Modal Logic

Propositional logic may be defined in a Hilbert style fashion

Propositional logic is a set H defined as smallest set X of formulas verifying the following properties:
1. if A, B, C are formulas then X contains the formulas (called axioms)
P1 $A \rightarrow (B \rightarrow A)$
P2 $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
P3 $((\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B))$
moreover
MP if $A \in X$ and $A \rightarrow B \in X$ then $B \in X$(modus ponens)

We write $\vdash_H A$ to denote that $A \in H$

If Ω is a finite set of formulas we write $\Omega \vdash_H A$ to denote that $\vdash_H \bigwedge \Omega \rightarrow A$

If Ω is an infinite set of formulas we write $\Omega \vdash_H A$ to denote that there is a finite subset Ω_0 of Ω s.t. $\Omega_0 \vdash_H A$.

language of modal logic

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\rightarrow, \perp$

(iii) modal operator $\Box$

(iv) auxiliary symbols : (,).

$AT = \{p_0, p_1, p_2, \ldots,\} \cup \{\perp\}$

The set WFF of (modal) formulas is the smallest set X with the properties

(i) $p_i \in X$ $(i \in N)$, $\perp \in X$,

(ii) $A, B \in X \Rightarrow (A \rightarrow B) \in X$,

(iii) $A \in X \Rightarrow (\neg A) \in X$

(iv) $A \in X \Rightarrow (\Box A) \in X$

Let **Z** be a set o formula.
The normal modal logic **L**[**Z**] is defined as smallest set X of formulas verifying the following properties:

1. **Z**$\subseteq$ **X**
2. if A, B, C are formulas then X contains the formulas (called axioms)
   P1 $A\rightarrow(B\rightarrow A)$
   P2 $(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C))$
   P3 $((\neg B\rightarrow\neg A)\rightarrow((\neg B\rightarrow A)\rightarrow B))$
   P4 $\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$
3. moreover
   MP if $A\in X$ and $A\rightarrow B\in X$ then $B\in X$(modus ponens)
   NEC if $A\in X$ then $\Box A\in X$ (necessitation)

We write $\vdash_{\mathbf{L[Z]}} A$ to denote that $A\in$**L**[**Z**]

If Ω is a finite set of formulas we write $\Omega \vdash_{\mathbf{L[Z]}} A$ to denote that $\vdash_{\mathbf{L[Z]}} \bigwedge\Omega \rightarrow A$

If Ω is an infinite set of formulas we write $\Omega \vdash_{\mathbf{L[Z]}} A$ to denote that there is a finite subset Ω_0 of Ω s.t. $\Omega_0 \vdash_{\mathbf{L[Z]}} A$.

$L[\varnothing]$ is called minimal normal modal logic and $L[\varnothing]$ is denoted simply by **K**

Abbreviations

The usual abbreviations of classical logic plus

$\Diamond A := \neg \Box \neg A$

If $N_1,..,N_k$ are names of schemas of formula the sequence $N_1..N_k$ is the set $N_1^* \cup \ldots \cup N_1^*$, where

$N_i^* = \{A$: A is an instance of the schema $N_i\}$

some schema

D. $\Box A \rightarrow \Diamond A$

T. $\Box A \rightarrow A$

4. $\Box A \rightarrow \Box \Box A$

B. $A \rightarrow \Box \Diamond A$

some modal logic

$T := L[T]$

$S4 := L[T4]$

$S5 := L[T4B]$

$KT := L[T]$

$K4 := L[4]$

Possible world semantics
or
Kripke semantics

Let Prop be the set of propositional symbols.

A **structure** $F = \langle U,R \rangle$, where U is a nonempty set and $R \subseteq U \mathrm{x} U$ is called **frame** ($\mathscr{F}$ is a graph).

A **valuation** on a frame $F=\langle U,R \rangle$ is a function $V : U \rightarrow 2^{Prop}$.

A **(Kripke) model** M is a frame plus a valuation V, $M = \langle U,R,V \rangle$

Let $M = \langle U,R,V \rangle$ a model,

the satisfiability relation $M \vDash \subseteq U x WFF$

is defined as

1. $M, w \vDash A \wedge B \Leftrightarrow M, w \vDash A$ AND $M, w \vDash B$

2. $M, w \vDash A \vee B \Leftrightarrow M, w \vDash A$ OR $M, w \vDash B$

3. $M, w \vDash \neg A \Leftrightarrow M, w \nvDash A$,

4. $M, w \vDash A \rightarrow B \Leftrightarrow (M, w \vDash A \Rightarrow M, w \vDash B)$,

5. $M, w \vDash \Box A \Leftrightarrow \forall u\ (wRu \Rightarrow M, u \vDash A)$

6. $M, w \vDash \Diamond A \Leftrightarrow \exists u\ (wRu$ AND $M, u \vDash A)$

7. $M, w \nvDash \bot$

8. $M, w \vDash p$ iff $p \in V(w)$

$let\ M\ be\ a\ model,\ M \models A\ iff\ for\ each\ u \in U\ we\ have\ M, u \models A$

$let\ M\ be\ a\ model\ and\ let\ \Sigma\ be\ a\ set\ of\ formulas,\ M \models \Sigma\ iff\ for\ each\ A \in \Sigma\ M \models A$

$\models A\ iff\ for\ each\ model\ M\ we\ have\ M \models A.$

$let\ F\ be\ a\ frame,\ F \models A\ iff\ for\ each\ valuation\ V,\ \langle F, V \rangle \models A$

$let\ F\ be\ a\ frame,\ F, w \models A\ iff\ for\ each\ valuation\ V,\ \langle F, V \rangle, w \models A$

$let\ M\ be\ a\ model,\ Th(M) = \{A : M \models A\}$

$let\ F\ be\ a\ ,\ Th(F) = \{A : F \models A\}$

$Md(A) = \{M : M is\ a\ model, M \models A\}$

$Md(\Sigma) = \{M : M is\ a\ model, M \models \Sigma\}$

$Fr(A) = \{F : F is\ a\ frame, F \models A\}$

$Fr(\Sigma) = \{F : F is\ a\ model, F \models \Sigma\}$

Theorem 1.2.2 (soundness) *Let* Σ *be a set of formulas and let* $M \in Md(\Sigma)$ *(*$F \in Fr(\Sigma)$*) then for each theorem* $A \in \mathbf{L}[\Sigma]$ *we have that* $M \models A$ *(*$F \models A$*).*

Modal definability

First order translation

- **Let us assume a modal language with a denumerable set Prop of propositional symbols.**
- **Let us consider a first order language *L,* with a denumerable set Π of unary predicate symbols, and a binary predicate symbol R.**
- **Let τ:Prop→Π a bijective map**
- **Let Form be the set of first order formula formulas in the language *L.***

*Given a **fixed variable x**, we define an injective mapping*

$$\mathbf{ST_x}: \ \mathit{WFF} \to \mathit{Form}$$

1. $\mathbf{ST_x}(p) = P(x)$ for $p \in$Prop and $P = \tau(p)$;
2. $\mathbf{ST_x}(\neg A) = \neg \mathbf{ST_x}(A)$
3. $\mathbf{ST_x}(A \to B) = \mathbf{ST_x}(A) \to \mathbf{ST_x}(B)$
4. $\mathbf{ST_x}(\Box A) = \forall y(xRy \to \{\mathbf{ST_x}(A)\}[x/y])$ where y does no occur in ST(A).

definability

Let A (Σ) be a formula (a set of formulas), we say that A (Σ) defines a first/second order property Φ in the language with (R, =), if for each F ($F \in Fr(A)$ ($F \in Fr(\Sigma)$) $\Longleftrightarrow F \models \Phi$)

If the set Σ defines the condition Φ then we say also that the logic L[Σ] defines Φ.

formula name	formula	first order property
D	$\Box A \to \Diamond A$	$\forall x \exists y . xRy$
T	$\Box A \to A$	$\forall x . xRx$
4	$\Box A \to \Box\Box A$	$\forall xyz . (xRy \wedge yRz \to xRz)$
B	$\Diamond\Box A \to A$	$\forall x \forall y . (xRy \to yRx)$
G	$\Diamond\Box A \to \Box\Diamond A$	$\forall xyz . ((xRy \wedge xRz) \to \exists w (yRw \wedge zRw))$

Proposition 1.3.7 $\Box\alpha \to \Box\Box\alpha$ *defines transitivity* $\forall xyz.(xRy \wedge yRz \to xRz)$

PROOF

Proposition 1.3.7 $\Box\alpha \to \Box\Box\alpha$ *defines transitivity* $\forall xyz.(xRy \wedge yRz \to xRz)$

Proof.

1. $F \models \forall xyz.(xRy \wedge yRz \to xRz) \Rightarrow F \models \Box\alpha \to \Box\Box\alpha$. Let $F, w \models \Box\alpha$, and w', w'' s.t. $wRw', w'Rw''$ then by transitivity we have that wRw'' and therefore $F, w'' \models \alpha$; namely $F, w' \models \Box\alpha$ and $F, w \models \Box\Box\alpha$.

2. $F \models \Box\alpha \to \Box\Box\alpha \Rightarrow F \models \forall xyz.(xRy \wedge yRz \to xRz)$. Let us suppose that $F, w \models \Box\alpha \to \Box\Box\alpha$; we fix the following assignment $V(\alpha) = \{v | wRv\}$. We have that $F, V, w \models \Box\alpha$ and by hypothesis $F, V, w \models \Box\Box\alpha$. Now for a generic $v \in V(\alpha)$ let w'' s.t. vRw''. As $F, V, w'' \models \alpha$, we must have that R is transitive.

■

Proposition 1.3.8 $\Diamond\Box\alpha\to\Box\Diamond\alpha$ *defines directness:*
$dir = \forall xyz((xRy \wedge xRz)\to\exists u(yRu \wedge zRu))$

Proposition 1.3.8 $\Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$ *defines directness:*
$dir = \forall xyz((xRy \wedge xRz) \rightarrow \exists u(yRu \wedge zRu))$

Proof

1. $F \models \forall xyz((xRy \wedge xRz) \rightarrow \exists u(yRu \wedge zRu)) \Rightarrow F \models \Diamond\Box\alpha \rightarrow \Box\Diamond\alpha$
   Let $w \in W$ and $F, w \models \Diamond\Box\alpha$ then $\exists w', wRw' s.t. \forall w'' w'Rw'' \Rightarrow w'' \models \alpha$.

   As dir holds we have that $\forall v wRv \exists s w'Rs, vRs$ as $F, s \models \alpha$ and therefore $F, w \models \Box\Diamond\alpha$

2. $F \models \Diamond\Box\alpha \supset \Box\Diamond\alpha \Rightarrow F \models \forall xyz((xRy \wedge xRz) \rightarrow \exists u(yRu \wedge zRu))$
   Let w, w', w'' s.t. wRw', wRw'' and let V the assignment s.t. $V(\alpha) = \{s : w'Rs\}$

   We have that $F, w' \models \Box\alpha$ and that $F, w' \models \Diamond\Box\alpha$. As $F \models G$ we have that $F, w \models \Box\Diamond\alpha$ and therefore $\forall v wRv \Rightarrow \exists t F, t \models \alpha \Rightarrow t \in V(\alpha) \Rightarrow F \models dir$

■

$$Fr(\mathbf{K}) = \{\langle U, R\rangle : R \text{ is a generic relation}\}$$
$$Fr(\mathbf{KD}) = \{\langle U, R\rangle : R \text{ is total}\}$$
$$Fr(\mathbf{KT}) = \{\langle U, R\rangle : R \text{ is reflexive}\}$$
$$Fr(\mathbf{S4}) = \{\langle U, R\rangle : R \text{ is a preorder}\}$$
$$Fr(\mathbf{S5}) = \{\langle U, R\rangle : R \text{ is an equivalence}\}$$

COMPLETENESS

L[Z] is defined as smallest set X of formulas verifying the following properties:

1. Z⊆ X

2. if A, B, C are formulas then X contains the formulas (called axioms)
   - **P1** $A\to(B\to A)$
   - **P2** $(A\to(B\to C))\to((A\to B)\to(A\to C))$
   - **P3** $((\neg B\to\neg A)\to((\neg B\to A)\to B))$
   - **P4** $\Box(A\to B)\to(\Box A\to\Box B)$

3. L is closed w.r.t. the following operation
   - **MP** if $A\in X$ and $A\to B\in X$ then $B\in X$(modus ponens)
   - **NEC** if $A\in X$ then $\Box A\in X$ (necessitation)

Given a set Z of modal fomulas the modal logic L[Z] is defined by means of the following axioms and inference rules plus a notion of derivation.

axioms

1. if A, B, C are formulas then the following are axioms

- **P1** $A\to(B\to A)$
- **P2** $(A\to(B\to C))\to((A\to B)\to(A\to C))$
- **P3** $((\neg B\to\neg A)\to((\neg B\to A)\to B))$
- **P4** $\Box(A\to B)\to(\Box A\to\Box B)$

2. if $A\in$ **Z** then A is an axiom

Inference rules

$$\frac{A \quad A\to B}{B}$$ **MP**

$$\frac{A}{\Box A}$$ **NEC**

Derivations

A derivation is a finite sequence $A_1, \ldots, A_n$ of formulas s.t. for each $i \in [1, n]$

A_i is an axiom; or

$A_i \equiv B$ and $\exists j, k < i$ s.t. $A_j \equiv A$, $A_k \equiv A \to B$;

$A_i \equiv \Box A$ and $\exists k < i$ s.t. $A_k \equiv A$;

We write $\vdash_{L[Z]} A$ to denote that there is a derivation $A_1,\ldots,A_n$ with $A_n \equiv A$

The construction of the canonical model

Maximal Consistent Sets

A set Γ of WFF is **consistent** if
$\Gamma \nvdash \bot$.
A set Γ of WFF is **inconsistent** if
$\Gamma \vdash \bot$.

A set Γ is maximally consistent iff
(a) Γ is consistent,
(b) $\Gamma \subseteq \Gamma'$ and Γ' consistent $\Rightarrow \Gamma = \Gamma'$.

If Γ is maximally consistent, then Γ is closed under derivability (i.e. $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$).

Theorem:
Each consistent set Γ is contained in a maximally consistent set Γ^*

1) enumerate all the formulas

$\varphi_0, \varphi_1, \varphi_2, \ldots\ldots$

2) define the non decreasing sequence:

$\Gamma_0 = \Gamma$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\phi_n\} \text{ if } \Gamma_n \cup \{\phi_n\} \text{ is consistent,} \\ \Gamma_n \text{ otherwise} \end{cases}$$

3) define

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma_n \, .$$

Propositional logic:

If Γ is consistent, then there exists a CANONICAL valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$.

Let L be a normal modal logic, a model $M = \langle U,R,V \rangle$ is called canonical iff

1. $U = \{w : w \text{ is maximal consistent}\}$
2. $R = \{(u,v) : \{A : \Box A \in u\} \subseteq v$
3. $u \in V(p) \Leftrightarrow p \in u$

A logic L is called **canonical** if, taken the canonical model $\langle U,R,V\rangle$, we have $\langle U,R\rangle \in Fr(L)$.

Theorem CM

Let $\langle U,R,V\rangle$ the canonical model of L

$\vdash_L \alpha \Leftrightarrow \langle U,R,V\rangle \vDash \alpha$

A normal modal logic L is said to be **model complete** if for each formula A:

$$\vdash_L A \Leftrightarrow \forall M \in Md(L)\ M \vDash A$$

Theorem

Each normal modal logic is model complete

Proof

$$(\Rightarrow)$$

$\vdash_L A \Rightarrow \forall M \in Md(L) M \vDash A$ by soundness

$$(\Leftarrow)$$

In order to prove

$\forall M \in Md(L) M \vDash A \Rightarrow \vdash_L A$ we use the canonical model.

If $\forall M \in Md(L)\ M \vDash A$ we have in particular that taken the canonical model $\langle U, R, V\rangle$ we have that $\langle U, R, V\rangle \vDash A$, and applying theorem CM we conclude.

A normal modal logic $L(\Sigma)$ is said to be **frame complete** if for each formula A:

$\vdash_L A \Leftrightarrow \forall F \in Fr(\Sigma)\ F \models A$

Theorem The logics K, KD, KT, S4, S5, are frame complete.

Proof

Let $L \in \{K, KD, KT, S4, S5\}$, it is sufficient to show that if $\langle U,R,V \rangle$ is the canonical model of L then the frame $\langle U,R \rangle \in Fr(L)$.

Let Σ be a set of formulas, and let $\mathscr{C} \subseteq \mathrm{Fr}(\Sigma)$ a set of frames; the modal logic L[Σ] is said to be **$\mathscr{C}$-complete** (complete w.r.t. the class $\mathscr{C}$ of frames) if

$A \in L(\Sigma) \Leftrightarrow \forall F \in C, F \models A$

Theorem

1. The logics K (KD) is complete with respect to the class of denumerable frames with irreflexive, asymmetric and intransitive (total) accessibility relation.
2. The logic S4 is complete w.r.t. the set of denumerable partial order.

Modal logic and intuitionism

Let us consider the following translation function []* from propositional formulas to modal ones.

$p^* = \Box p$ (p is a propositional symbol)

$[A \wedge B]^* = [A]^* \wedge [B]^*$

$[A \vee B]^* = [A]^* \vee [B]^*$

$[A \rightarrow B]^* = \Box([A]^* \rightarrow [B]^*)$

$[\neg A]^* = \Box(\neg [A]^*)$

Lemma

Let $\langle W,R,V_i\rangle$ be an intuitionistic model and $\langle W,R,V_{S4}\rangle$ be a partial order model of S4 s.t. for each propositional symbol p,

$w \Vdash_i p$ iff $w \vDash_{S4} \Box p$,

then for each propositional formula A, $w \Vdash_i A$ iff $w \vDash_{s4} A^*$

Lemma

Let $M_i=\langle W,R,V_i\rangle$ be an intuitionistic model and $M_{S4}=\langle W,R,V_{S4}\rangle$ be a partial order model of S4 s.t. for each propositional symbol p,

$w \Vdash_i p$ iff $w \vDash_{S4} \Box p$,

then for each propositional formula A, $M_i \Vdash_i A$ iff $M_{S4} \vDash_{s4} A^*$

Theorem

$\vdash_i A \Leftrightarrow \vdash_{S4} A^*$

natural deduction?

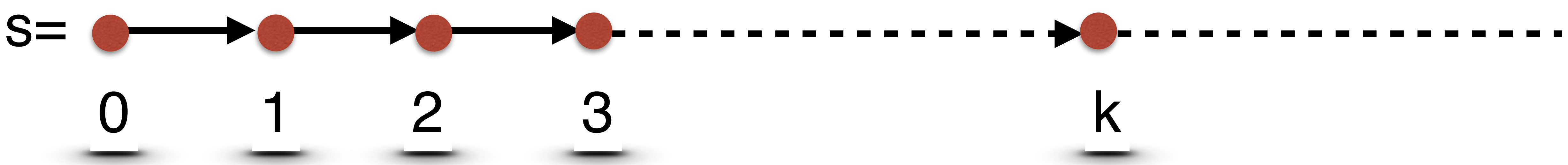
TEMPORAL LOGIC

LTL: Linear Temporal Logic

timeline/computation/fullpath

Kripke frame is **Nat**$=\langle \mathbb{N}, \sigma, \leq \rangle$

(as usual $\sigma(n)$ will be written as $n+1$)



each natural number identifies an temporal instant

A Linear Time Kripke model **M** (or, simply, a model) is a frame plus a valuation of propositional symbols, namely

M$= \langle$**Nat**$, V: \mathbb{N} \rightarrow 2^{Prop} \rangle$

σ induces the accessibility relation

$\mathcal{N} \subseteq \mathbb{N} \mathbf{x} \mathbb{N}$

$n \mathcal{N} m \Longleftrightarrow m=n+1$

language of linear temporal logic

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\rightarrow, \perp$

(iii) modal operator $\bigcirc$, $\mathcal{U}$,

(iv) auxiliary symbols : (,).

$AT = \{p_0, p_1, p_2, \ldots,\} \cup \{\perp\}$

The set WFF of (modal) formulas is the smallest set X with the properties

(i) $p_i \in X$ ($i \in N$), $\perp \in X$,

(ii) $A, B \in X \Rightarrow (A \rightarrow B) \in X$,

(iii) $A \in X \Rightarrow (\neg A) \in X$

(iv) $A \in X \Rightarrow (\bigcirc A) \in X$

(v) $A, B \in X \Rightarrow (A \,\mathcal{U}\, B) \in X$,

abbreviations:

$\Diamond A := (\neg\perp) \mathcal{U} A$

$\Box A := \neg \Diamond \neg A$

Let $\mathbf{M} = \langle \mathbf{Nat}, V \rangle$ a model,

the satisfiability relation $\mathbf{M} \vDash \subseteq \mathbb{N} \times WFF$

is defined as

1. $\mathbf{M}, n \vDash A \wedge B \Leftrightarrow M, n \vDash A \ \& \ M, n \vDash B$

2. $M, n \vDash A \vee B \Leftrightarrow M, n \vDash A$ OR $M, n \vDash B$

3. $M, n \vDash \neg A \Leftrightarrow M, n \nvDash A$,

4. $M, n \vDash A \rightarrow B \Leftrightarrow (M, n \vDash A \Rightarrow M, n \vDash B)$,

5. $M, n \vDash A \mathcal{U} B \Leftrightarrow \exists\, m(n \leq m \ \& \ (M, m \vDash B \ \& \ \forall j (j \in [n, m-1] \Rightarrow M, j \vDash A)))$

6. $M, n \vDash \Box A \Leftrightarrow \ \forall\, m\ (n \leq m \Rightarrow M, m \vDash A)$

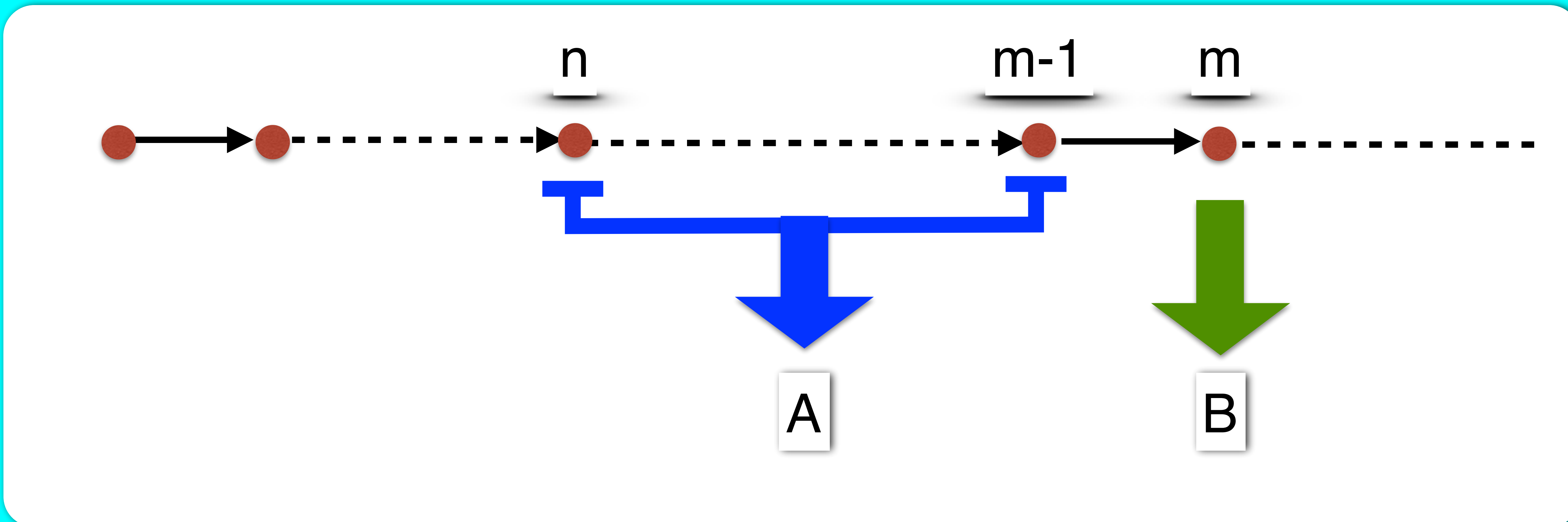
7. $M, n \vDash \Diamond A \Leftrightarrow \ \exists\, m\ (n \leq m \ \& \ M, m \vDash A)$

8. $M, n \vDash \bigcirc A \Leftrightarrow \ M, n+1 \vDash A)$

9. $M, n \nvDash \bot$

10. $M, n \vDash p$ iff $p \in V(n)$

$M,n \vDash A\,\mathcal{U}\,B \Leftrightarrow \exists\ m \geq n\ M,m \vDash B\ \&\ \forall j \in [n,m-1]\ M,j \vDash A$



Sometimes in literature a model is given by

K=$\langle T, s:\mathbb{N}\rightarrow T, V\rangle$

where

T is a denumerable set of temporal instants

s is a bijection and

$V:T\rightarrow 2^{Prop}$ is a valuation

these models are completely equivalent to the models previously introduced.

Let **K=**$\langle T, s:\mathbb{N}\rightarrow T, V\rangle$,
the satisfiability relation **K** $\models \subseteq$ TxWFF
is defined as

$M, s_k \models A\rightarrow B \Leftrightarrow (M, s_k \models A \Rightarrow M, s_k \models B)$,

$M, s_n \models A\mathscr{U}B \Leftrightarrow \exists\, m(n\leq m \,\&\, (M, s_m \models B \,\&\, \forall j(j\in[n, m-1]\Rightarrow M, s_j \models A)))$

$M, s_n \models \bigcirc A \Leftrightarrow M, s_{n+1} \models A)$

$M, s_n \not\models \perp$

$M, s_n \models p$ iff $p\in V(s_n)$

$$M \models A \iff \forall n\ M,n \models A$$

$$\models A \iff \forall M.\ M \models A$$

A0 All temporal instances of propositional classical tautologies.

A1 $\circ(A \to B) \to (\circ A \to \circ B)$

A2 $\neg \circ A \to \circ \neg A$

A3 $\Box(A \to B) \to (\Box A \to \Box B)$

A4 $\Box A \to A$

A5 $\Box A \to \Box \Box A$

A6 $\Box A \to \circ A$

A7 $\Box A \to \circ \Box A$

A8 $A \wedge \Box(A \to \circ A) \to \Box A$

MP $\dfrac{A \quad A \to B}{B}$

Gen$\Box$ $\dfrac{A}{\Box A}$

Gen$\circ$ $\dfrac{A}{\circ A}$

temporal induction

$$A \wedge \Box(A \to \circ A) \to \Box A$$

$$0 \models A \wedge \Box(A \to \circ A) \to \Box A$$

$$\Longleftrightarrow$$

$$(0 \models A \;\&\; \forall n(n \models A \Rightarrow n+1 \models A)) \Rightarrow \forall n\,(n \models A)$$

Let $\alpha(x)$ be the property $x \models A$

$$0 \models A \wedge \Box(A \to \circ A) \to \Box A$$

$$\Longleftrightarrow$$

$$(\alpha(0) \;\&\; \forall n(\alpha(n) \Rightarrow \alpha(n+1))) \Rightarrow \forall n\,(\alpha(n))$$

$$k \models A \wedge \Box(A \to \circ A) \to \Box A$$

$$\Longleftrightarrow$$

$$(\alpha(k) \;\&\; \forall n \geq k(\alpha(n) \Rightarrow \alpha(n+1))) \Rightarrow \forall n \geq k\,(\alpha(n))$$

SOUNDNESS

$$\vdash A \;\Rightarrow\; \models A$$

(A simple induction on derivations: exercise)

COMPLETENESS

$$\models A \;\Rightarrow\; \vdash A$$

Difficult: the canonical kripke model is not a temporal model

BRANCHING TIME

INTUITIVE IDEA: TREES/GRAPHS instead of COMPUTATIONS

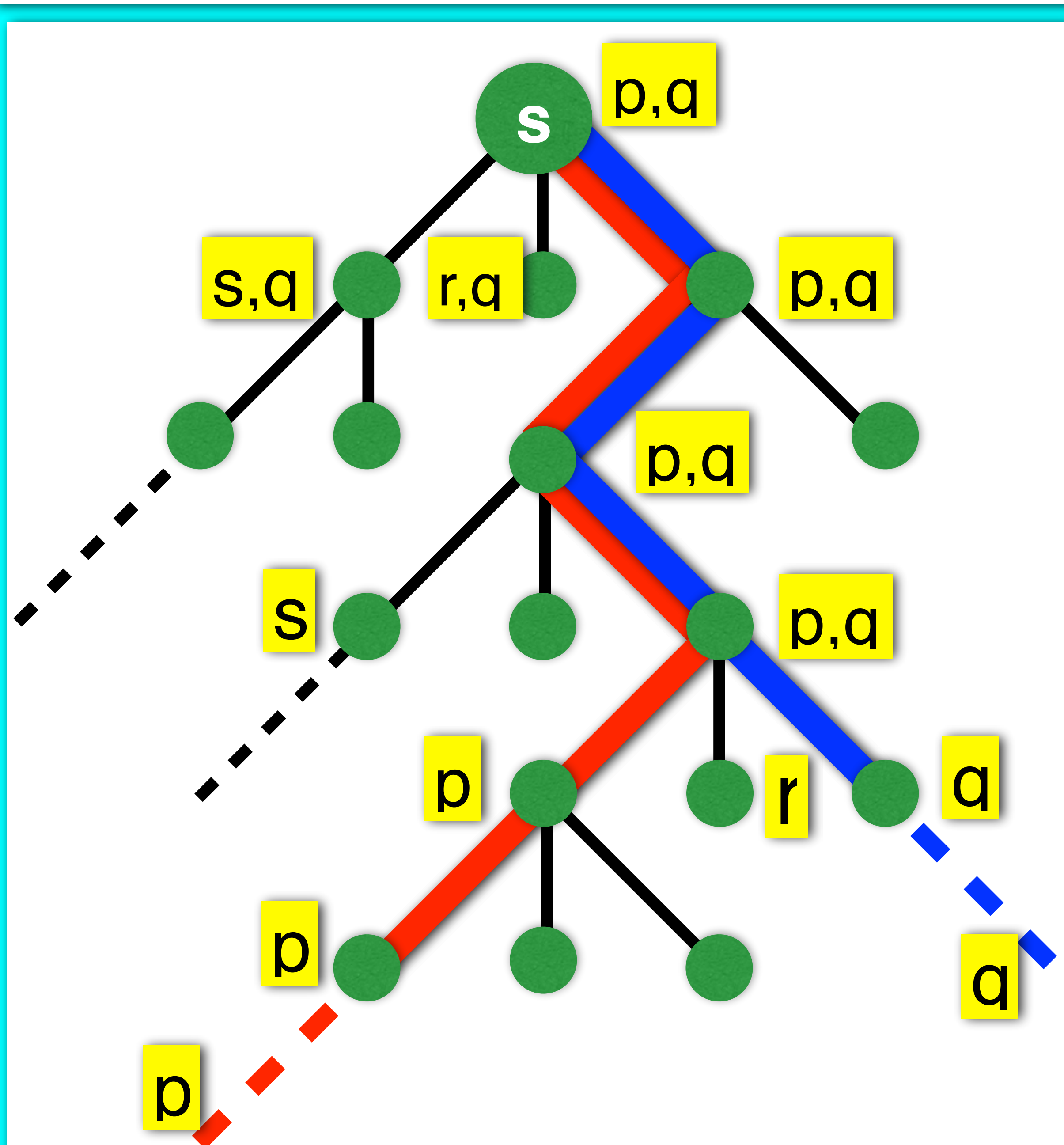
$\forall\bigcirc$ =for each next time; $\exists\bigcirc$= there exists a next time such that

$\forall\Box$= for each computation and for each state in it

$\forall\Diamond$= for each computation there exists a state in it such that

$\exists\Box$= there exists a computation such that for each state in it

$\exists\Diamond$=there exists a computation and a state in it such that



$s\models \forall\bigcirc\, q$

$s\models \exists\bigcirc\, s \wedge \exists\bigcirc p$

$s\models \exists\Box\, p$

$s\models \exists\Box\, q$

$s\models \exists\Diamond\, r$

$s\models \exists\Diamond\, (s \wedge \neg q)$

language of UB

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\rightarrow, \perp$

(iii) modal operator $\forall\bigcirc, \forall\Box, \forall\Diamond$

(iv) auxiliary symbols : (,).

$AT = \{p_0, p_1, p_2, \ldots,\} \cup \{\perp\}$

The set WFF of (modal) formulas is the smallest set X with the properties

(i) $p_i \in X$ $(i \in N)$, $\perp \in X$,

(ii) $A, B \in X \Rightarrow (A \rightarrow B) \in X$,

(iii) $A \in X \Rightarrow (\neg A) \in X$

(iv) $A \in X \Rightarrow (\forall\Diamond\ A), (\forall\Box\ A), (\forall\bigcirc\ A) \in X$

abbreviations:

$\exists\Box\ A := \neg\forall\Diamond\neg\ A$

$\exists\Diamond\ A := \neg\forall\Box\neg\ A$

$\exists\bigcirc\ A := \neg\forall\bigcirc\neg\ A$

Semantics

an (UB-)frame is a graph

$$\langle S,N\rangle$$

where $N \subseteq S \times S$ is total ($\forall s \exists s'\ sNs'$)

An s-branch/s-computation is a sequence $b_s=(s_i)_{i<\omega}$ s.t. $s=s_0$ & $\forall\ i\in\mathbb{N}\ s_iNs_{i+1}$

if $b_s=(s_i)_{i<\omega}$ with $b_s[k]$ we denote s_k and with $s'\in b_s$ we mean that $\exists k$ s.t. $s'= b_s[k]$

an (UB-)model is a pair

$$\langle F,V\rangle$$

where F is a frame

and $V:S\rightarrow 2^{Prop}$

is a valuation

Let M= $\langle S,N,V\rangle$ a model,

the satisfiability relation $M \models \subseteq S \times WFF$

is defined as

1. $M, s \not\models \bot$
2. $M, s \models p$ iff $p \in V(s)$
3. $M, s \models A \wedge B \Leftrightarrow M,s \models A$ & $M,s \models B$
4. $M, s \models A \vee B \Leftrightarrow M,s \models A$ OR $M,s \models B$
5. $M,s \models \neg A \Leftrightarrow M,s \not\models A$,
6. $M,s \models A \rightarrow B \Leftrightarrow (M,s \models A \Rightarrow M,s \models B)$,
7. $M,s \models \forall\Box A \Leftrightarrow \forall b_s \forall s' \in b_s\ M,s' \models A$
8. $M,s \models \forall\Diamond A \Leftrightarrow \forall b_s \exists s' \in b_s\ M,s' \models A$
9. $M,s \models \exists\Box A \Leftrightarrow \exists b_s \forall s' \in b_s\ M,s' \models A$
10. $M,s \models \exists\Diamond A \Leftrightarrow \exists b_s \exists s' \in b_s\ M,s' \models A$
11. $M,s \models \forall\bigcirc A \Leftrightarrow \forall s'\ (sNs' \Rightarrow M,s' \models A)$
12. $M,s \models \exists\bigcirc A \Leftrightarrow \exists s'\ (sNs'\ \&\ M,s' \models A)$

AXIOMATIZATION ($\mathscr{U}$-free fragment)

A0 All temporal instances of propositional classical tautologies.

- (*A1*) $\forall\Box(A\to B)\supset(\forall\Box A\to\forall\Box B)$
- (*A2*) $\forall\bigcirc(A\to B)\supset(\forall\bigcirc A\to\forall\bigcirc B)$
- (*A3*) $\forall\Box A\to(\quad A\wedge\forall\bigcirc\forall\Box A)$
- (*A4*) $A\wedge\forall\Box(A\to\forall\bigcirc A)\to\forall\Box A)$
- (*E1*) $\forall\Box(A\to B)\supset(\exists\Box A\to\exists\Box B)$
- (*E2*) $\exists\Box A\to(A\wedge\exists\bigcirc\exists\Box A)$
- (*E3*) $\forall\Box A\to\exists\Box A$
- (*E4*) $A\wedge\forall\Box(A\to\exists\bigcirc A)\to\exists\Box A$

$$\text{MP}\quad\frac{A\quad A\to B}{B}$$

$$\text{Gen}\quad\frac{A}{\forall\Box A}$$

SOUNDNESS

$$\vdash A \;\Rightarrow\; \models A$$

(A simple induction on derivations: exercise)

COMPLETENESS

$$\models A \;\Rightarrow\; \vdash A$$

Difficult: the canonical kripke model is not an UB-model

The Logic CTL

CTL= UB+$\mathcal{U}$*ntil*

language of CTL

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\rightarrow, \perp$

(iii) modal operator $\forall\bigcirc, \forall\mathcal{U}, \exists\mathcal{U}$

(iv) auxiliary symbols : (,).

$AT=\{p_0, p_1, p_2, \ldots,\}\cup\{\perp\}$

The set WFF of (modal) formulas is the smallest set X with the properties

(i) $p_i \in X$ ($i\in N$), $\perp\in X$,

(ii) $A,B\in X \Rightarrow (A\rightarrow B)\in X$,

(iii) $A\in X \Rightarrow (\neg A)\in X$

(iv) $A,B\in X \Rightarrow (\forall\bigcirc A), (A \forall\mathcal{U} B) \in X$

abbreviations:

$\exists\bigcirc A = \neg\forall\bigcirc\neg A$

$\exists\Box A=\neg\forall\Diamond\neg A$ $\forall\Box A=\neg\exists\Diamond\neg A$ $\exists\Diamond\alpha \equiv true\ \exists\mathcal{U}\ A$ $\forall\Diamond A \equiv true\ \forall\mathcal{U}\ A$

NOTATION: if $b_s=(s_i)_{i<\omega}$ with $b_s[k]$ we denote s_k

$M,s \vDash B\ \exists\mathscr{U}\ A$

$\Leftrightarrow$

$\exists b_s\ \exists k\ (\ M,b_s[k] \vDash A\ \&\ \forall j\in[0,k-1]\ b_s[j] \vDash B$

$M,s \vDash B\ \forall\mathscr{U}\ A$

$\Leftrightarrow$

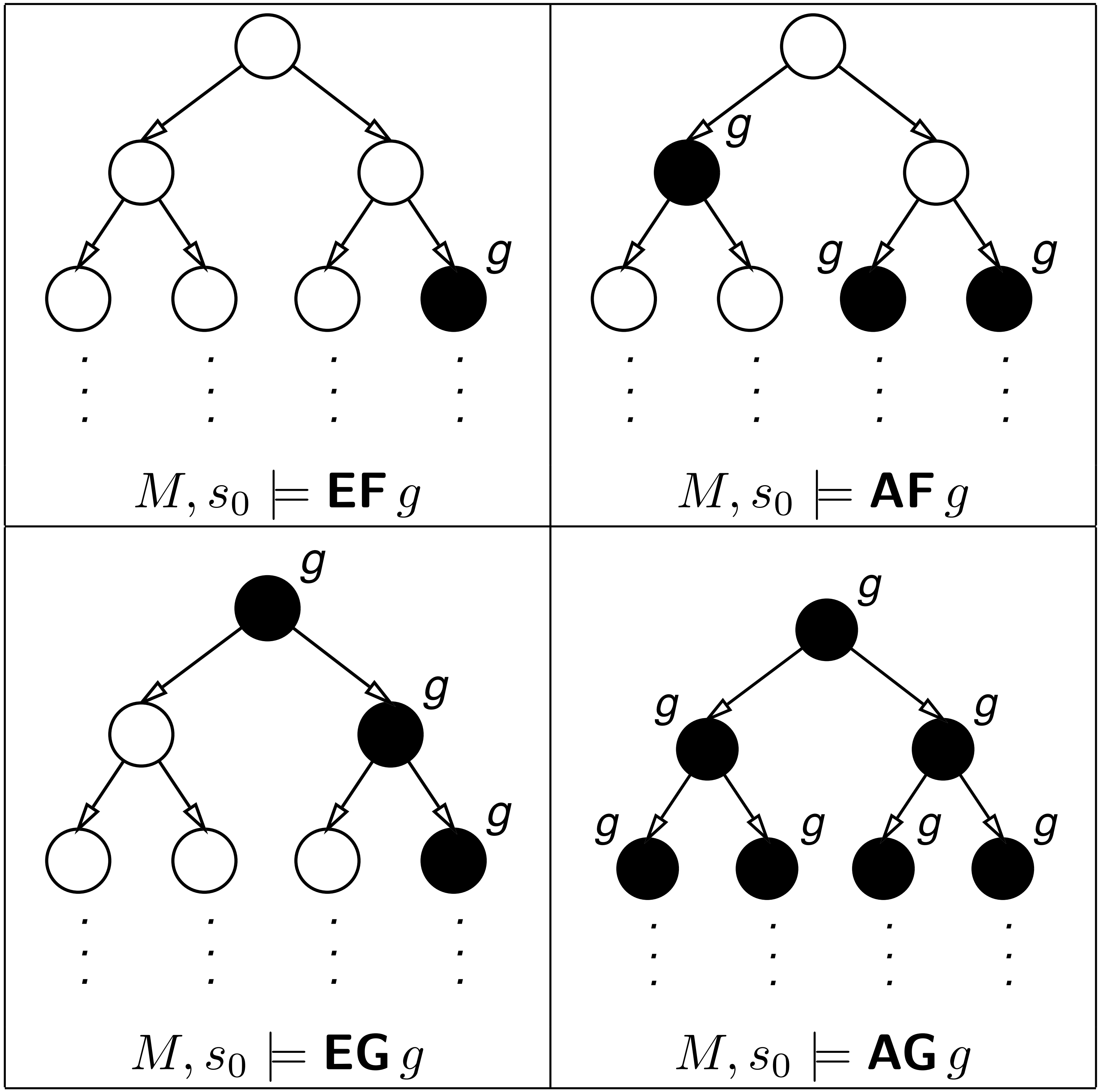
$\forall b_s\ \exists k\ (\ M,b_s[k] \vDash A\ \&\ \forall j\in[0,k-1]\ b_s[j] \vDash B$

in order to axiomatize CTL we add to the axioms od UB the following

$\forall\Box(C\rightarrow(\neg B\wedge(A\rightarrow\forall\bigcirc C))\rightarrow(C\rightarrow\neg(A\exists\mathcal{U}B))$

$\forall\Box(C\rightarrow(\neg B\wedge\exists\bigcirc C))\rightarrow(C\rightarrow\neg(A\forall\mathcal{U}B))$

The four most widely used CTL operators are illustrated here.
Each computation tree has the state s_0 as its root.



SOUNDNESS

$\vdash A \;\Rightarrow\; \models A$

(A simple induction on derivations: exercise)

COMPLETENESS

$\models A \;\Rightarrow\; \vdash A$

Difficult: the canonical kripke model is not CTL-model

CTL*

The computation tree logic CTL* (pronounced "CTL star") combines both branching-time and linear-time operators.

In this logic a path quantifier can prefix an assertion composed of arbitrary combinations of the usual linear-time operators.

1. Path quantifiers:
   - **A** — "for every path"
   - **E** — "there exists a path"
2. Linear-time operators:
   - **X** p — p holds true *next* time.
   - **F** p — p holds true sometime in the *future*
   - **G** p — p holds true *globally* in the future
   - p **U** q — p holds true *until* q holds true

For a path $\pi = (s_0, s_1, \ldots)$, state s_0 is considered to be at the present time.

The syntax of state formulas is given by the following rules:

- If p is an atomic proposition, then p is a state formula.
- If f and g are state formulas, then $\neg f$ and $f \vee g$ are state formulas.
- If f is a path formula, then **E**(f) and **A**(f) are state formulas.

Two additional rules are needed to specify the syntax of path formulas:

- If f is a state formula, then f is also a path formula.
  (A state formula f is true for a path π if the f is true in the initial state of the path π.)
- If f and g are path formulas, then $\neg f$, $f \vee g$, $\mathbf{X}\, f$, $\mathbf{F}\, f$, $\mathbf{G}\, f$, and $f\, \mathbf{U}\, g$ are path formulas.

If f is a state formula, the notation $\boxed{M, s \models f}$ means that f holds at state s in the Kripke structure M.

Assume f_1 and f_2 are state formulas and g is a path formula. The relation $M, s \models f$ is defined inductively as follows:

1. $s \models p$ $\Leftrightarrow$ atomic proposition p is true in s .
2. $s \models \neg f_1$ $\Leftrightarrow$ $s \not\models f_1$.
3. $s \models f_1 \vee f_2$ $\Leftrightarrow$ $s \models f_1$ or $s \models f_2$.
4. $s \models \textbf{E}(g)$ $\Leftrightarrow$ g holds true for some path π starting with s
4. $s \models \textbf{A}(g)$ $\Leftrightarrow$ g holds true for every path π starting with s

If f is a path formula, the notation $\boxed{M, \pi \models f}$ means that f holds true for path π in Kripke structure M.

Assume g_1 and g_2 are path formulas and f is a state formula. The relation $M, \pi \models f$ is defined inductively as follows:

1. $\pi \models f \quad \Leftrightarrow \quad s$ is the first state of π and $s \models f$.
2. $\pi \models \neg g_1 \quad \Leftrightarrow \quad \pi \not\models g_1$.
3. $\pi \models g_1 \vee g_2 \quad \Leftrightarrow \quad \pi \models g_1$ or $\pi \models g_2$.
4. $\pi \models \mathbf{X}\, g_1 \quad \Leftrightarrow \quad \pi^1 \models g_1$.
5. $\pi \models \mathbf{F}\, g_1 \quad \Leftrightarrow \quad \pi^k \models g_1$ for some $k \geq 0$
6. $\pi \models \mathbf{G}\, g_1 \quad \Leftrightarrow \quad \pi^k \models g_1$ for every $k \geq 0$
7. $\pi \models g_1\, \mathbf{U}\, g_2 \quad \Leftrightarrow \quad$ there exists a $k \geq 0$ such that $\pi^k \models g_2$ and $\pi^j \models g_1$ for $0 \leq j < k$.

Recall: For $\pi = (s_0, s_1, \ldots)$, we write π^i to denote the suffix starting with s_i.

Notice that $\mathbf{F}p$, $\mathbf{F}\mathbf{F}p$, $\mathbf{F}\mathbf{F}\mathbf{F}p$, etc., hold true for a path π even if p holds true at only the initial state in the path π.

Note the following:

- $\mathbf{A}(f) \equiv \neg\,\mathbf{E}(\neg f)$
- $\mathbf{F}\,f \equiv (true\,\mathbf{U}\,f)$

  (Recall: $\pi \models g_1\,\mathbf{U}\,g_2 \Leftrightarrow$ there exists a $k \geq 0$ such that $\pi^k \models g_2$ and $\pi^j \models g_1$ for $0 \leq j < k$.)
- $\mathbf{G}\,f \equiv \neg\,\mathbf{F}\,\neg f$

So, given any CTL* formula, we can rewrite it without using the operators **A**, **F**, or **G**.

- **EF**$(Started \wedge \neg Ready)$: It is possible to get to a state where *Started* holds but *Ready* does not hold.
- **AG**$(Req \rightarrow$ **AF** $Ack)$: If a request occurs, then it will be eventually acknowledged.
- **AG**(**AF** $DeviceEnabled)$: The proposition *DeviceEnabled* holds infinitely often on every computation path.
- **AG**(**EF** $Restart)$: From any state it is possible to get to the *Restart* state.
- **A**(**GF** $enabled \Rightarrow$ **GF** $executed)$: if a process is infinitely-often *enabled*, then it is infinitely-often *executed*.

Note that the first four formulas are CTL formulas.

SOUNDNESS

$\vdash$?

COMPLETENESS

????

It can be shown that the three logics discussed in this section have different expressive powers.

For example, there is no CTL formula that is equivalent to the LTL formula $\mathbf{A}(\mathbf{FG}\,p)$.

Likewise, there is no LTL formula that is equivalent to the CTL formula $\mathbf{AG}(\mathbf{EF}\,p)$.

The disjunction $\mathbf{A}(\mathbf{FG}\,p) \vee \mathbf{AG}(\mathbf{EF}\,p)$ is a CTL* formula that is not expressible in either CTL or LTL.

Model Checking

Given a model M and a formula A

$M \models A$?

model checking is important for verification of properties of concurrent and distribute systems.
M represent the computational space and A the property to be verified

Theorem
The model checking problem for CTL is in deterministic polynomial time

Theorem
The model checking problem for LTL is PSPACE-complete

Modal Deductive Systems

NORMALIZATION

we consider a logic without the absurdum connective, and with propositional connective only implication and conjunction

$$
\begin{prooftree}
\AxiomC{$[\sigma \wedge \varphi]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{φ}
\AxiomC{$[\varphi \to \psi]^1$}
\RightLabel{$\to E$}
\BinaryInfC{ψ}
\AxiomC{$[\sigma \wedge \varphi]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{σ}
\RightLabel{$\to I$}
\UnaryInfC{$\psi \to \sigma$}
\RightLabel{$\to E$}
\BinaryInfC{σ}
\RightLabel{$\to I_1$}
\UnaryInfC{$(\varphi \to \psi) \to \sigma$}
\RightLabel{$\to I_2$}
\UnaryInfC{$(\sigma \wedge \varphi) \to ((\varphi \to \psi) \to \sigma)$}
\end{prooftree}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sigma \wedge \varphi]^2}{\varphi}\wedge E \qquad [\varphi \to \psi]^1}{\psi}\to E \qquad \dfrac{\dfrac{[\sigma \wedge \varphi]^2}{\sigma}\wedge E}{\psi \to \sigma}\to I}{\sigma}\to E}{(\varphi \to \psi) \to \sigma}\to I_1}{(\sigma \wedge \varphi) \to ((\varphi \to \psi) \to \sigma)}\to I_2$$

$$\dfrac{\dfrac{\dfrac{[\sigma \wedge \varphi]^1}{\sigma}\wedge E}{(\varphi \to \psi) \to \sigma}\to I}{(\sigma \wedge \varphi) \to ((\varphi \to \psi) \to \sigma)}\to I_1$$

$$
\dfrac{\dfrac{\dfrac{\sigma \qquad \sigma \rightarrow \varphi}{\varphi} \qquad \sigma}{(\varphi \wedge \sigma)} \qquad \dfrac{\dfrac{\dfrac{[\varphi \wedge \sigma]}{\varphi}}{\alpha \rightarrow \varphi}}{(\varphi \wedge \sigma) \rightarrow (\alpha \rightarrow \varphi)}}{\alpha \rightarrow \varphi}
$$

$$\dfrac{\dfrac{\dfrac{\sigma \qquad \sigma\to\varphi}{\varphi} \qquad \sigma}{(\varphi\wedge\sigma)} \qquad \dfrac{\dfrac{\dfrac{[\varphi\wedge\sigma]}{\varphi}}{\alpha\to\varphi}}{(\varphi\wedge\sigma)\to(\alpha\to\varphi)}}{\alpha\to\varphi}$$

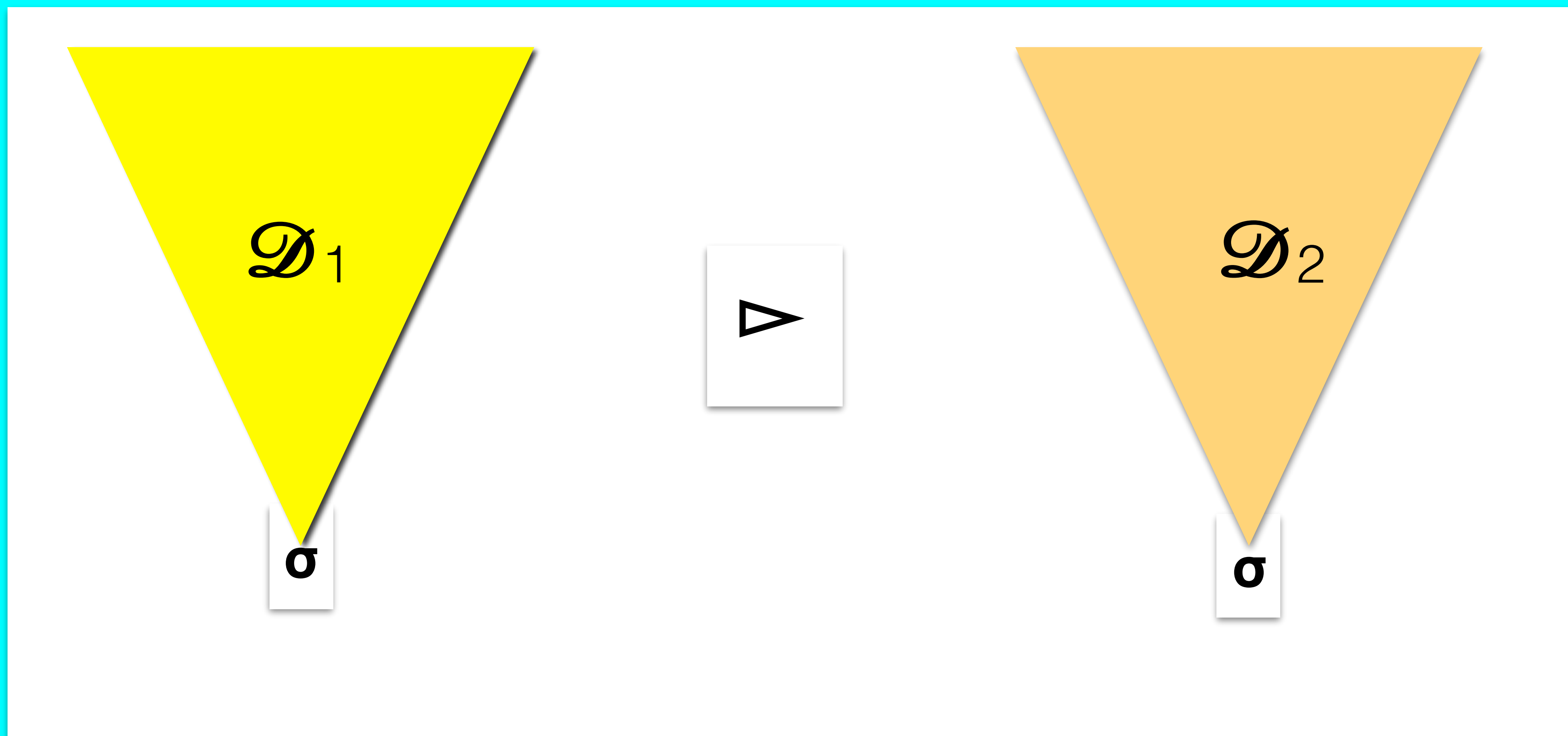
$$\dfrac{\dfrac{\dfrac{\dfrac{\sigma \qquad \sigma\to\varphi}{\varphi} \qquad \sigma}{(\varphi\wedge\sigma)}}{\varphi}}{\alpha\to\varphi}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\sigma \qquad \sigma\to\varphi}{\varphi} \qquad \sigma}{(\varphi\wedge\sigma)}}{\varphi}}{\alpha\to\varphi}$$

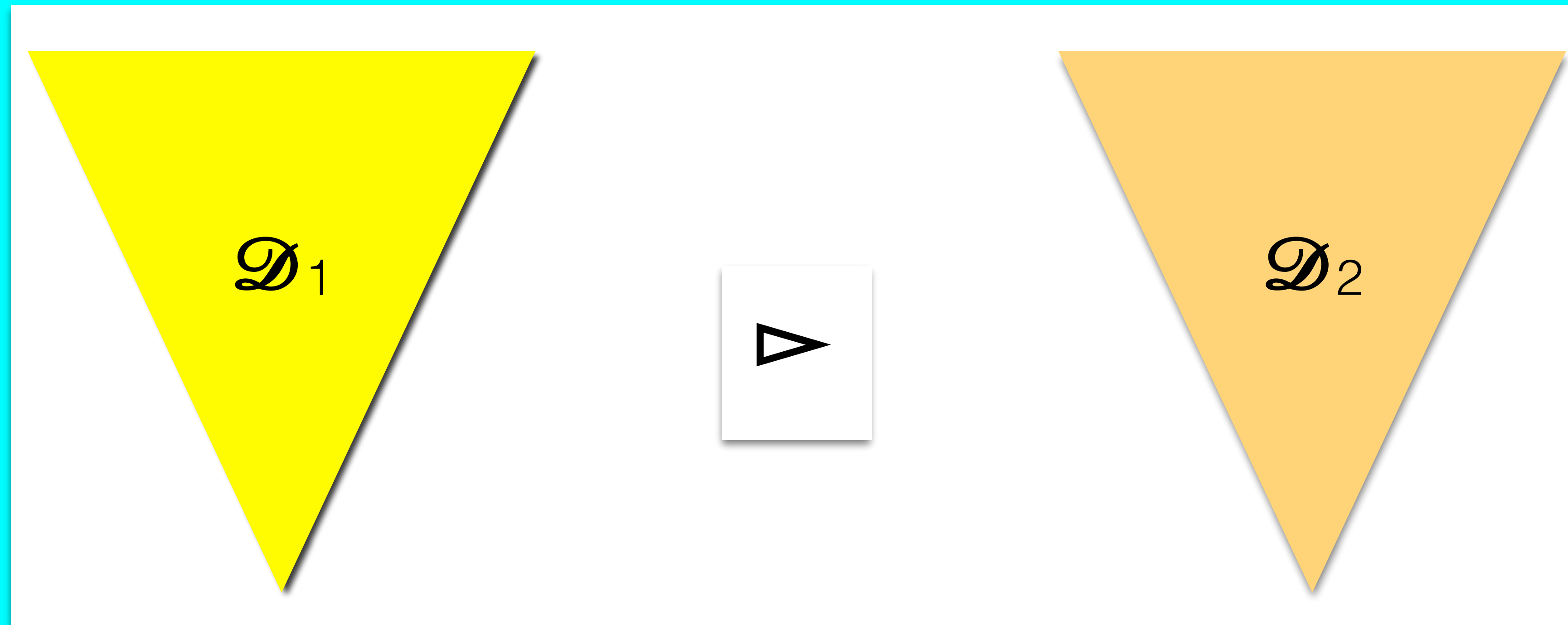
conversions

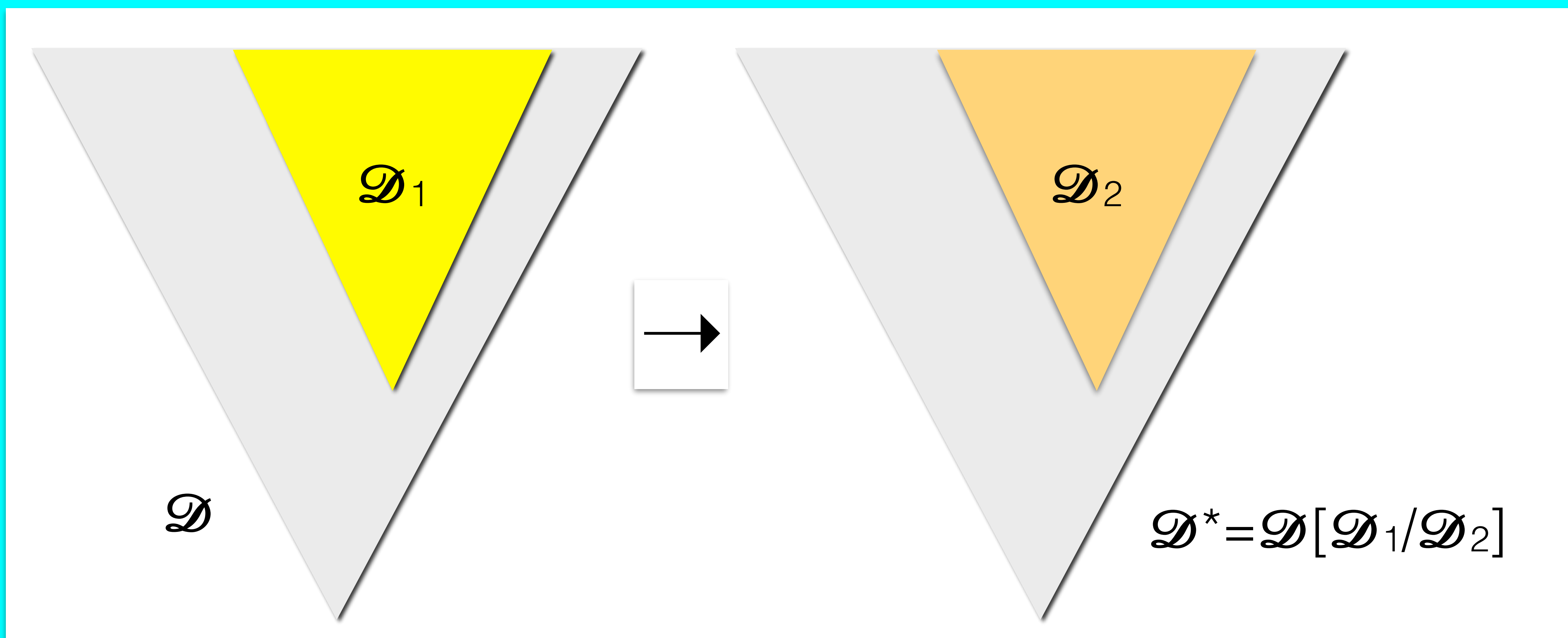
$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi\end{array} \qquad \dfrac{\begin{array}{c}[\psi]\\ \mathcal{D}_2\\ \varphi\end{array}}{\psi \to \varphi}\to I}{\varphi}\to E \qquad \rhd \qquad \begin{array}{c}\mathcal{D}_1\\ \psi\\ \mathcal{D}_2\\ \varphi\end{array}$$

Redex/cut: sequence →I , →E



$\mathrm{hp}\mathscr{D}_2 \subseteq \mathrm{hp}\mathscr{D}_1$





$$\mathscr{D} \to \mathscr{D}^*$$

$\mathscr{D} \to \mathscr{D}^*$ (**$\mathscr{D}$ 1-step reduces to $\mathscr{D}^*$**): $\mathscr{D}^*$ is obtained by applying a conversion to a subderivation of $\mathscr{D}$

$\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ (**$\mathscr{D}$ reduces to $\mathscr{D}^*$**): $\exists\ \mathscr{D}_1 \ldots \mathscr{D}_n$ s.t. $\mathscr{D}=\mathscr{D}_1$, $\mathscr{D}^*=\mathscr{D}_n$, $\mathscr{D}_1 \to \ldots \to \mathscr{D}_n$

$\twoheadrightarrow$ is the reflexive and transitive closure of $\to$

$\mathscr{D}$ is in normal form (irreducible) if $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ implies that $\mathscr{D}=\mathscr{D}^*$

$\mathscr{D}$ is in normal form (irreducible) if there is no $\mathscr{D}^*$ s.t. $\mathscr{D} \to \mathscr{D}^*$

Theorem (weak normalisation)

for each $\mathscr{D}$ there is $\mathscr{D}^*$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}^*$ is in normal form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi\end{array} \qquad \dfrac{\begin{array}{c}[\psi]\\ \mathcal{D}_2\\ \varphi\end{array}}{\psi \to \varphi}\to I}{\varphi}\to E \qquad \rhd \qquad \begin{array}{c}\mathcal{D}_1\\ \psi\\ \mathcal{D}_2\\ \varphi\end{array}$$

cut formula

conversion with cut formula $\psi\to\varphi$

$d(\phi)$= size of ϕ

ϕ is maximal in a derivation $\mathscr{D}$ if:

1. ϕ is a cut formula
2. $d(\phi)=\max\{d(\delta): \delta \text{ is a cut formula in } \mathscr{D}\}$

Theorem (weak normalisation)

for each $\mathscr{D}$ there is $\mathscr{D}^*$ s.t. $\mathscr{D} \twoheadrightarrow \mathscr{D}^*$ and $\mathscr{D}^*$ is in normal form

$d=\max\{d(\delta): \delta$ is a cut formula in $\mathscr{D}\}$

$n=\#\{\delta : \delta$ is an occurrence of a maximal cut$\}$

Let call $R(\mathscr{D})$ the pair (d,n) of $\mathscr{D}$.

Let us assume the lexicographic well order $<$ for pairs of natural numbers: $(d,n) < (d',n')$ iff $d < d'$ or $d=d'$ and $n < n'$.

The proof is by induction on $R(\mathscr{D})$.

Base: if $R(\mathscr{D})=(0,0)$ then $\mathscr{D}$ is in normal form;

Induction step: let us suppose that $R(\mathscr{D})=(d,n)$.

Make a reduction with a maximal cut formula δ: $\mathscr{D} \rightarrow \mathscr{D}^*$, with $R(\mathscr{D}^*)=(d^*,n^*)$

Now observe that $(d^*,n^*)<(d,n)$ (if $n>1$ then $d^*=d$ and $n^*=n-1$, if $n=1$, then $d^*<d$)

By induction hypothesis $\mathscr{D}^* \twoheadrightarrow \mathscr{D}^\circ$

Since $\mathscr{D} \rightarrow \mathscr{D}^*$ and $\mathscr{D}^* \twoheadrightarrow \mathscr{D}^\circ$ we have the thesis.

There is no general way of giving a proof theory for modal logics.

The case of S4

$$\dfrac{\begin{array}{c}\Box\Gamma, \neg\Diamond\Gamma' \\ \mathcal{D} \\ A\end{array}}{\Box A}\;\Box I \qquad \dfrac{\begin{array}{c}\mathcal{D} \\ \Box A\end{array}}{A}\;\Box E$$

$$\dfrac{\begin{array}{c}\mathcal{D} \\ A\end{array}}{\Diamond A}\;\Diamond I \qquad \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Diamond B\end{array} \qquad \begin{array}{c}\Box\Gamma, \neg\Diamond\Gamma', [\![B]\!] \\ \mathcal{D}_2 \\ C\end{array}}{C}\;\Diamond E$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{\Box A}\ \Box I$$

$C \in hp\,\mathcal{D} \Leftrightarrow C$ *has the shape either* $\Box B$ *or* $\neg\Diamond B$

failure of normalisation

$$
\begin{prooftree}
\AxiomC{$\Box A \wedge \Box B$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\Box A$}
\AxiomC{$[\Box A]^1$}
\RightLabel{$\Box E$}
\UnaryInfC{A}
\AxiomC{$[\Box B\,]$}
\RightLabel{$\Box E$}
\UnaryInfC{B}
\RightLabel{$\wedge I$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$\Box I$}
\UnaryInfC{$\Box(A \wedge B)$}
\RightLabel{$\supset I$}
\UnaryInfC{$\Box B \supset \Box(A \wedge B)$}
\RightLabel{$\supset I$ (1)}
\UnaryInfC{$\Box A \supset (\Box B \supset \Box(A \wedge B))$}
\RightLabel{$\supset E$}
\BinaryInfC{$\Box B \supset \Box(A \wedge B)$}
\end{prooftree}
$$

failure of normalisation

$$\dfrac{\dfrac{\Box A\wedge\Box B}{\Box A}\,\wedge E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box A]^1}{A}\,\Box E \quad \dfrac{[\Box B]}{B}\,\Box E}{A\wedge B}\,\wedge I}{\Box(A\wedge B)}\,\Box I}{\Box B\supset\Box(A\wedge B)}\,\supset I}{\Box A\supset(\Box B\supset\Box(A\wedge B))}\,\supset I\ (1)}{\Box B\supset\Box(A\wedge B)}\,\supset E$$

$$\triangleright$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box A\wedge\Box B}{\Box A}\,\wedge E}{A}\,\Box E \quad \dfrac{[\Box B]}{B}\,\Box E}{A\wedge B}\,\wedge I}{\Box(A\wedge B)}\,\Box I}{\Box B\supset\Box(A\wedge B)}\,\supset I$$

The solution proposed by Prawitz

$$\dfrac{\begin{matrix} \mathcal{D}_1 & & \mathcal{D}_n \\ [\Box\tau_1 & \ldots & \Box\tau_n] \\ & \mathcal{D} & \\ & \sigma & \end{matrix}}{\Box\sigma}\ \Box I$$

Γ=hp of the derivation,
#Γ=max{i: $\alpha^i \in \Gamma$}

$$\dfrac{\begin{matrix}\Gamma\\ \vdots \\ \sigma^j\end{matrix}}{\Box\sigma^{j-1}}\,\Box\mathcal{I} \qquad j > \#\Gamma \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots \\ \Box\sigma^j\end{matrix}}{\sigma^{j+1}}\,\Box\mathcal{E}$$

$$
\begin{prooftree}
\AxiomC{$[\Box\alpha^{1}]$}
\UnaryInfC{α^{2}}
\AxiomC{$[\Box(\alpha \to \beta)^{1}]$}
\UnaryInfC{$\alpha \to \beta^{2}$}
\BinaryInfC{β^{2}}
\UnaryInfC{$\Box\beta^{1}$}
\UnaryInfC{$\Box\alpha \to \Box\beta^{1}$}
\UnaryInfC{$\Box(\alpha \to \beta) \multimap (\Box\alpha \to \Box\beta)^{1}$}
\end{prooftree}
$$

T

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \square\sigma^{j}\end{array}}{\sigma^{k}}\square\mathcal{E} \qquad k\in\{j,j+1\}$$

$$\dfrac{\dfrac{[\square\alpha^{1}]}{\alpha^{1}}}{\square\alpha \rightarrow \alpha^{1}}.$$

$$\begin{array}{c}\Gamma\\ \vdots\\ \dfrac{\Box\sigma^{j}}{\sigma^{k}}\,\Box\mathcal{E}\end{array}\qquad k>j$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box\alpha^{1}]}{\alpha^{3}}\,!\mathcal{E}}{\Box\alpha^{2}}\,!\mathcal{I}}{\Box\Box\alpha^{1}}\,!\mathcal{I}}{\Box\alpha\rightarrow\Box\Box\,\alpha^{1}}\multimap\mathcal{I}$$

S4

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \Box\sigma^{j}\end{array}}{\sigma^{k}}\,\Box\mathcal{E} \qquad k \geq j$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \alpha^{k}\end{array}}{\Box\alpha^{k-1}}\,\Box\mathcal{I}}{\alpha^{k-1+j}}\,\Box\mathcal{E} \quad \rhd \quad \begin{array}{c}[j-1]_k\mathcal{D}\\ \alpha^{k-1+j}\end{array}, \text{ where: } \begin{cases} j=1 & \text{in } L^{\ell}K \\ 0 \leq j \leq 1 & \text{in } L^{\ell}KT \\ 1 \leq j & \text{in } L^{\ell}K4 \\ 0 \leq j & \text{in } L^{\ell}KT4 \end{cases}$$

$$\sigma^k$$

$$\frac{\begin{array}{cc}\Gamma & \Delta \\ \vdots & \vdots \\ \sigma^k & \tau^k\end{array}}{\sigma \wedge \tau^k}\wedge\mathcal{I} \qquad \frac{\begin{array}{c}\Gamma \\ \vdots \\ \sigma \wedge \tau^k\end{array}}{\sigma^k}\wedge\mathcal{E}_l \qquad \frac{\begin{array}{c}\Gamma \\ \vdots \\ \sigma \wedge \tau^k\end{array}}{\tau^k}\wedge\mathcal{E}_r$$

$$\frac{\begin{array}{cc}\Gamma & [\sigma^k] \\ & \vdots \\ & \tau^k\end{array}}{\sigma \supset \tau^k}\supset\mathcal{I} \qquad \frac{\begin{array}{cc}\Gamma & \Delta \\ \vdots & \vdots \\ \sigma \supset \tau^k & \sigma^k\end{array}}{\tau^k}\supset\mathcal{E}$$

$$\frac{\begin{array}{c}\Gamma \\ \vdots \\ \sigma^k\end{array}}{\Box\sigma^{k-1}}\Box\mathcal{I} \quad \| > \# - \qquad \frac{\begin{array}{c}\Gamma \\ \vdots \\ \Box\sigma^j\end{array}}{\sigma^k}\Box\mathcal{E}^\dagger$$

$\dagger$ $k = j + 1$ in $\lambda^\ell K$;

$k \in \{j, j+1\}$ in $\lambda^\ell KT$;

$k \geq j + 1$ in $\lambda^\ell K4$; $k \geq j$ in $\lambda^\ell S4$.

Arithmetic in a Natural deduction setting

Language $\mathbf{L_0}$:

one unary function symbol S

two binary function symbol +, •

two predicate symbols =, ≤

$$\dfrac{A(0) \qquad \begin{matrix}[A(x)] \\ \mathscr{D} \\ A(S(x))\end{matrix}}{A(t)}$$

$x \notin FV(hp(\mathscr{D}) - \{A(x)\}$

$$\begin{array}{c} \vdots \\ \dfrac{A(x)}{\forall x A(x)} \quad (\forall I) \end{array}$$

the variable x does not occur free in any of the undischarged assumptions on which the deduction of A(x) depends.

∀I can be reformulated by saying that A(x) does not interact, with respect to the variable x, with any of the undischarged assumptions on which the deduction of A(x) depends.

$$\frac{\begin{matrix}\vdots \\ A\end{matrix}}{\Box A}$$

A does not interact with any of the undischarged assumptions on which the deduction of A depends

????

the basic idea is to add a component to a propositional modal formula: a space position s

We write A^s to say that formula A has (is in) position s. We call A^s a position formula

positions allow a clear denition of modal interaction between formulas

Only modal operators and an induction rule can change the position of a formula

$$\dfrac{\begin{matrix}[A^s] \\ \vdots \\ B^s\end{matrix}}{A \to B^s} \quad (\to I)$$

The nature of positions and the treatment of modal operators

As already said, a position is a space coordinate

a position

is an ordered pair $\langle$**n;S**$\rangle$ where n is a natural number and S is a nite set of tokens

We intuitively think of $\langle$n + 1; S$\rangle$ as the successor position of $\langle$n; S$\rangle$

$$\frac{\begin{matrix}\vdots \\ A^{\langle n+1,S\rangle}\end{matrix}}{\circ A^{\langle n,S\rangle}}\ (\circ I) \qquad \frac{\begin{matrix}\vdots \\ \circ A^{\langle n,S\rangle}\end{matrix}}{A^{\langle n+1,S\rangle}}\ (\circ E)$$

In analogy with the first-order case, we say that formula $A^{\langle n,S\rangle}$ *interacts with* formula $B^{\langle m,T\rangle}$ *with respect to the token* x if $x \in S \cap T$.

$$\frac{\begin{array}{c}\vdots\\ A^{\langle n,S\rangle}\end{array}}{\Box A^{\langle n,S\setminus\{x\}\rangle}}\ (\Box I)$$

with the restriction that x∈S and x does not occur in any of the undischarged assumptions on which the deduction of A⟨n;S⟩ depends.

with the restriction that $x \in S$ and x does not occur in any of the undischarged assumptions on which the deduction of $A^{\langle n;S \rangle}$ depends.

$$\dfrac{\begin{array}{c}\vdots\\ \forall x A(x)\end{array}}{A(t)}$$

with the restriction that term t is free for variable x in $A(x)$.

Intuitively, after the elimination of from A^s, formula A can be in any position that is "reachable" from position s.

Reachable positions from s = ⟨n; S⟩ are those obtained by "adding" an arbitrary position to s, namely those of the form ⟨n + m; S ∪ T ⟩, where ⟨m;T⟩ is any position.

$$\dfrac{\begin{array}{c}\vdots\\ \Box A^{\langle n,S\rangle}\end{array}}{A^{\langle n+m,S\cup T\rangle}},$$

$$\Box A \rightarrow \Box\Box A^{\langle 0, \emptyset \rangle}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box A^{\langle 0,\emptyset\rangle}]}{A^{\langle 0,\{x,y\}\rangle}}\ \Box E}{\Box A^{\langle 0,\{x\}\rangle}}\ \Box I}{\Box\Box A^{\langle 0,\emptyset\rangle}}\ \Box I}{\Box A \longrightarrow \Box\Box A^{\langle 0,\emptyset\rangle}}\ \longrightarrow I$$

INDUCTION

$$(A \wedge \Box(A \to \circ A)) \to \Box A.$$

$$\dfrac{\begin{matrix} & [A(x)] \\ \vdots & \vdots \\ A(0) & A(x+1) \end{matrix}}{A(t)}$$

$$\dfrac{A(0) \qquad \begin{array}{c}[A(x)]\\ \vdots\\ A(x+1)\end{array}}{A(t)}$$

$$\dfrac{\begin{array}{c}\\ \vdots\\ A^{\langle n,S\rangle}\end{array} \qquad \begin{array}{c}[A^{\langle n,S\cup\{x\}\rangle}]\\ \vdots\\ A^{\langle n+1,S\cup\{x\}\rangle}\end{array}}{A^{\langle n+m,S\cup T\rangle}}\ (\mathrm{IND})$$

with the restriction that x *does not occur in* S *or in any of the assumptions on which the deduction of* $A^{\langle n+1,S\cup\{x\}\rangle}$ *depends, with the exception of the discharged assumptions* $A^{\langle n,S\cup\{x\}\rangle}$.

Modal formulas have an alphabet consisting of:

- denumerably many *proposition symbols* $p_0, p_1, \ldots,$
- the symbol $\perp$ for *absurdum*,
- the *propositional connectives* $\vee, \wedge, \rightarrow$,
- the *modal operators* $\circ, \Box, \Diamond$, and
- the *auxiliary symbols* (and).

Definition 2.1. The set of *modal formulas* is the least set that contains $\perp$, the proposition symbols and is closed under applications of the prepositional connectives and the modal operators.

Definition 2.2. The set of *positions* is the set of all pairs $\langle n, S\rangle$ where n is a natural number and S is a finite set of tokens from a denumerable set $T = \{x_0, x_1, \dots\}$.

Definition 2.3. A *position formula* (briefly: *formula*) is an expression of the form A^s, where A is a modal formula and s is a position.

Let $s = \langle n, S\rangle$ and $t = \langle m, T\rangle$ be positions. For sake of simplicity we introduce the following notation:

- $s \oplus t$ for $\langle n+m, S \cup T\rangle$,
- if $T = \emptyset$ we write $s \oplus m$ for $s \oplus t$,
- if $t = \langle 0, \{x\}\rangle$ we write $s \oplus x$ for $s \oplus t$,
- we let $s[t/x] = \begin{cases} \langle n+m, (S \backslash \{x\}) \cup T\rangle & \text{if } x \in S, \\ s & \text{otherwise.} \end{cases}$

$$\dfrac{\begin{matrix}\vdots & \vdots \\ A^s & B^s\end{matrix}}{A \wedge B^s}\ (\wedge I) \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B^s\end{matrix}}{A^s}\ (\wedge_1 E) \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B^s\end{matrix}}{B^s}\ (\wedge_2 E)$$

$$\dfrac{\begin{matrix}\vdots \\ A^s\end{matrix}}{A \vee B^s}\ (\vee_1 I) \qquad \dfrac{\begin{matrix}\vdots \\ B^s\end{matrix}}{A \vee B^s}\ (\vee_2 I) \qquad \dfrac{\begin{matrix} & [A^s] & [B^s] \\ \vdots & \vdots & \vdots \\ A \vee B^s & C^t & C^t\end{matrix}}{C^t}\ (\vee E)$$

$$\dfrac{\begin{matrix}[A^s] \\ \vdots \\ B^s\end{matrix}}{A \rightarrow B^s}\ (\rightarrow I) \qquad \dfrac{\begin{matrix}\vdots & \vdots \\ A \rightarrow B^s & A^s\end{matrix}}{B^s}\ (\rightarrow E)$$

$$\dfrac{\begin{matrix}[\neg A^s] \\ \vdots \\ \perp^t\end{matrix}}{A^s}\ (\perp_c) \qquad \dfrac{\begin{matrix}\vdots \\ \perp^t\end{matrix}}{A^s}\ (\perp_i)$$

$$\dfrac{\begin{matrix}\vdots\\ A^{s\oplus 1}\end{matrix}}{\circ A^{s}}\ (\circ I) \qquad \dfrac{\begin{matrix}\vdots\\ \circ A^{s}\end{matrix}}{A^{s\oplus 1}}\ (\circ E)$$

$$\dfrac{\begin{matrix}\vdots\\ A^{s\oplus x}\end{matrix}}{\Box A^{s}}\ (\Box I) \qquad \dfrac{\begin{matrix}\vdots\\ \Box A^{s}\end{matrix}}{A^{s\oplus t}}\ (\Box E)$$

In the rule $\Box I$ the token x does not occur in s or in the set of assumptions on which $A^{s\oplus x}$ depends.

$$\dfrac{\begin{array}{c}\vdots\\ A^{s\oplus t}\end{array}}{\Diamond A^{s}}\ (\Diamond I) \qquad \dfrac{\begin{array}{c}\vdots\\ \Diamond A^{s}\end{array} \quad \begin{array}{c}[A^{s\oplus x}]\\ \vdots\\ C^{t}\end{array}}{C^{t}}\ (\Diamond E)$$

In the rule $\Diamond E$ the token x does not occur in s, t or in the set of assumptions on which C^t depends, with the exception of the discharged assumptions $A^{s \oplus x}$.

$$\dfrac{\begin{array}{c}\vdots\\ A^{s}\end{array} \quad \begin{array}{c}[A^{s\oplus x}]\\ \vdots\\ A^{s\oplus\langle 1,x\rangle}\end{array}}{A^{s\oplus t}}\ (\mathsf{IND})$$

In the rule IND, the token x does not occur in s or in any of the assumptions on which $A^{s\oplus\langle 1,x\rangle}$ depends, with the exception of the discharged assumptions $A^{s\oplus x}$.

We write $\vdash A^s$ to say that there exists a deduction of A^s in the system whose (undischarged) assumptions belongs to .

$$\neg \circ A \rightarrow \circ \neg A$$

$\neg\circ A \to \circ\neg A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A^{s\oplus 1}]}{\circ A^{s}} \quad \neg\circ A^{s}}{\bot^{s}}}{\bot^{s\oplus 1}}}{\neg A^{s\oplus 1}}}{\circ\neg A^{s}}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[A \wedge \Box(A \to \circ A)^s]}{A^s} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A \wedge \Box(A \to \circ A)^s]}{\Box(A \to \circ A)^s}}{A \to \circ A^{s\oplus x}} \qquad [A^{s\oplus x}]}{\circ A^{s\oplus x}}}{A^{s\oplus x\oplus 1}}}{A^{s\oplus x}}\ \mathrm{IND}}{\Box A^s}}{A \wedge \Box(A \to \circ A) \to \Box A^s}
$$

Towards Strong Normalisation (intuitionistic system) Very Difficult

Proposition 2.4. *Let $\Gamma \vdash_N A^s$, where N is* PNK or PNJ. *Then there exists a deduction of A^s from Γ in the system N such that*

1. *each proper token is the proper token of exactly one instance of $\Box I$ or $\Diamond E$* or IND *rule*;
2. *the proper token of any instance of $\Box I$ rule occurs only in the sub-derivation above that instance of the rule*;
3. *the proper token of any instance of $\Diamond E$ or* IND *rule occurs only in the sub-derivation above the minor premiss of that instance of the rule.*

We denote by $\Pi[t/x]$ the tree obtained by replacing each position s in a deduction Π with s$[t/x]$.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A^s\end{matrix}\qquad\begin{matrix}\Pi_2\\ B^s\end{matrix}}{A\wedge B^s}}{A^s}\quad\rhd\quad\begin{matrix}\Pi_1\\ A^s\end{matrix}
\qquad\qquad
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A^s\end{matrix}\qquad\begin{matrix}\Pi_2\\ B^s\end{matrix}}{A\wedge B^s}}{B^s}\quad\rhd\quad\begin{matrix}\Pi_2\\ B^s\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A^s\end{matrix}}{A\vee B^s}\qquad\begin{matrix}[A^s]\\ \Pi_2\\ C^t\end{matrix}\qquad\begin{matrix}[B^s]\\ \Pi_3\\ C^t\end{matrix}}{C^t}\quad\rhd\quad\begin{matrix}\Pi_1\\ A^s\\ \Pi_2\\ C^t\end{matrix}
\qquad
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ B^s\end{matrix}}{A\vee B^s}\qquad\begin{matrix}[A^s]\\ \Pi_2\\ C^t\end{matrix}\qquad\begin{matrix}[B^s]\\ \Pi_3\\ C^t\end{matrix}}{C^t}\quad\rhd\quad\begin{matrix}\Pi_1\\ B^s\\ \Pi_3\\ C^t\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[A^s]\\ \Pi_1\\ B^s\end{matrix}}{A\to B^s}\qquad\begin{matrix}\Pi_2\\ A^s\end{matrix}}{B^s}\quad\rhd\quad\begin{matrix}\Pi_2\\ A^s\\ \Pi_1\\ B^s\end{matrix}
\qquad\qquad
\dfrac{\dfrac{\begin{matrix}\Pi\\ A^{s\oplus 1}\end{matrix}}{\circ A^s}}{A^{s\oplus 1}}\quad\rhd\quad\begin{matrix}\Pi\\ A^{s\oplus 1}\end{matrix}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ A^{s\oplus x}\end{array}}{\Box A^{s}}}{A^{s\oplus t}} \quad \rhd \quad \begin{array}{c}\Pi[t/x]\\ A^{s\oplus t}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A^{s\oplus t}\end{array}}{\Diamond A^{s}} \qquad \begin{array}{c}[A^{s\oplus x}]\\ \Pi_2\\ C^{t}\end{array}}{C^{t}} \quad \rhd \quad \begin{array}{c}\Pi_1\\ A^{s\oplus t}\\ \Pi_2[t/x]\\ C^{t}\end{array}$$

$$\dfrac{\begin{array}{c}\Pi_1\\ A^{s}\end{array} \qquad \begin{array}{c}[A^{s\oplus x}]\\ \Pi_2\\ A^{s\oplus\langle 1,x\rangle}\end{array}}{A^{s}} \quad \rhd \quad \begin{array}{c}\Pi_1\\ A^{s}\end{array}$$

$$\dfrac{\begin{array}{c}\Pi_1\\ A^{s}\end{array} \qquad \begin{array}{c}[A^{s\oplus x}]\\ \Pi_2\\ A^{s\oplus\langle 1,x\rangle}\end{array}}{A^{s\oplus(t\oplus 1)}} \quad \rhd \quad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A^{s}\end{array} \qquad \begin{array}{c}[A^{s\oplus x}]\\ \Pi_2\\ A^{s\oplus\langle 1,x\rangle}\end{array}}{A^{s\oplus t}}\\ \Pi_2[t/x]\\ A^{s\oplus(t\oplus 1)}\end{array}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\vee B^s\end{matrix}\qquad\begin{matrix}[A^s]\\ \Pi_2\\ C^t\end{matrix}\qquad\begin{matrix}[B^s]\\ \Pi_3\\ C^t\end{matrix}}{C^t}\qquad \Pi_4}{D^r}\,R \quad\triangleright\quad \dfrac{\begin{matrix}\Pi_1\\ A\vee B^s\end{matrix}\qquad \dfrac{\begin{matrix}[A^s]\\ \Pi_2\\ C^t\end{matrix}\qquad \Pi_4}{D^r}\,R\qquad \dfrac{\begin{matrix}[B^s]\\ \Pi_3\\ C^t\end{matrix}\qquad \Pi_4}{D^r}\,R}{D^r}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \Diamond A^s\end{matrix}\qquad\begin{matrix}[A^{s\oplus x}]\\ \Pi_2\\ C^t\end{matrix}}{C^t}\qquad \Pi_3}{D^r}\,R \quad\triangleright\quad \dfrac{\begin{matrix}\Pi_1\\ \Diamond A^s\end{matrix}\qquad \dfrac{\begin{matrix}[A^{s\oplus x}]\\ \Pi_2\\ C^t\end{matrix}\qquad \Pi_3}{D^r}\,R}{D^r}$$

The system is strong normalising !!!
(10 pagine di dimostrazione)

BCTL*

2. Frames

Frames are tree–like triples $\mathbf{S} = (S, \prec, D)$ where

(1) S is a nonempty set (elements of S will be called *time instants* or *nodes*);
(2) $\prec$ is a discrete strict partial ordering on S that satisfies the following:
  (a) for all $s \in S$ there exists $s' \in S$ such that $s \prec s'$.
  (b) for all $s, s' \in S$, the set $\{x \in S : s \preceq x \preceq s'\}$ is finite and linearly ordered.
(3) $D \subseteq {}^{\omega}S$ satisfies the property that, for each $(s_i)_{i\in\omega} \in D$, s_{i+1} is an immediate successor of s_i, for all $i \in \omega$. Elements of D will be called *(time)–directions*.
(4) D is
  (a) *suffix–closed*, namely whenever $(s_i)_{i\in\omega} \in D$, then $(s_{i+1})_{i\in\omega} \in D$;
  (b) *fusion–closed*, namely whenever $(s_i)_{i\in\omega} \in D, (t_i)_{i\in\omega} \in D$ and $s_n = t_n$ for some n, then $(s_0, \dots, s_n, t_{n+1}, t_{n+2}, \dots) \in D$.

Let d be a direction. The n-th node of direction d will be denoted by d_n (d_0 thus being the *initial instant* of d). We denote by d^n the direction $(d_n, d_{n+1}, \dots)$. Notations like d^n_m have the expected meaning: $d^n_m = d_{n+m}$.

SYNTAX

(1) a countable set $V_0 = \{x_i : i \in \omega\}$ of variables ranging over natural numbers, called *tokens*. We shall use metavariables x, y, z for tokens.
(2) a countable set $\Delta = \{\delta_i : i \in \omega\}$ of variables ranging over directions, called *path-variables*. We shall shall use γ, δ, η as metavariables for path-variables.
(3) a countable set V_1 of *propositional variables (letters)*.

A position p is a finite (possibly empty) string of tokens and occurrences of the digit 1 with the property that each token has at most one occurrence in p.

Intuitively, position $p = p_1 \dots p_n$ stands for the algebraic expression $\sum p = \sum_{i=1}^{n} p_i$. We regard two positions p, q as equal if $\sum p = \sum q$ is an algebraic identity. So, formally speaking, we deal with equivalence classes of positions. For better readability, we shall abbreviate $\underbrace{1 \dots 1}_{n \text{ times}}$ with the natural number n. We denote by P the set of positions.

Definition 2. *The set Σ of strings is the least set X such that*

(1) $\Delta \cup P \cup \{\epsilon\} \subseteq X$;
(2) if $\sigma \in X$ *then* $\sigma\delta \in X$ *for all* $\delta \in \Delta$ *not occurring in* σ;
(3) if $\sigma \in X$ *then* $\sigma p \in X$ *for all* $p \in P$ *such that no token in* p *occurs in* σ.

Definition 3. *The set Π of terms is the collection of strings whose leftmost element is a path-variable.*

Definition 4. *The set of pre–formulas is the least set X such that*

(1) $\perp \in X$ *and* $A \in X$ *whenever* A *is a propositional letter;*
(2) $(A \wedge B) \in X$ *whenever* $A, B \in X$;
(3) $(A \rightarrow B) \in X$ *whenever* $A, B \in X$;
(4) $(\bigcirc A)$, $(\Box A)$ *and* $(\forall A)$ *are in* X *whenever* $A \in X$.

We regard $\neg A$ and $\exists A$ as abbreviations for $A \rightarrow \perp$ and $\neg\forall\neg A$ respectively.

Definition 5. *A labelled temporal formula (briefly: formula) is a string of the form A^π where A is a pre–formula and $\pi \in \Pi$. Formula A^π is atomic when A is atomic.*

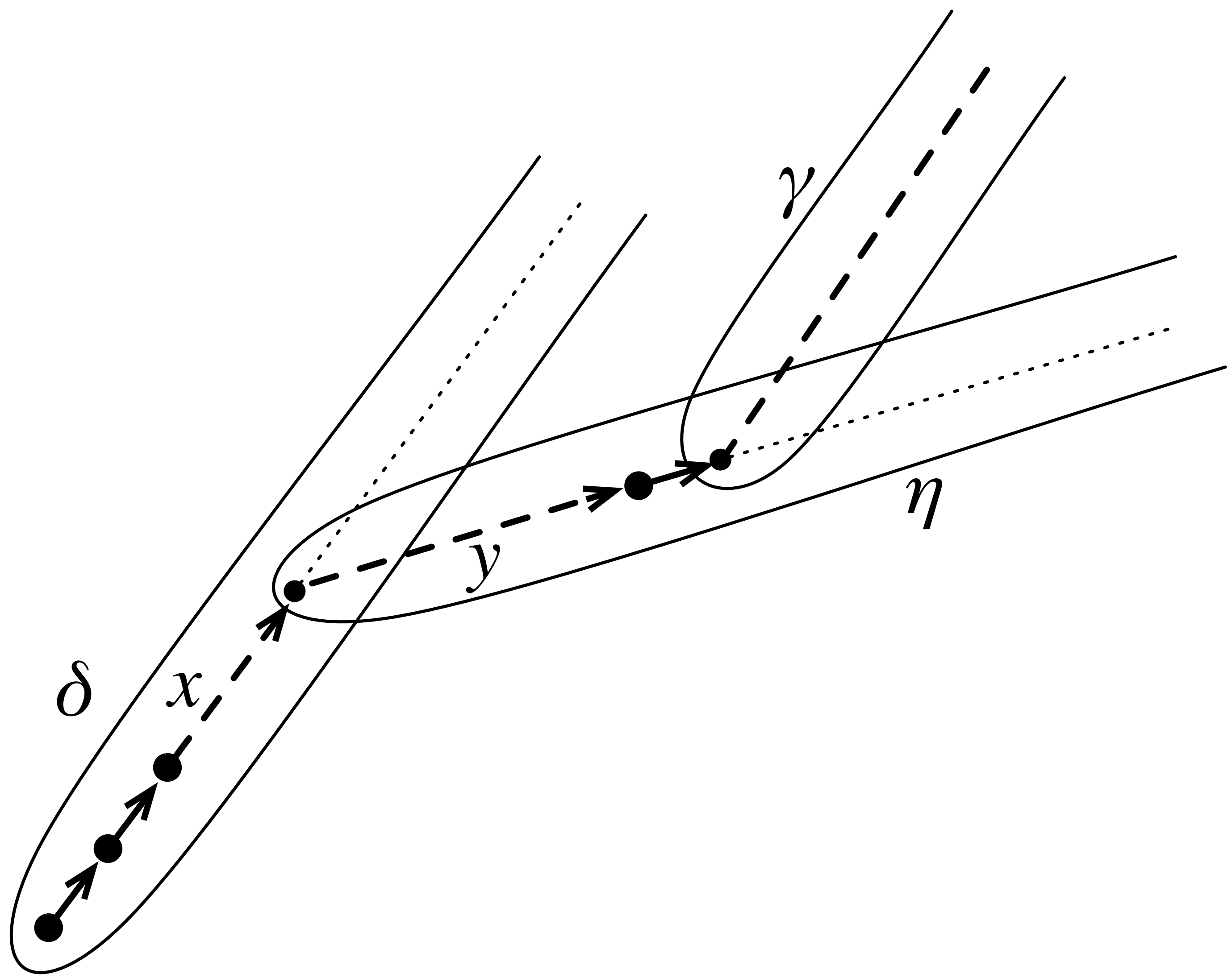


Figure 1. Graphical representation of term $\delta 2 x \eta y 1 \gamma$.

$$\frac{A_1^\pi \quad A_2^\pi}{(A_1 \wedge A_2)^\pi} \quad \wedge I \qquad\qquad \frac{\begin{array}{c}[A_1^\pi] \\ \vdots \\ A_2^\pi\end{array}}{(A_1 \to A_2)^\pi} \quad \to I$$

$$\frac{(A_1 \wedge A_2)^\pi}{A_i^\pi} \quad (i = 1, 2) \quad \wedge E_i \qquad\qquad \frac{(A_1 \to A_2)^\pi \;\; A_1^\pi}{A_2^\pi} \quad \to E$$

$$\frac{\begin{array}{c}[\neg A^\pi] \\ \vdots \\ \perp^\rho\end{array}}{A^\pi} \quad \perp_c$$

$$\frac{A^{\pi x}}{(\Box A)^{\pi}} \quad \Box\,\mathrm{I}$$

where x *does not occur in any of the assumptions on which the proof of the premiss depends.*

$$\frac{A^{\pi 1}}{(\bigcirc A)^{\pi}} \quad \bigcirc\,\mathrm{I} \tag{5}$$

$$\frac{A^{\gamma}}{(\forall A)^{\gamma}} \quad \forall\,\mathrm{I1} \qquad \frac{A^{\pi\gamma}}{(\forall A)^{\pi\gamma}} \quad \forall\,\mathrm{I2} \qquad \frac{A^{\pi\gamma}}{(\forall A)^{\pi}} \quad \forall\,\mathrm{I3} \tag{6}$$

where γ *does not occur in any of the assumptions on which the proof of the premiss depends.*

$$\frac{(\Box A)^{\pi}}{A^{\pi p}} \quad \Box\,\mathrm{E} \tag{7}$$

when p is the empty position, we just write A^{π} in the conclusion.

$$\frac{(\bigcirc A)^{\pi}}{A^{\pi 1}} \quad \bigcirc\,\mathrm{E} \tag{8}$$

$$\frac{(\forall A)^{\pi}}{A^{\pi}} \quad \forall\,\mathrm{E1} \qquad \frac{(\forall A)^{\pi}}{A^{\pi\gamma}} \quad \forall\,\mathrm{E2} \qquad \frac{(\forall A)^{\pi\gamma}}{A^{\pi}} \quad \forall\,\mathrm{E3} \tag{9}$$

$$\dfrac{A^{\pi} \quad \begin{array}{c}[A^{\pi x}] \\ \vdots \\ A^{\pi x 1}\end{array}}{A^{\pi p}} \quad \text{IND} \tag{10}$$

where token x does not occur free in any of the assumptions different from $A^{\pi x}$ on which the proof $A^{\pi x 1}$ depends.

$$\frac{A^{\pi}}{A^{\pi\delta}} \quad \text{At} \tag{11}$$

where A is a propositional letter.

$$\frac{B^{\pi\delta 1\sigma}}{B^{\pi 1\gamma\sigma}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\Box A)^{\delta}]}{A^{\delta 1x}}}{(\Box A)^{\delta 1}}}{(\bigcirc\Box A)^{\delta}}}{(\Box A \to \bigcirc\Box A)^{\delta}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[(\forall A)^{\delta}]}{A^{\delta\gamma}}}{(\forall A)^{\delta\gamma}}}{(\forall\forall A)^{\delta}}}{(\forall A \to \forall\forall A)^{\delta}}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\forall\neg A)^{\delta\gamma}]}{(\neg A)^{\delta}} \qquad A^{\delta}}{\bot^{\delta}}}{\bot^{\delta\gamma}}}{(\neg\forall\neg A)^{\delta\gamma}}}{(\forall\exists A)^{\delta}}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\forall \bigcirc A)^{\delta}]}{(\bigcirc A)^{\delta\gamma}}}{A^{\delta\gamma 1}}}{A^{\delta 1\gamma}}}{(\forall A)^{\delta 1}}}{(\bigcirc \forall A)^{\delta}}}{(\forall \bigcirc A \rightarrow \bigcirc \forall A)^{\delta}}$$

$$\vdash_{\mathcal{H}} A \Longleftrightarrow \vdash A^{\pi}$$

$$\models A^{\pi} \Longleftrightarrow \vdash A^{\pi}$$