per ogni } x, y \in [0,1]\big\},$$

is a compact set in $C([0,1])$, when endowed with the superior norm.

Pb 4. Let $f : [0,+\infty) \to \mathbb{R}$ be a continuous function such that $f(x) \to 0$ as $x \to +\infty$. Prove that if $f \in L^1(0,+\infty)$, then $f \in L^p(0,+\infty)$, for every $p > 1$.

Pb 5. Detect a norm on $C([0,1])$ such that $Tf = f(0) + 2f(1)$ is continuous. Detect a norm on $C^1([0,1])$ such that $Tf = f(0) + f'(1)$ is continuous. Estimate the norms of these operators.

Pb 6. Let for every $j \in \mathbb{N}$

$$C_j = \big\{f \in C([0,1]) : \text{there exists } x \text{ such that } |f(x) - f(y)| \leq j|x - y| \text{ for every } y \in [0,1]\big\}.$$

Prove that each C_j is closed in $C([0,1])$, and that setting $A_j = C([0,1]) \setminus C_j$ if $f \in \cap_{j\in\mathbb{N}} A_j$, than f is *nowhere* differentiable.

Pb 7. Let, for all $j \geq 1$

$$C_j = \Big\{f \in L^2(0,1) : \int_0^1 |f|^2 \leq j\Big\}.$$

Prove that there exists j_0 such that C_{j_0} has *nonempty* interior.

Pb 8. Let, for all $j \geq 1$

$$Q_j = \Big\{f \in L^2(0,1) : \int_0^{\frac{1}{j}} |f|^2 \geq 1\Big\}.$$

Is it $\bigcap_{j=1}^\infty Q_j \neq \emptyset$ or $\bigcap_{j=1}^\infty Q_j = \emptyset$?

Pb 9. Let $\varphi \in L^{3/2}([0,1])$ and $\psi \in L^m([0,1])$ for $m > 1$. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions such that $|f_n| \leq \varphi\psi$ and $f_n \to f$ as $n \to \infty$ pointwise. For which values of m can one conclude that $\int_0^1 f_n \to \int_0^1 f$ as $n \to \infty$.

Pb 10. Let $\varphi \in C_c^\infty(\mathbb{R})$. Compute $\int_{\mathbb{R}} \sin(nx)\varphi(x)$ as $n \to \infty$. And if $\varphi \in L^2(0,1)$?

Pb 11. Let

$$c_0 = \Big\{ x = (\xi_1, \dots, \xi_n, \dots) : \xi_j \in \mathbb{C},\ \xi_j \to 0 \text{ per } j \to \infty \Big\}.$$

Prove that c_0 is closed in ℓ_∞,

$$\ell_\infty = \Big\{ x = (\xi_1, \dots, \xi_n, \dots) : \xi_j \in \mathbb{C},\ \|x\|_\infty = \sup_{j \in \mathbb{N}} |\xi_j| < \infty \Big\}.$$

when endowed with the norm $\|\cdot\|_\infty$.

Pb 12. Let

$$c_{00} = \Big\{ x = (\xi_1, \dots, \xi_n, \dots) : \xi_j \in \mathbb{C},\ \exists K = K(x) :\ \xi_j = 0 \text{ per } j \geq K \Big\}.$$

Prove that c_{00} is not complete when endowed with the norm $\|\cdot\|_\infty$.

Pb 13. Let X, Y be normed spaces with X reflexive and $T : X \to Y$ linear such that $Tx_j \rightharpoonup 0$ as $j \to \infty$ if $x_j \rightharpoonup 0$ as $j \to \infty$. Show that T is continuous.

Pb 14. Let X be a reflexive Banach space and let $(T_j) \subset X'$. Suppose that T_j converges to T weakly$*$ and that $x_j \to x$ in X as $j \to \infty$. Prove that $\langle T_j, x_j \rangle \to \langle T, x \rangle$ as $j \to \infty$.

Pb 15. Suppose that $(\alpha_n) \subset \mathbb{R}$ is a sequence such that

$$\sum_{n=1}^{\infty} |\beta_n \alpha_n| < +\infty$$

for every sequence $(\beta_n) \subset \mathbb{R}$ converging to zero. Then

$$\sum_{n=1}^{\infty} |\alpha_n| < +\infty.$$

Pb 16. Let X be a Banach space, Y be a normed space, $\mathcal{T} \subset \mathcal{L}(X, Y)$ such that

$$\sup_{T \in \mathcal{T}} |\langle \varphi, Tx \rangle| < +\infty, \qquad \forall x \in X,\ \varphi \in X'.$$

Prove that $\mathcal{T}$ is bounded in $\mathcal{L}(X, Y)$.

Pb 17. Let X be a Banach space and let $B : X \times X \to \mathbb{R}$ be a bilinear form such that the maps $x \mapsto B(x, y)$ and $y \mapsto B(x, y)$ are continuous. Prove that B is continuous.

Pb 18. let $L : \ell^2 \to \ell^2$ be the left shift operator, that is

$$Lx = (x_2, x_3, \dots), \qquad x = (x_1, x_2, x_3, \dots)$$

and the right shift, namely

$$Rx = (0, x_1, x_2, \dots), \qquad x = (x_1, x_2, x_3, \dots).$$

Study the kernel and the image of L and R. Prove that R is an isometry, while L is not.

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