Introduction to Wavelets

Discrete Wavelet Transform

- A wavelet is a function of zero average centered in the neighborhood of t=0 and is normalized

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0$$

$$\|\psi\| = 1$$

- The translations and dilations of the wavelet generate a family of functions over which the signal is projected

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

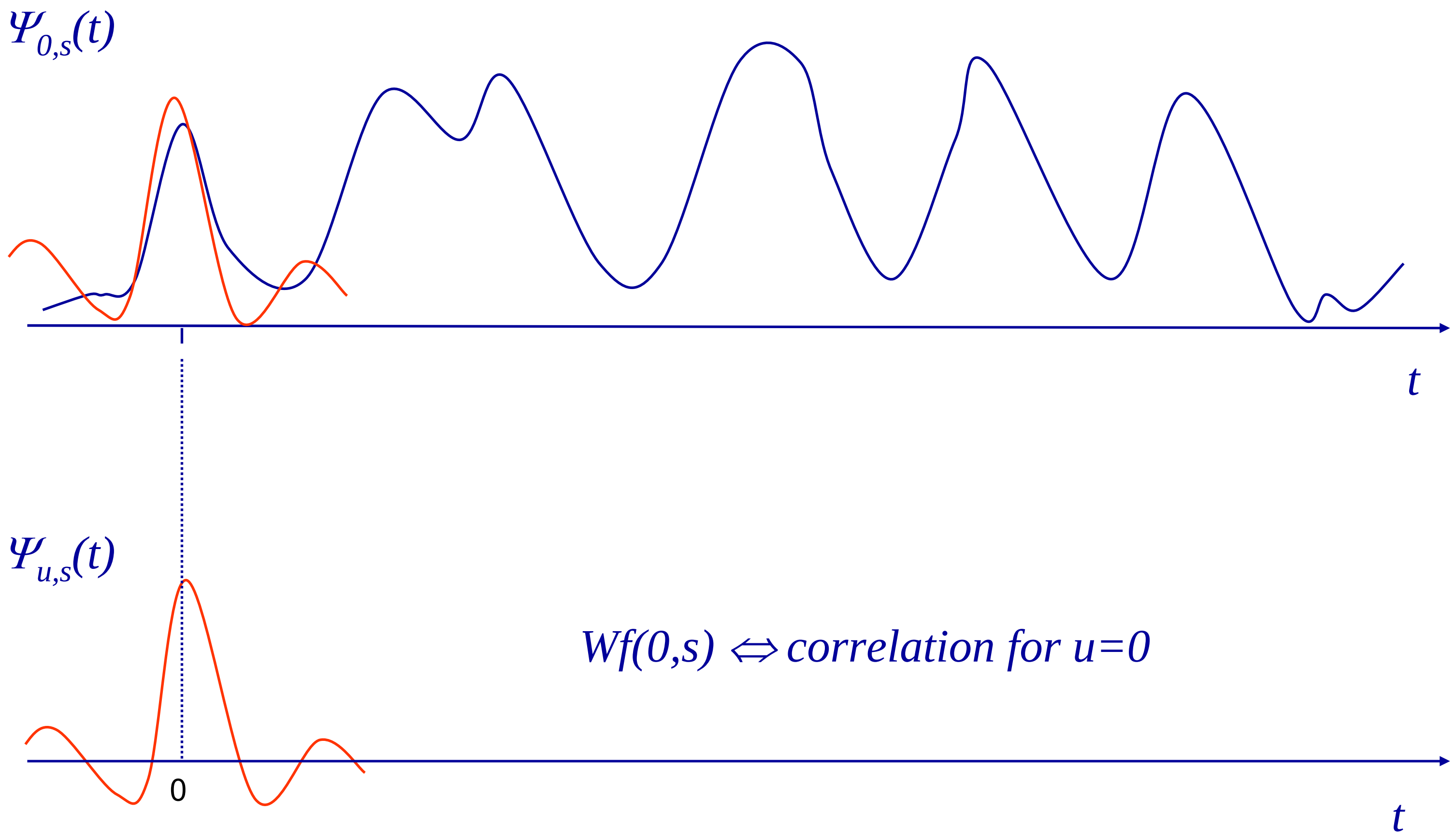
- Wavelet transform of f in $L^2(R)$ at position u and scale s is

$$Wf(u,s) = \left\langle f, \psi_{u,s} \right\rangle = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt$$

$$s = 2^j$$

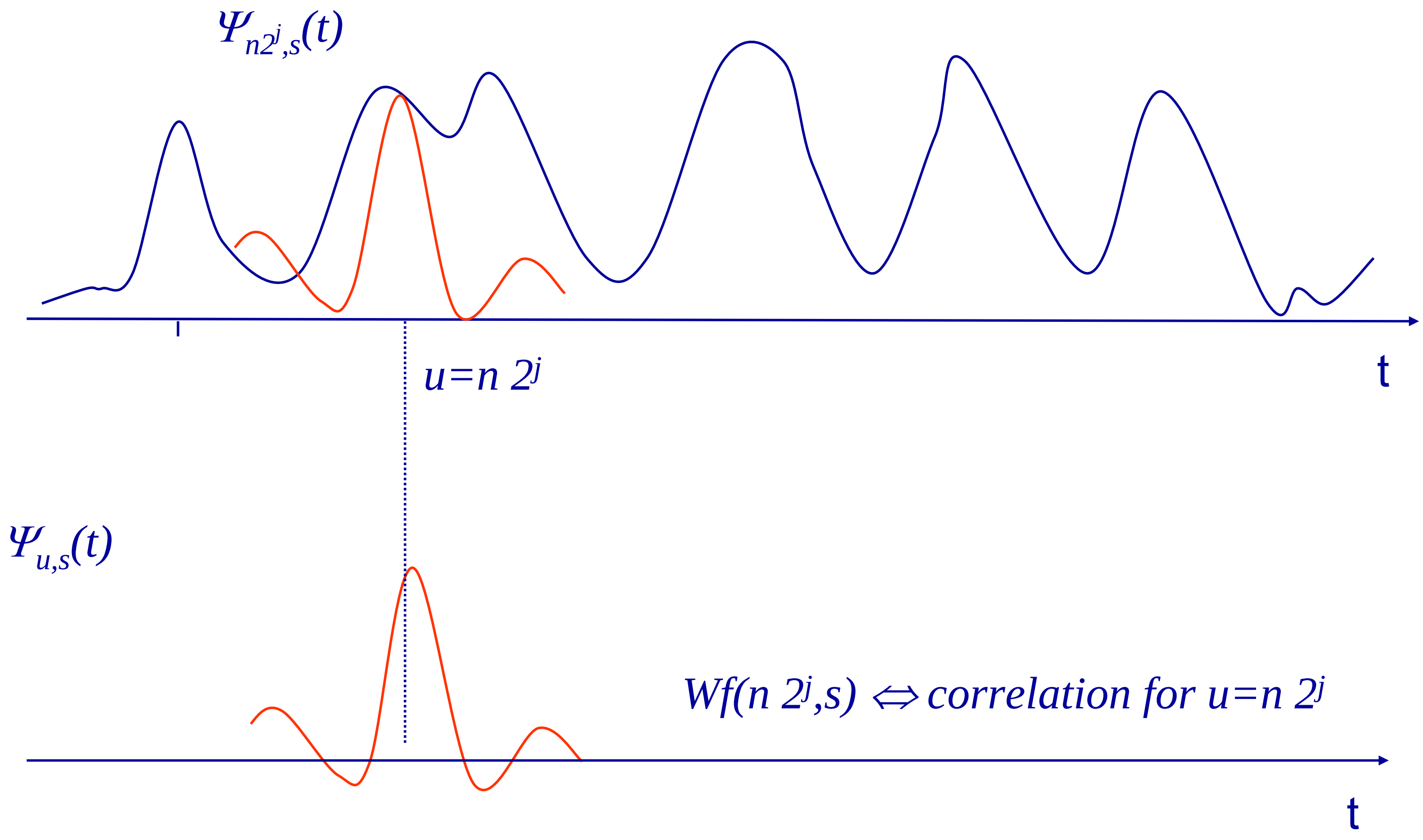
$$u = k \cdot 2^j$$

Wavelet transform

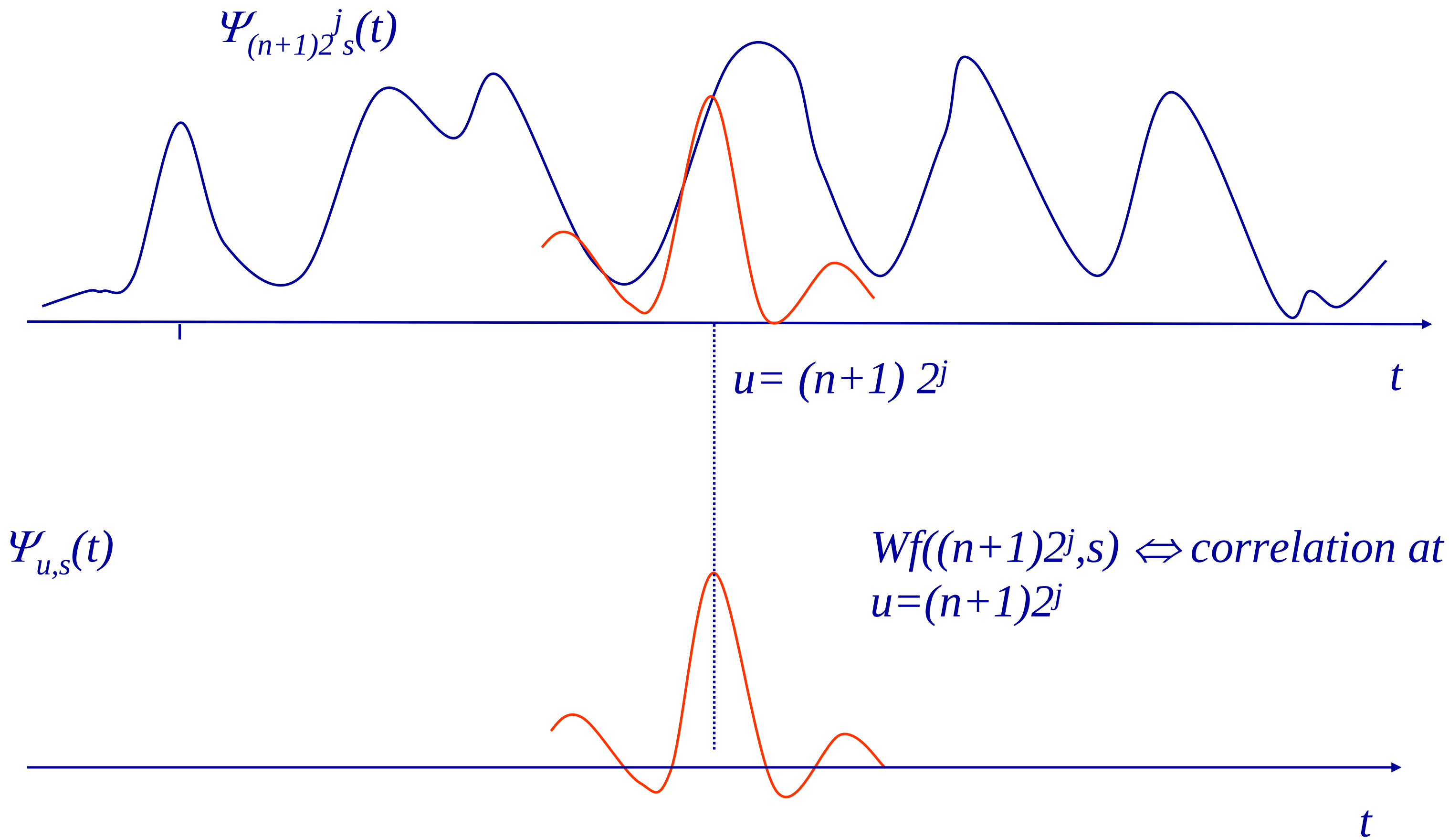


Wf(0,s) $\Leftrightarrow$ *correlation for u=0*

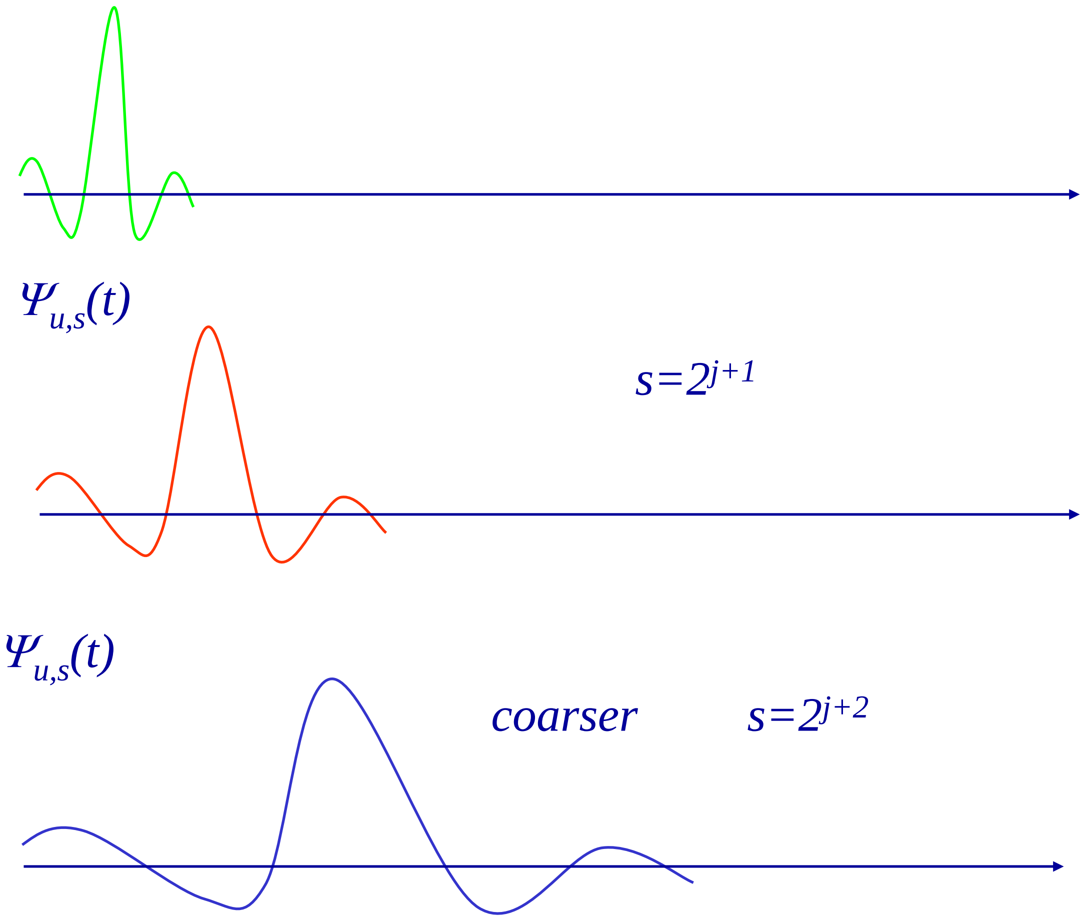
Wavelet transform



Wavelet transform



Changing the *scale*

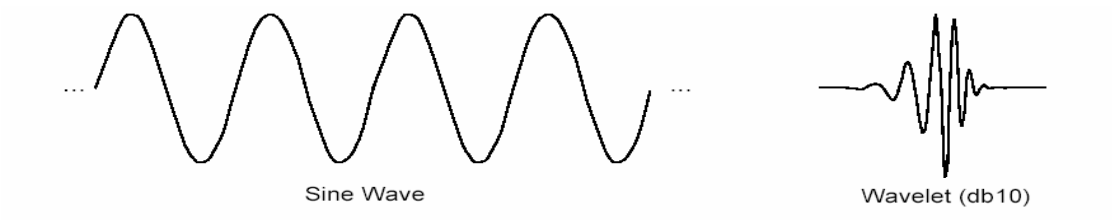


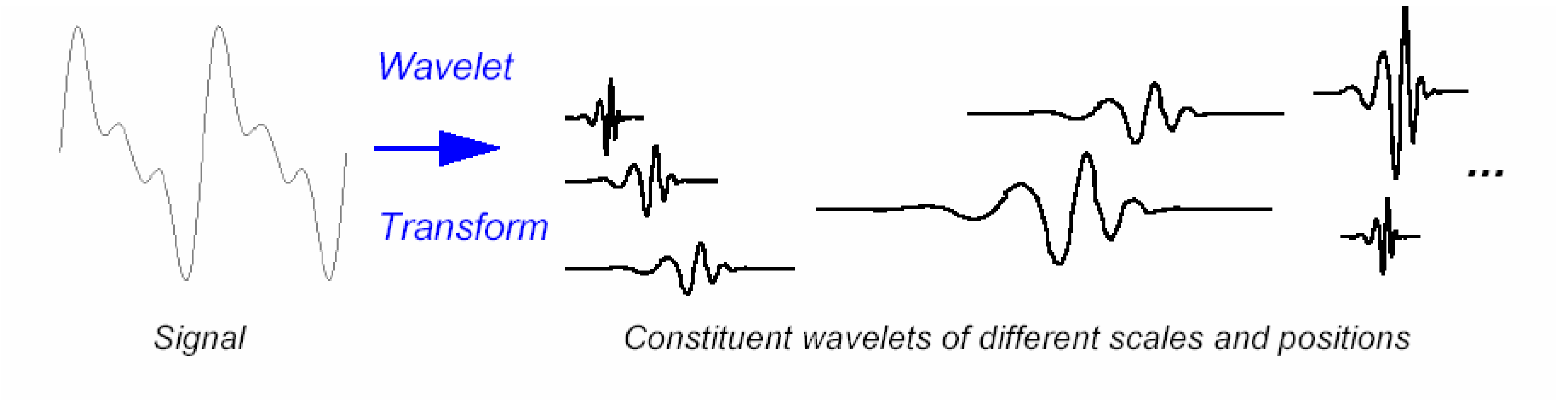
multiresolution

Fourier versus Wavelets

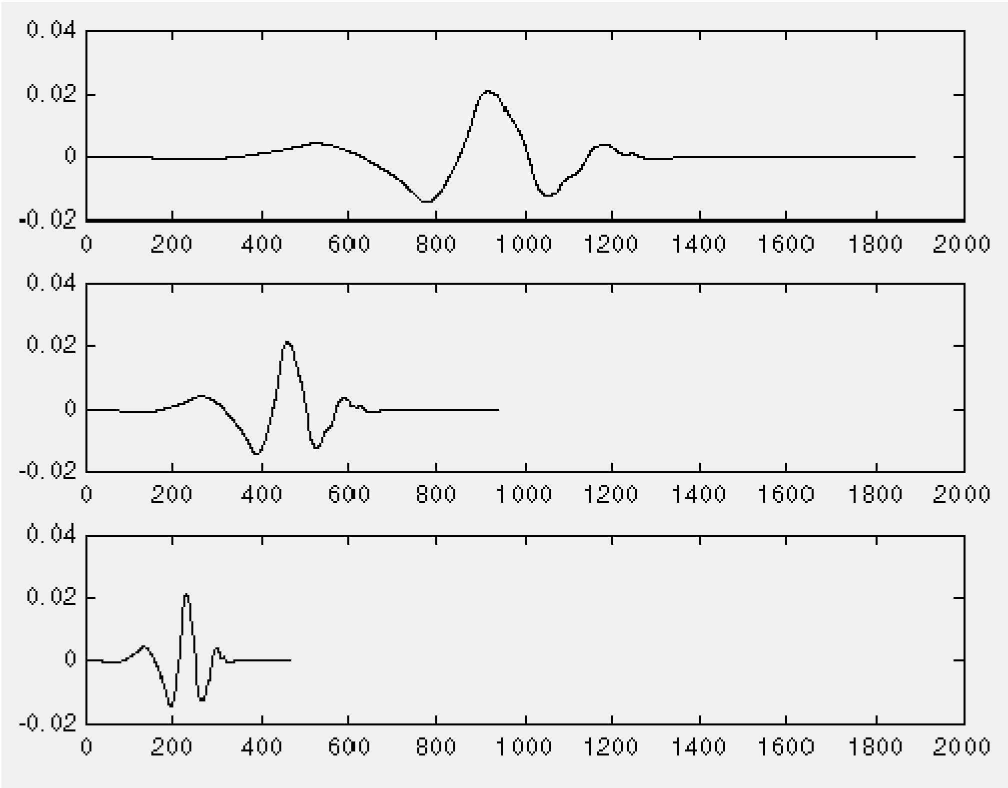
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$C(scale, position) = \int_{-\infty}^{\infty} f(t)\psi(scale, position, t)dt$$





Scaling

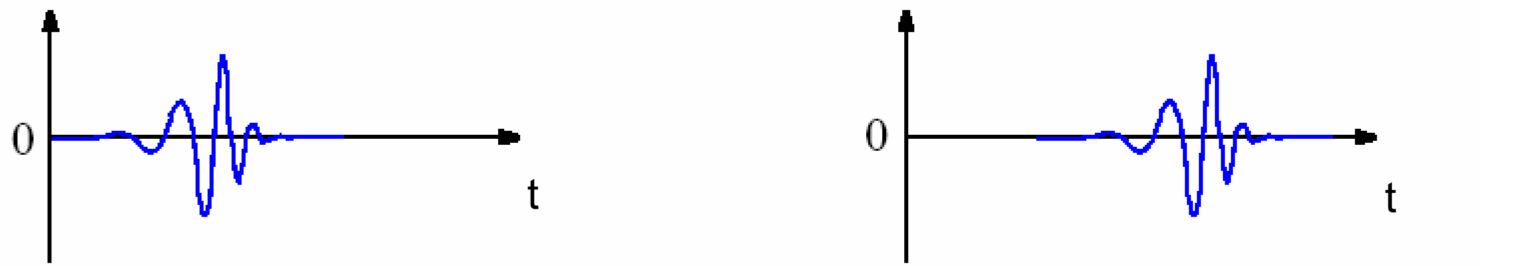


$$f(t) = \psi(t) \quad ; \quad a = 1$$

$$f(t) = \psi(2t) \quad ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \quad ; \quad a = \frac{1}{4}$$

Shifting



Wavelet function
$\psi(t)$

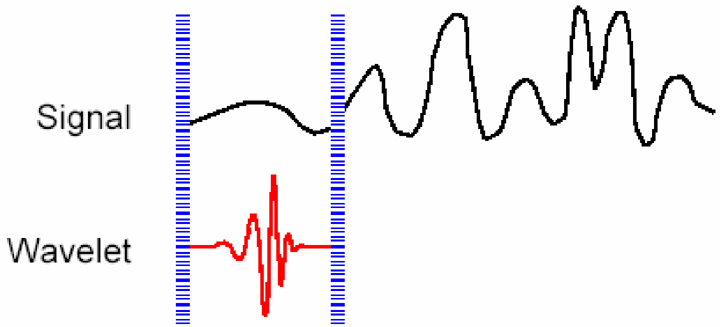
Shifted wavelet function
$\psi(t-k)$

Recipe

1 Take a wavelet and compare it to a section at the start of the original signal.

2 Calculate a number, `C`, that represents how closely correlated the wavelet is with this section of the signal. The higher `C` is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one, `C` may be interpreted as a correlation coefficient.

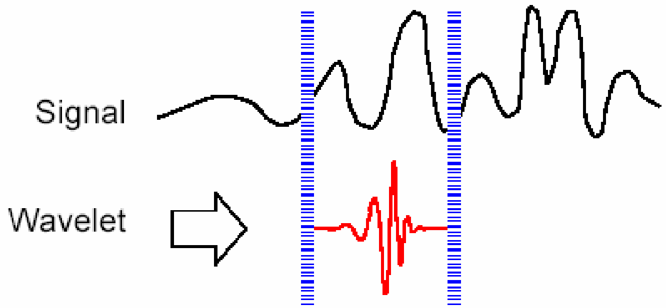
Note that the results will depend on the shape of the wavelet you choose.



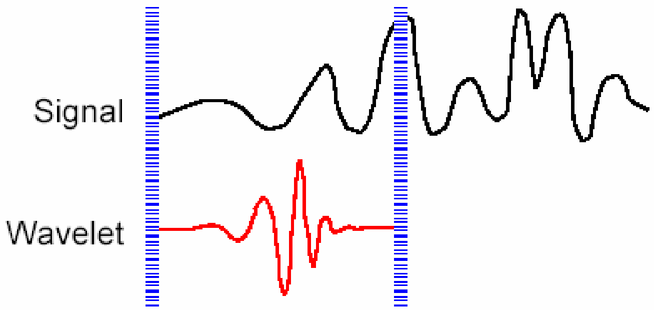
C = 0.0102

Recipe

3 Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



4 Scale (stretch) the wavelet and repeat steps 1 through 3.

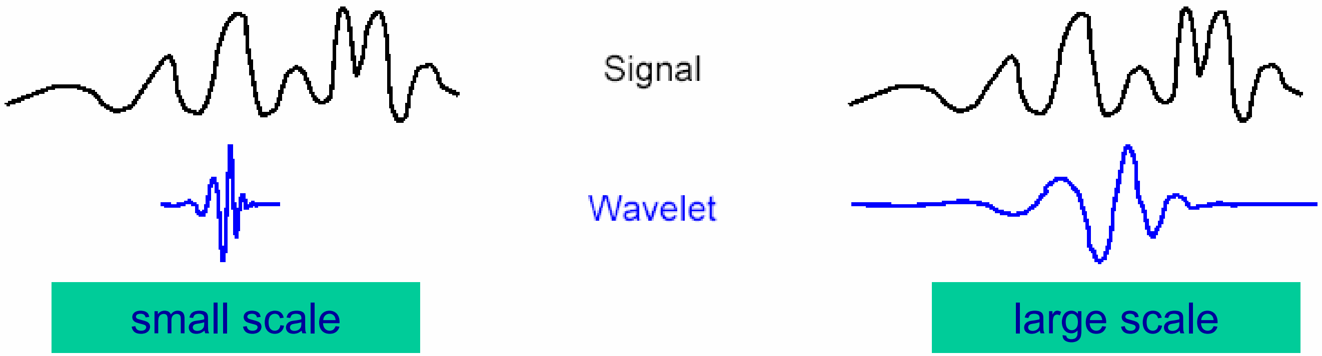


C = 0.2247

5 Repeat steps 1 through 4 for all scales.

Wavelet Zoom

- WT at position u and scale s measures the local correlation between the signal and the wavelet



Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:

(small) • Low scale $a \Rightarrow$ Compressed wavelet $\Rightarrow$ Rapidly changing details $\Rightarrow$ High frequency ω.

(large) • High scale $a \Rightarrow$ Stretched wavelet $\Rightarrow$ Slowly changing, coarse features $\Rightarrow$ Low frequency ω.

Frequency domain

- Parseval $$Wf(u,s) = \int_{-\infty}^{+\infty} f(t)\psi^*{}_{u,s}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\Psi^*{}_{u,s}(\omega)d\omega$$

  The wavelet coefficients Wf(u,s) depend on the values of f(t) (and F(ω)) in the time-frequency region where the energy of the corresponding wavelet function (respectively, its transform) is concentrated

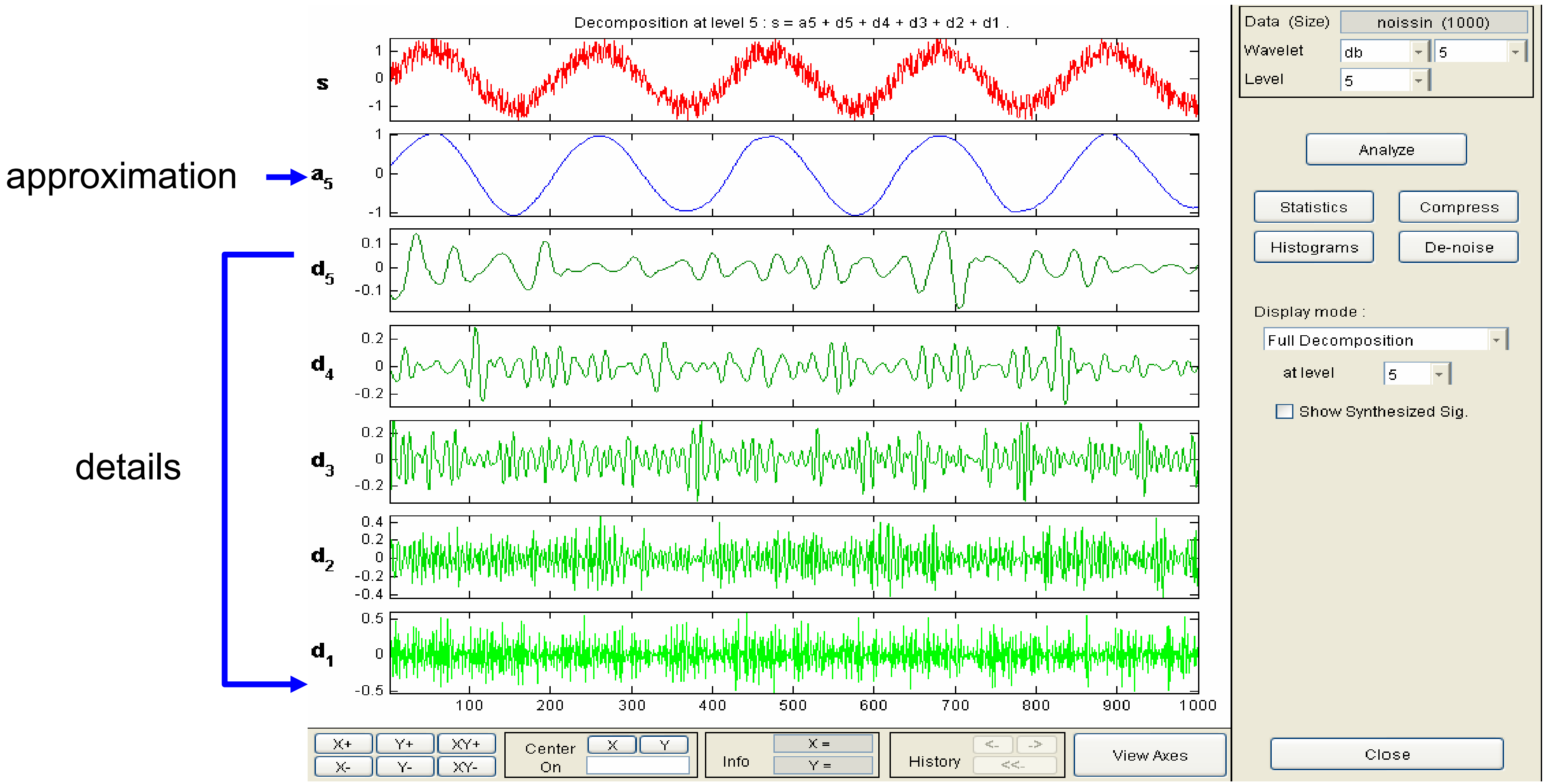
  - *time/frequency localization*
  - The *position and scale* of high amplitude coefficients allow to characterize the *temporal evolution* of the signal

  - Time domain signals (1D) : Temporal evolution
  - Spatial domain signals (2D) : Localize and characterize spatial singularities

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \Leftrightarrow \Psi_{u,s}(\omega) = \sqrt{s}\,\Psi(s\omega)e^{-j\omega s}$$

Stratching in time ↔ Shrinking in frequency (and viceversa)

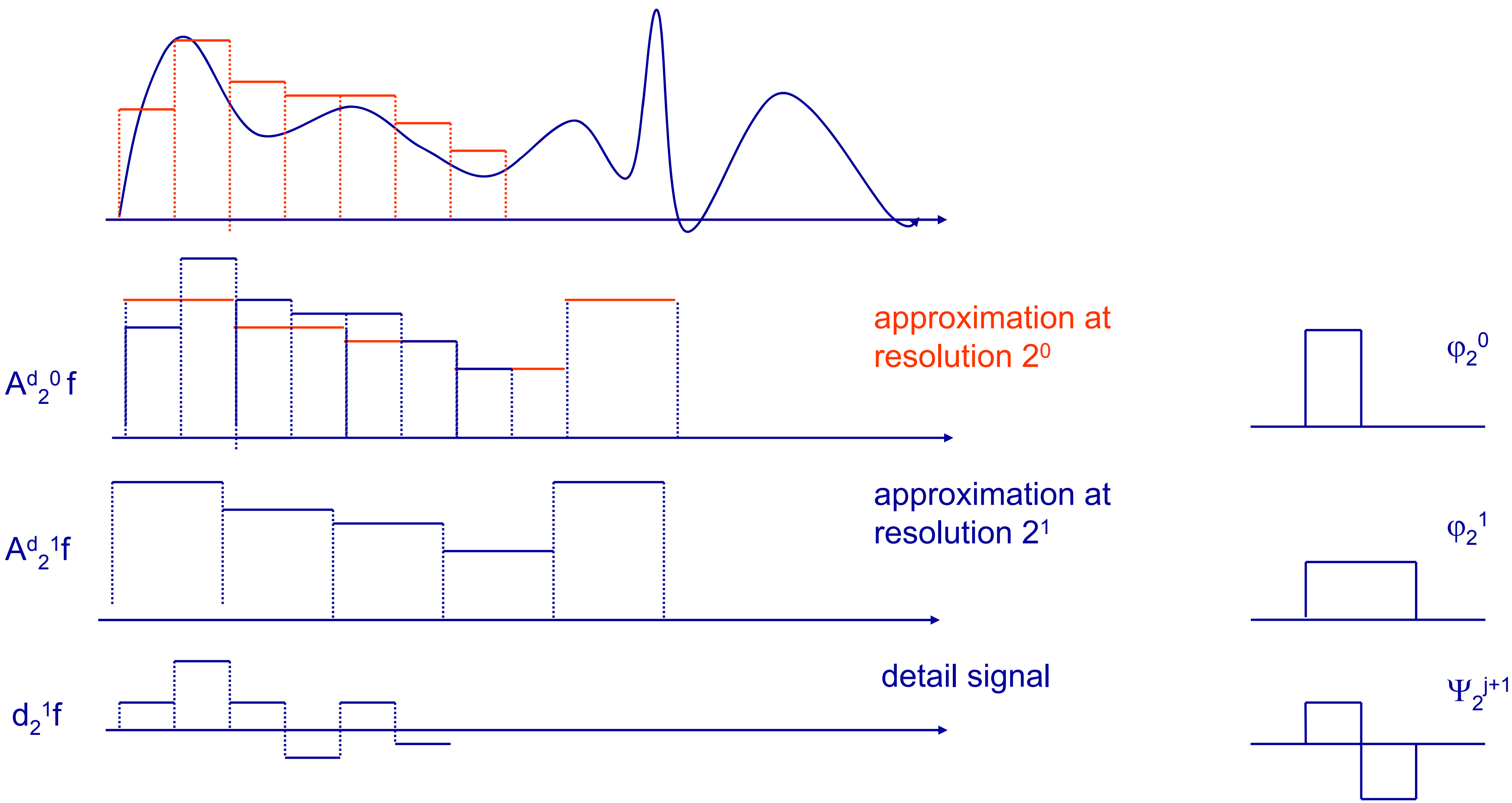
Example



Wavelet representation = approximation + details

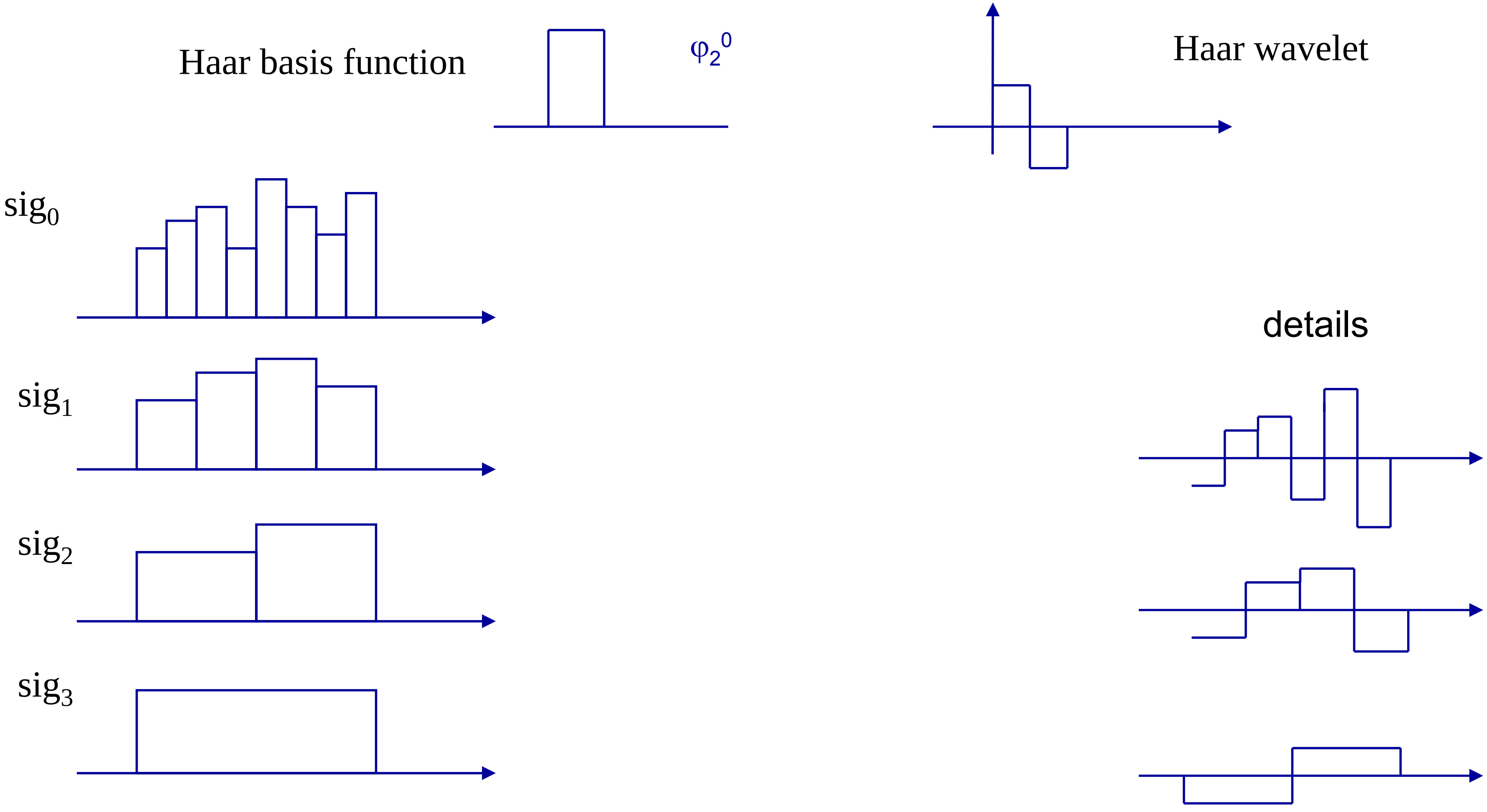
approximation ↔ scaling function
details ↔ wavelets

A different perspective



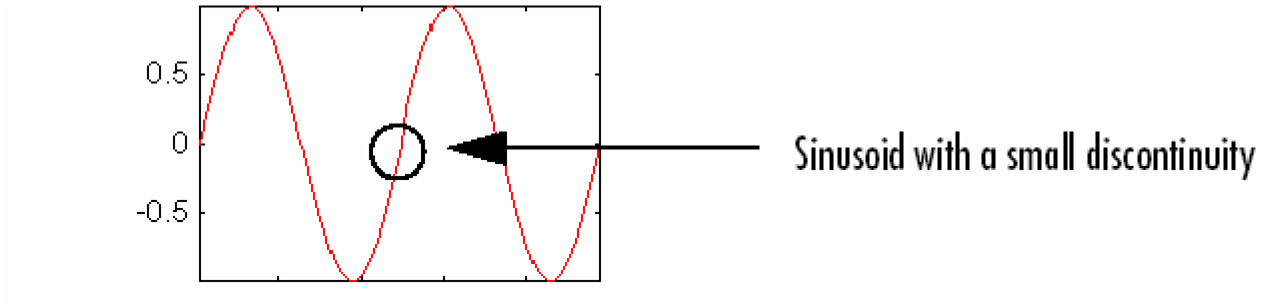
$A^d_{2^j} f = A^d_{2^{j+1}} f + d_{2^{j+1}} f$

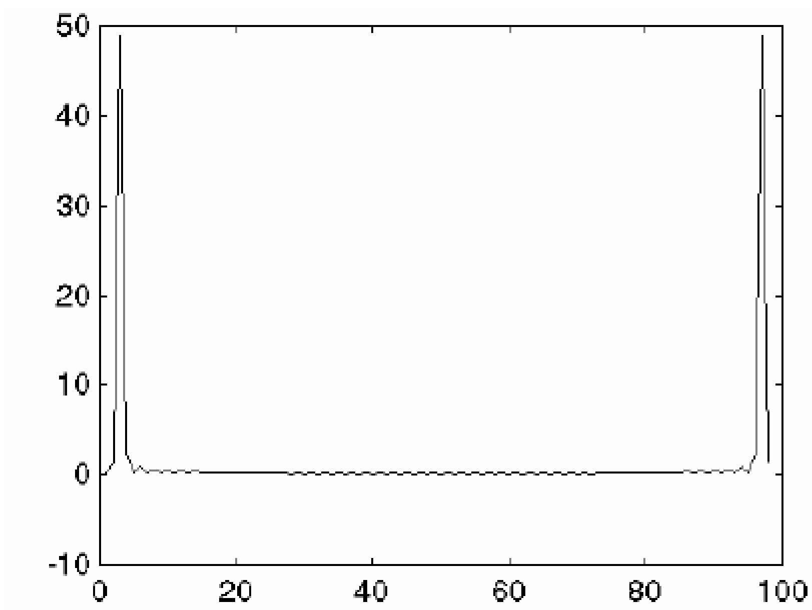
Haar pyramid [Haar 1910]



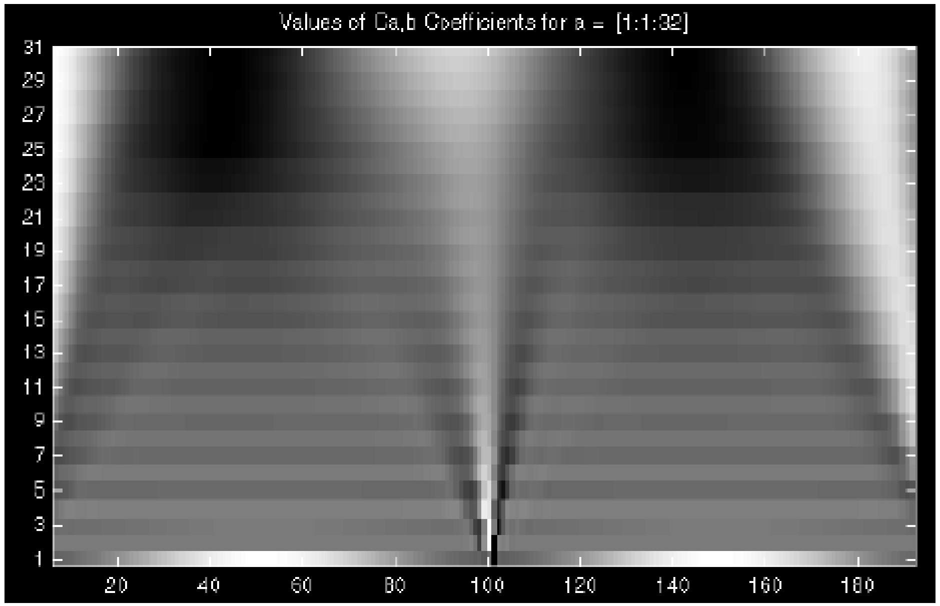
signal=approximation at scale n + details at scales 1 to n

What wavelets can do?





Fourier Coefficients



Wavelet Coefficients

Wavelets and linear filtering

- The WT can be rewritten as a convolution product and thus the transform can be interpreted as a linear filtering operation

$$Wf(u,s) = \left\langle f, \psi_{u,s} \right\rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt = f * \bar{\psi}_s(u)$$

$$\bar{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi^* \left(\frac{-t}{s} \right)$$

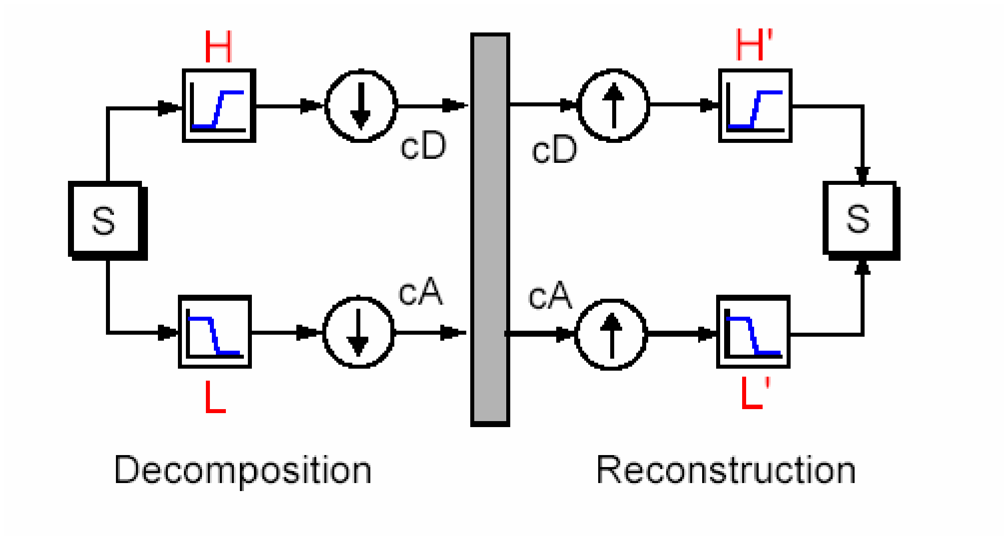
$$\hat{\bar{\psi}}_s(\omega) = \sqrt{s}\,\hat{\psi}^*(s\omega)$$

$$\hat{\psi}(0) = 0$$

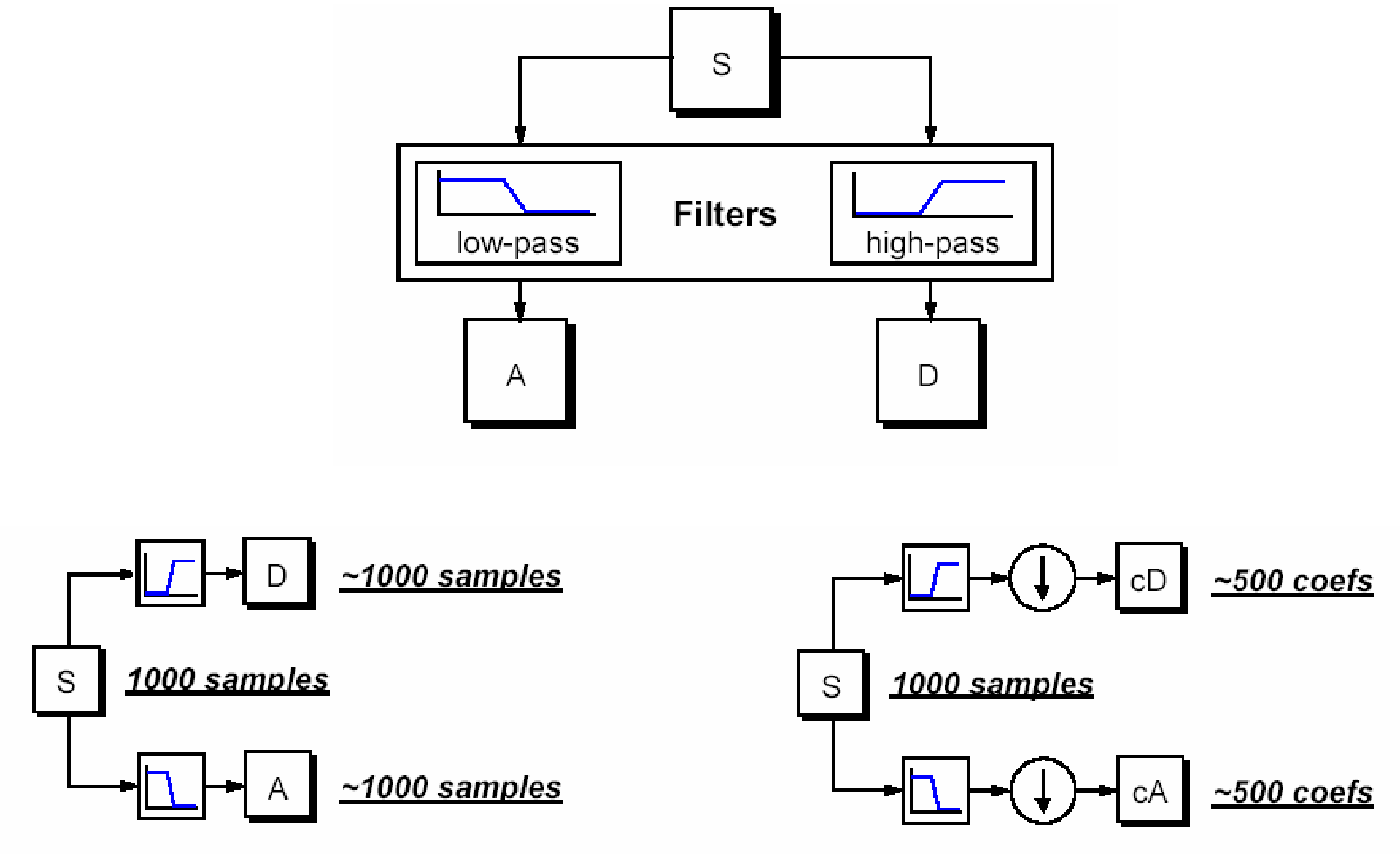
$\rightarrow$ band-pass filter

Wavelets & filterbanks

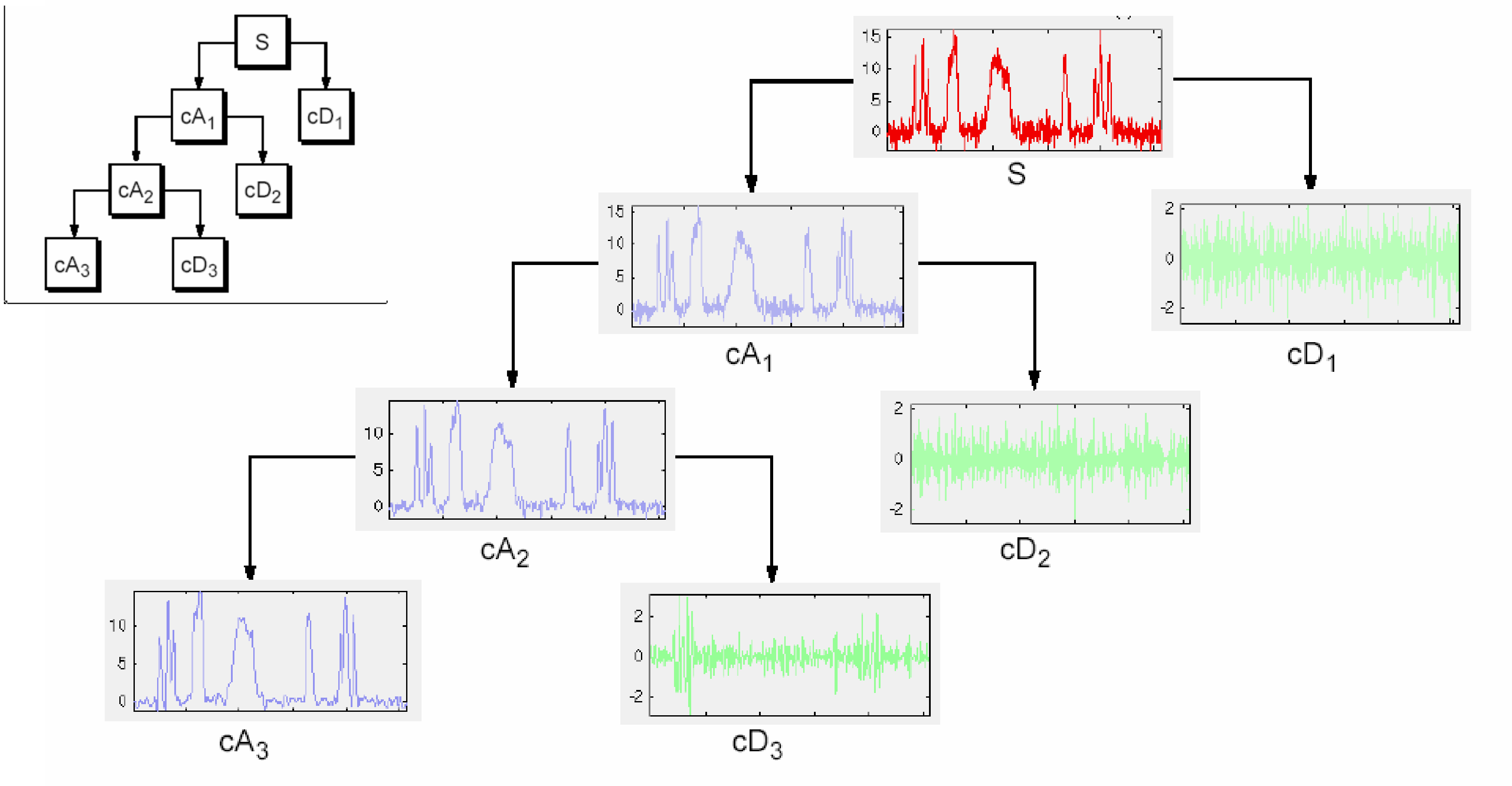
Quadrature Mirror Filter (QMF)



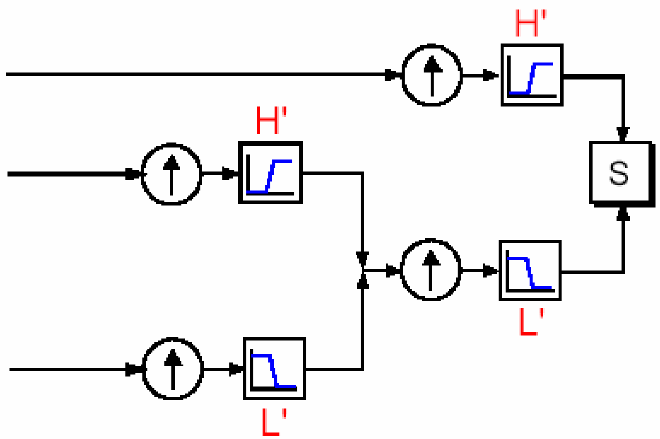
Analysis or decomposition



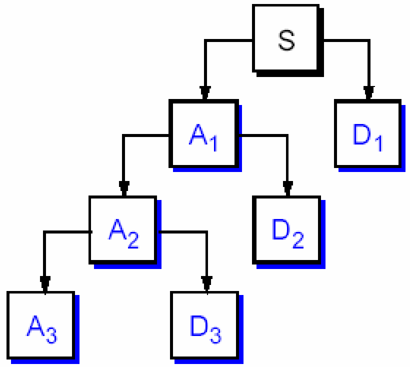
Analysis or decomposition



Synthesis or reconstruction

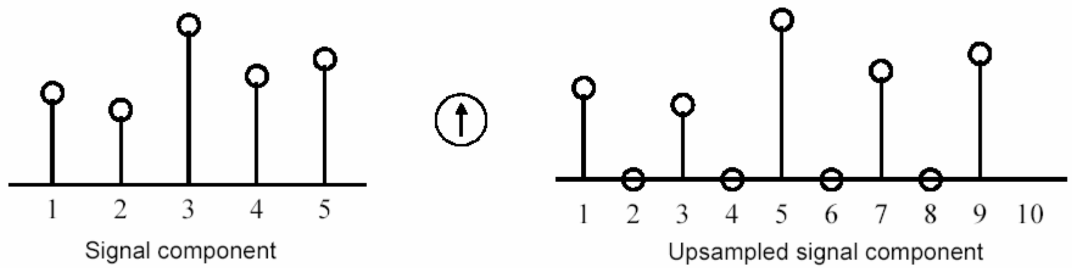


Reconstructed
Signal
Components

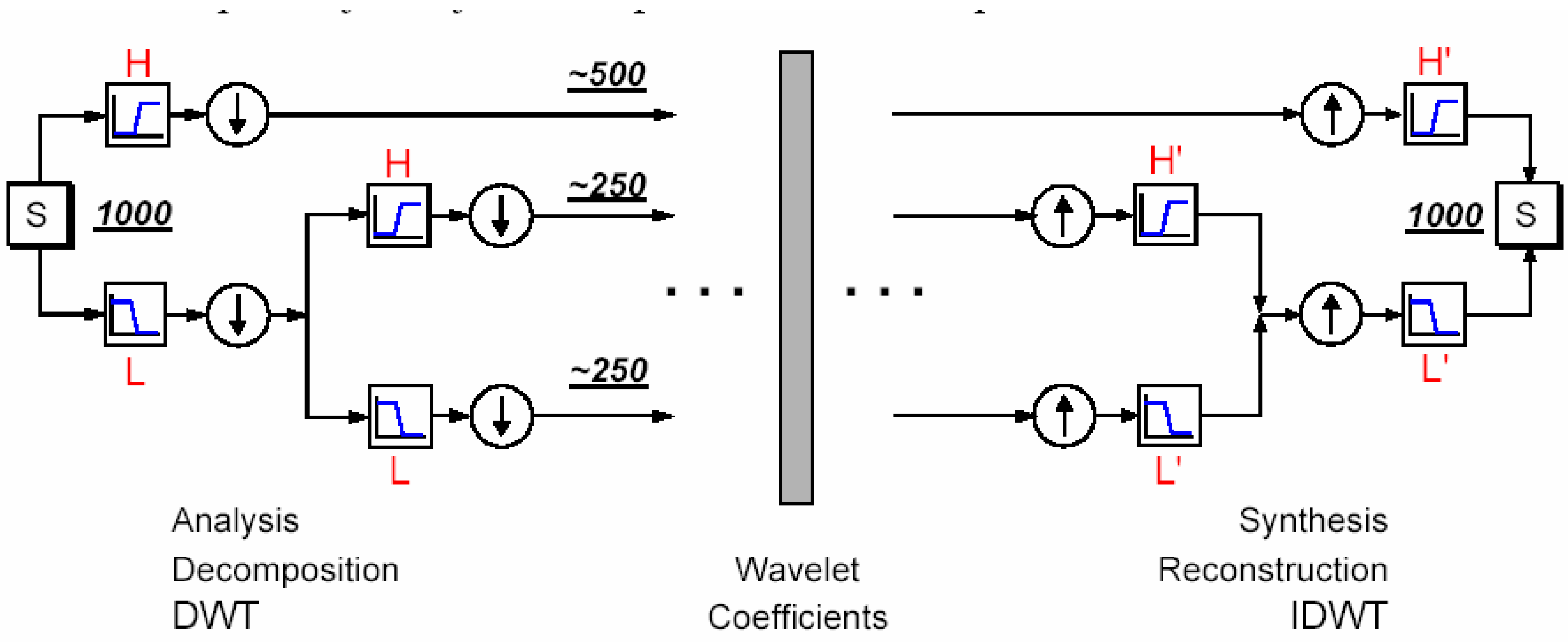


$$S = A_1 + D_1$$
$$= A_2 + D_2 + D_1$$
$$= A_3 + D_3 + D_2 + D_1$$

upsampling

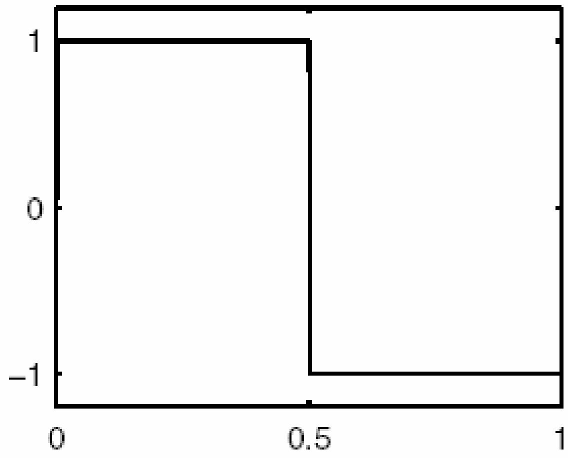


Multi-scale analysis



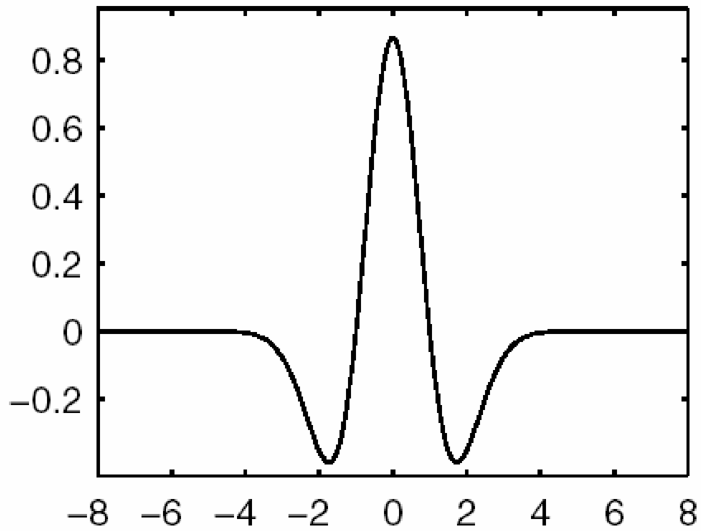
Famous wavelets

Haar



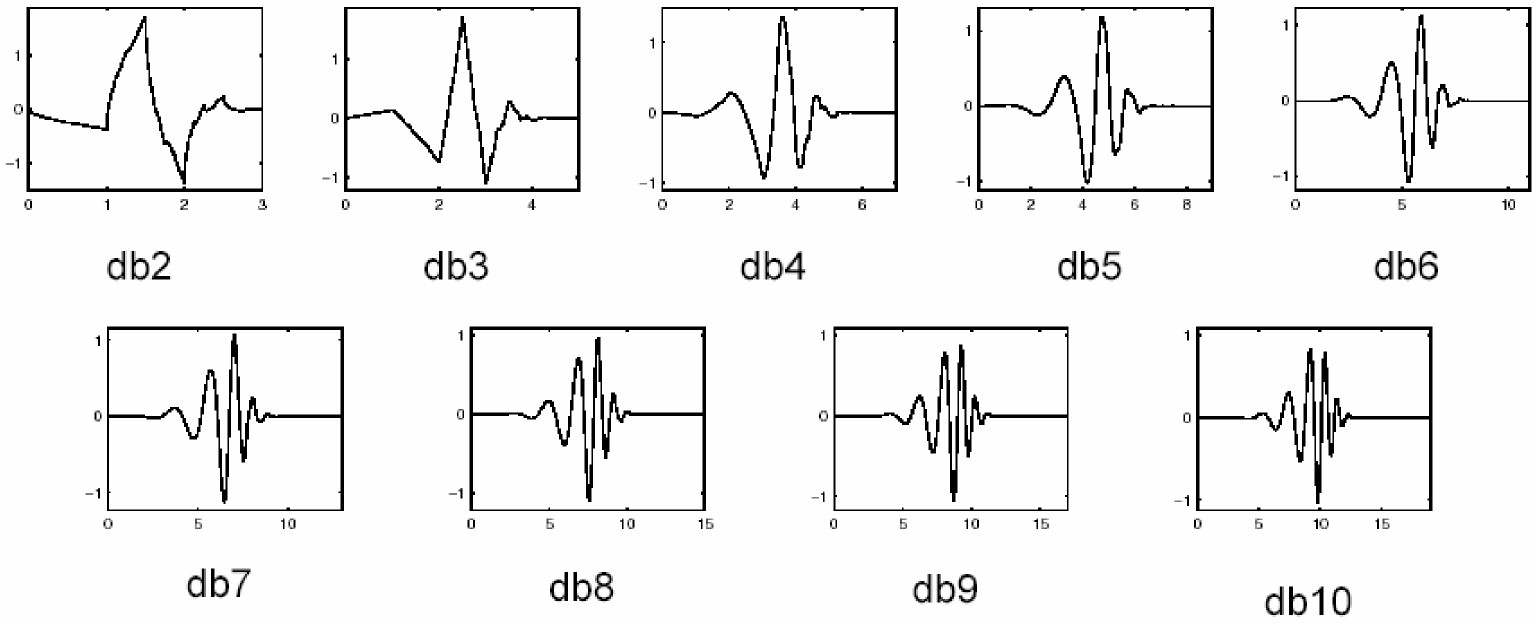
Wavelet function psi

Mexican hat



Wavelet function psi

Daubechie's



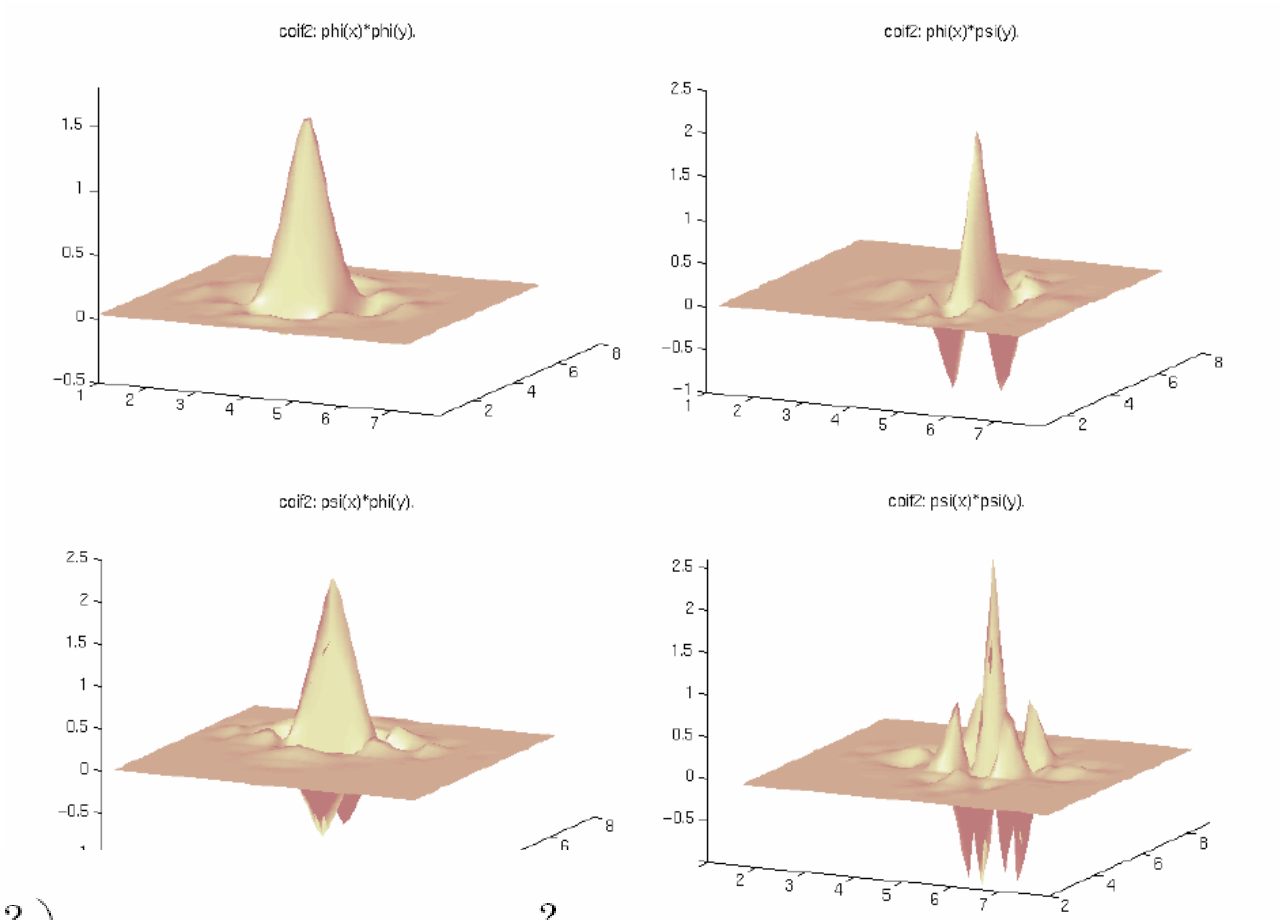
Bi-dimensional wavelets

$$\varphi(x,y) = \varphi(x)\varphi(y)$$

$$\psi^1(x,y) = \varphi(x)\psi(y)$$

$$\psi^2(x,y) = \psi(x)\varphi(y)$$

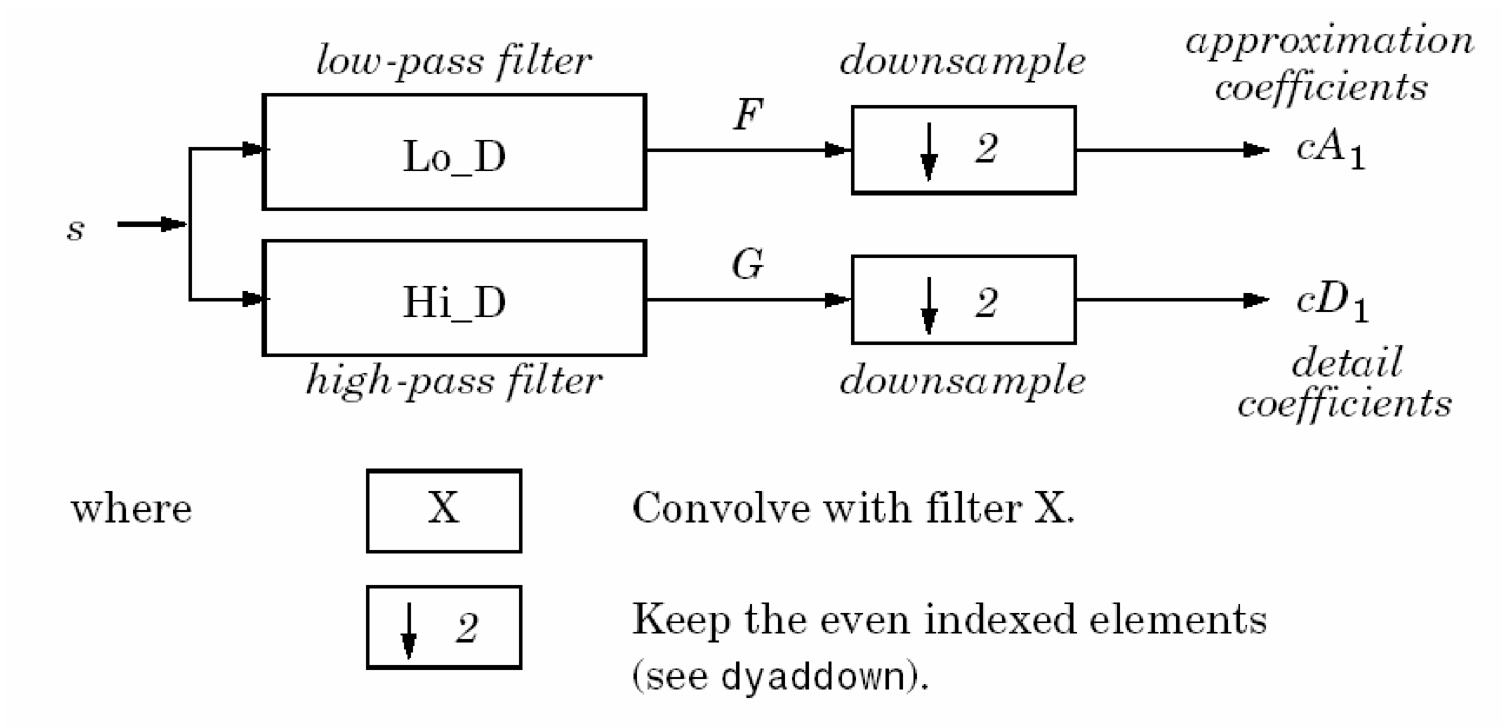
$$\psi^3(x,y) = \psi(x)\psi(y)$$



$$\frac{1}{\sqrt{a_1 a_2}}\psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}\right) \text{ where } (x = (x_1,x_2) \in R^2)$$

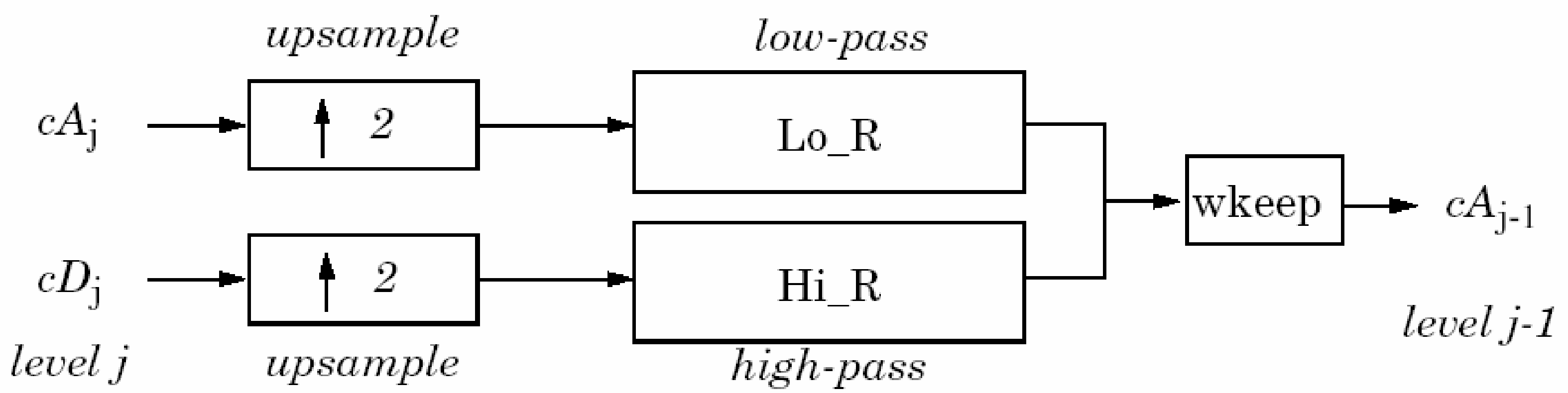
Fast wavelet transform algorithm (DWT)

Decomposition step



Fast wavelet transform algorithm (DWT)

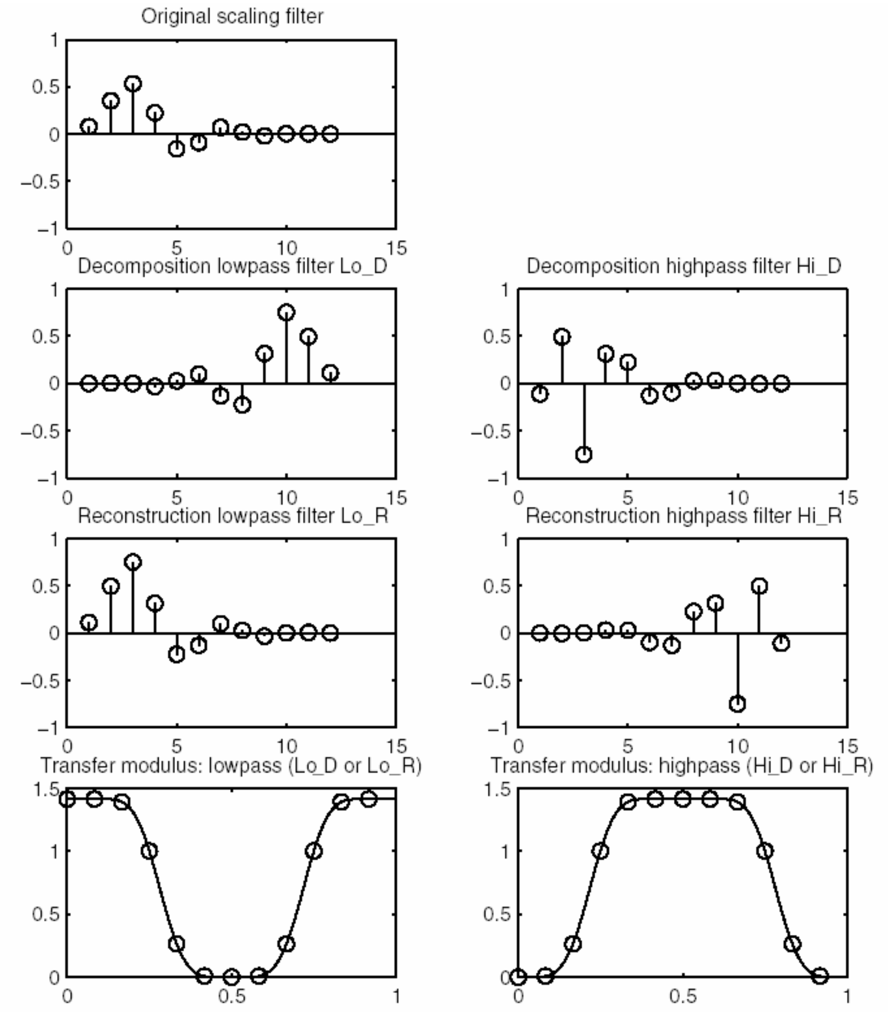
Reconstruction Step



where

$\uparrow 2$	Insert zeros at odd-indexed elements.
X	Convolve with filter X.
wkeep	Take the central part of U with the convenient length.

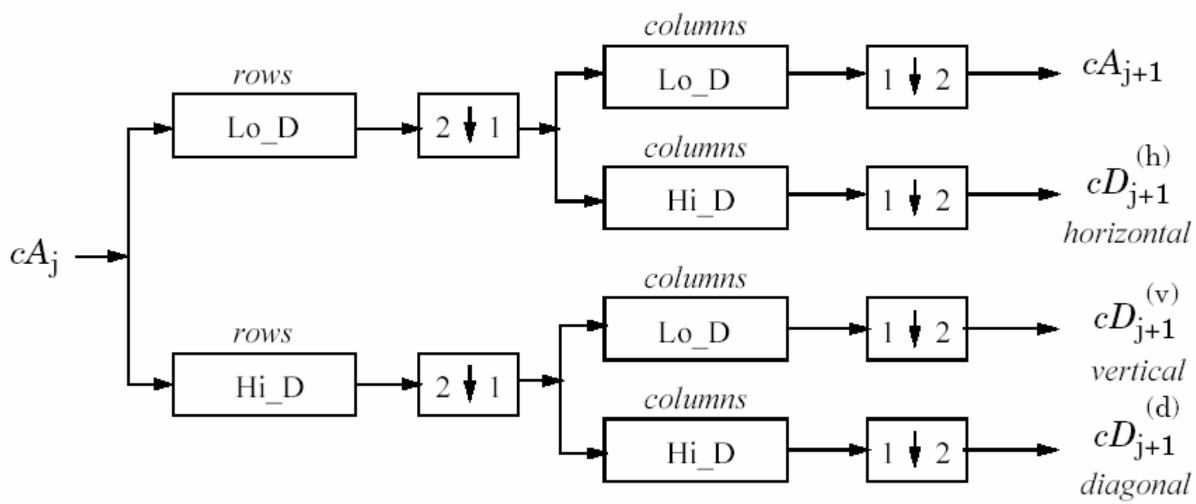
Filters



Original scaling filter
Decomposition lowpass filter Lo_D
Decomposition highpass filter Hi_D
Reconstruction lowpass filter Lo_R
Reconstruction highpass filter Hi_R
Transfer modulus: lowpass (Lo_D or Lo_R)
Transfer modulus: highpass (Hi_D or Hi_R)

Fast DWT for images

Decomposition Step

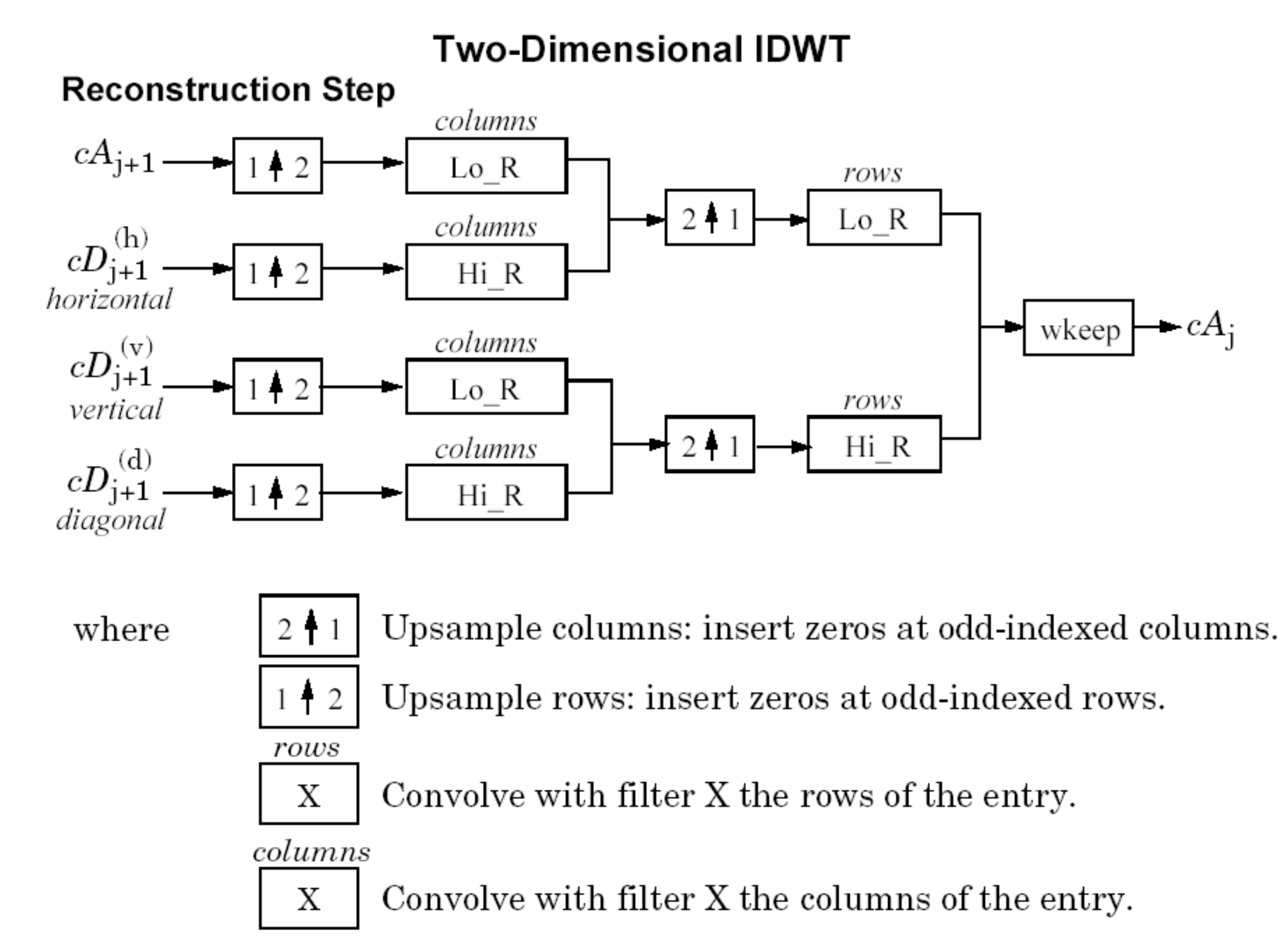


where

- $\boxed{2 \downarrow 1}$ Downsample columns: keep the even indexed columns.
- $\boxed{1 \downarrow 2}$ Downsample rows: keep the even indexed rows.
- rows $\boxed{X}$ Convolve with filter X the rows of the entry.
- columns $\boxed{X}$ Convolve with filter X the columns of the entry.

Initialization $CA_0 = s$ for the decomposition initialization.

Fast DWT for images



Subband structure for images