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MASTER'S DEGREE IN MATHEMATICS

COURSE OF OPTIMIZATION

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Warm up exercises

These exercises are intended as a tool to test the knowledge of some prerequisites of the course. The theory and the tools needed to solve them have been presented in other courses, and so they will not be given during the course of Optimization. In case of difficulties in solving these exercises, please report to the Representative of Students Dr. Li Vigni.

1. Mathematical Analysis 2, Dynamical System, Mathematical Models for Biology

Exercise 1.1. Find the general solution of this system of first-order ordinary differential equation with constant coefficients:

$$\begin{cases} x'(t) &= 4x(t) + 2y(t) + t, \\ y'(t) &= x(t) + 3y(t). \end{cases}$$

Determine the solution satisfying $(x(0), y(0)) = (1, 2)$, and draw a qualitative picture of the phase portrait of this solution in the xy plane for $t > 0$. What can be said of the behaviour of the solution for large t?

Exercise 1.2. Given $\alpha \in \mathbb{R}$, determine for which values of α the following functions admit maximum in $\mathbb{R}$. In the affirmative case, find the value of that maximum as a function of α.

(1) $f_\alpha(x) := \alpha x - |2x - 3|^2$;
(2) $g_\alpha(x) := \alpha x - e^x$;
(3) $h_\alpha(x) := \alpha x + \log |x|$.

2. Functional Analysis

Exercise 2.1. Show that $C^0(\mathbb{R})$ equipped with the norm of $L^2(\mathbb{R})$ is not an Hilbert space.

Exercise 2.2. Show that in a Hilbert space every closed convex set admits an element of minimum norm, and characterize it.

Exercise 2.3. Consider a separable Hilbert space of infinite dimension with countable orthonormal base $\{e_j\}_{j\in\mathbb{N}}$.

(1) Prove that the strong limit of the sequence $\{e_j\}_{j\in\mathbb{N}}$ does not exists;
(2) Find the weak limit of the sequence $\{e_j\}_{j\in\mathbb{N}}$;
(3) Use the previous results to show that in general closed and bounded subsets of H are not strongly compact. Which is the condition granting strong compactness in an infinite-dimensional Banach space?

Exercise 2.4. Consider the set:

$$\overline{B_\infty(0,1)} := \{f \in L^\infty(\mathbb{R}) : \|f\|_{L^\infty} \leq 1\}.$$

Determine if $\overline{B_\infty(0,1)}$ is compact or not in the w^*-topology of L^∞ induced by the duality with L^1. Give a proof or a counterexample.