Differential Geometry and Topology

Exercises by topic

1 Exterior algebra

Exercise 1. Suppose that $T \in \Lambda^p(V^*)$ and $v_1, \dots, v_p \in V$ are linearly dependent. Prove that $T(v_1, \dots, v_p) = 0$ for all $T \in \Lambda^p(V*)$.

Exercise 2. For a $k \times k$ matrix A, let A^t denote the transpose matrix. Using the fact that $\det A$ is multilinear in both rows and columns of A, prove that $\det(A^t) = \det(A)$. [Hint: Use $\dim \Lambda^k(\mathbb{R}^{k^*}) = 1$.]

Exercise 3.

1. Let T be a nonzero element of $\Lambda^k(V^*)$, where $\dim V = k$. Prove that two ordered bases $\{v_1, \dots, v_k\}$ and $\{v_1', \dots, v_k'\}$ for V are equivalently oriented if and only if $T(v_1, \dots, v_k)$ and $T(v_1', \dots, v_k')$ have the same sign.[Hint: determinant theorem.]

2. Suppose that V is oriented. Show that the one-dimensional vector space $\Lambda^k(V^*)$ acquires a natural orientation, by defining the sign of any positively oriented element $T \in \Lambda^k(V^*)$ to be the sign of $T(v_1, \dots, v_k)$ for any positively oriented ordered basis $\{v_1, \dots, v_k\}$ for V.

3. Conversely, show that an orientation of $\Lambda^k(V^*)$ naturally defines an orientation on V by reversing the above.

2 Differential forms

Exercise 4. Suppose that $\phi_1, \dots, \phi_m$ are differential forms on $\mathbb{R}^k$, with deg $\phi_i = p_i$, and $f(y_1, \dots, y_k)$ a 0-form (i.e. smooth function). Thus $f d\phi_1 \wedge d\phi_2 \wedge \dots \wedge d\phi_m$ is a $(p_1 + \dots + p_m + m)$-form. Show that

$$d\left(f d\phi_1 \wedge \dots \wedge d\phi_m\right) = df \wedge d\phi_1 \wedge \dots \wedge d\phi_m$$

Exercise 5. Let $f : X \to Y$ be a smooth map of manifolds, and let ϕ be a smooth function on Y. Then

$$f^*(d\phi) = d(f^*\phi).$$

Exercise 6. Let Z be a finite set of points in X, considered as a 0-manifold. Fix an orientation of Z, an assignment of orientation numbers $\sigma(z) = \pm 1$ to each $z \in Z$. Let f be any function on X, considered as a 0-form, and check that

$$\int_Z f = \sum_{z \in Z} \sigma(z) f(z).$$

Exercise 7. Suppose that the 1-form ω on X is the differential of a function, $\omega = df$. Prove that $\oint_\gamma \omega = 0$ for all closed curves γ on X.

Exercise 8. Define a 1-form ω on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

1. Calculate $\int_C \omega$ for any circle C of radius r around the origin.

2. Prove that in the half-plane $\{x > 0\}$, ω is the differential of a function. [Hint: try $\arctan(y/x)$ as a random possibility.]

3. Why isn't ω the differential of a function globally on $\mathbb{R}^2 \setminus 0$?

Exercise 9. Suppose that ω is a 1-form on the connected manifold X, with the property that $\oint_\gamma \omega = 0$ for all closed curves γ. Then if $p, q \in X$, define $\int_p^q \omega$ to be $\int_0^1 c^*\omega$ for a curve $c : [0, 1] \to X$ with $c(0) = p, c(1) = q$. Show that this is well-defined (i.e. independent of the choice of c.)

3 Stokes' theorem

Exercise 10. The Divergence theorem in electrostatics. Let D be a compact region in $\mathbb{R}^3$ with a smooth boundary S. Assume $0 \in Int(D)$. If an electric charge of magnitude q is placed at 0, the resulting force field is $q\mathbf{r}/r^3$, where $\mathbf{r}(x)$ is the vector to a point x from 0 and $r(x)$ is its magnitude. Show that the amount of charge q can be determined from the force on the boundary by proving Gauss's law:

$$\int_S \mathbf{F} \cdot \mathbf{n} dA = 4\pi q.$$

[Hint: apply the Div. Thm. to a region consisting of D minus a small ball around the origin.]

Exercise 11. Suppose that $X = \partial W$, W is compact, and $f : X \to Y$ is a smooth map. Let ω be a closed k-form on Y, where $k = \dim X$. Prove that if f extends to all of W, then $\int_X f^*\omega = 0$.

Exercise 12. Suppose that $f_0, f_1 : X \to Y$ are homotopic maps and that the compact, boundaryless manifold X has dimension k. Prove that for all closed k-forms ω on Y,

$$\int_X f_0^*\omega = \int_X f_1^*\omega.$$

Exercise 13. Show that if X is a simply connected manifold, then $\oint_\gamma \omega = 0$ for all closed 1-forms ω on X and all closed curves γ in X.

4 Homotopy invariance of de Rham cohomology

Exercise 14. Let M and N be manifolds and suppose that $N \subset M$ and the inclusion map $i : N \to M$ is smooth. A *deformation retract* of M into N is a smooth map $r : M \to N$ such that $r \circ i = Id_N$ and $i \circ r$ is homotopic to Id_M. Prove that M and N have the same de Rham cohomology.

Exercise 15. Show that the de Rham cohomology of the open Möbius strip = the de Rham cohomology of the punctured plane $\mathbb{C} \setminus \{0\}$.

5 Homological algebra and exact sequences

Exercise 16. Show that if $0 \to A \to B \to C \to 0$ is an exact sequence of finite dimensional vector spaces, then $\dim B = \dim A + \dim C$.

Exercise 17. Prove the Five Lemma: given a commutative diagram of abelian groups and group homomorphisms

$$\begin{array}{ccccccccc} A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \xrightarrow{f_3} & D & \xrightarrow{f_4} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A' & \xrightarrow{g_1} & B' & \xrightarrow{g_2} & C' & \xrightarrow{g_3} & D' & \xrightarrow{g_4} & E' \end{array}$$

in which the rows are exact, if the four outer maps α, β, δ and ϵ are isomorphisms then so is γ.

Exercise 18. Suppose that $0 \to V_1 \to V_2 \to \ldots \to V_n \to 0$ is an exact sequence of finite dimensional vector spaces. Show that $\sum_{i=1}^{n} (-1)^{i-1} \dim V_i = 0$.

6 Mayer-Vietoris, Poincaré duality, Künneth formula

Exercise 19. Compute the de Rham cohomology of the sphere S^2 with n holes (or equivalently the plane $\mathbb{R}^2$ with $n-1$ holes).

Exercise 20. Compute the de Rham cohomology of a compact oriented surface of genus 2 (also known as the two-holed donut). E.g. break it up into a sphere with 4 holes with two cylinders attached to close up the holes.

Exercise 21. Let M be a compact, orientable n manifold. Show that n odd $\implies \chi(M) = 0$. So, for example, all odd-dimensional spheres $S^k = \{\mathbf{x} \in \mathbb{R}^{k+1} | \|\mathbf{x}\| = 1\}$ have Euler characteristic equal to zero.

Exercise 22. Show that the n-torus $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$ has $B^k(T^n) = \binom{n}{k}$.

Exercise 23. Use the Kunneth formula to calculate $H^*_{dR}(S^2 \times S^2)$.

7 Compactly supported cohomology and Poincaré duality

Exercise 24. Compute the de Rham cohomology of the punctured torus $\Sigma = T^2 \setminus \{x\}$ by the following steps.

1. Find $H^0_{dR}(\Sigma)$.
2. Find $H^0_c(\Sigma)$.
3. By Poincaré duality this gives $H^2_{dR}(\Sigma)$.
4. Let D be an open disk containing the point x, so $T^2 = \Sigma \cup D$. Given that $H^0_{dR}(T^2) \cong \mathbb{R} \cong H^2_{dR}(T^2)$, $H^1_{dR}(T^2) \cong \mathbb{R}^2$, use the Mayer-Vietoris sequence to compute $H^1_{dR}(\Sigma)$.

8 Hodge theory

Exercise 25. Let H be the hyperbolic plane, i.e. the upper half plane $\{(x,y) \in \mathbb{R}^2 | y > 0\}$, equipped with the hyperbolic metric $g(x,y) = (dx \otimes dx + dy \otimes dy)/y^2$. Let $\Omega(x,y) = dx \wedge dy$ be the standard volume form. Find explicit expressions for

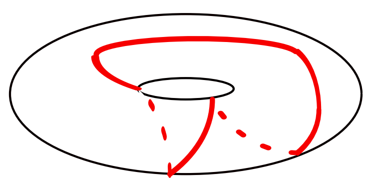
1. the Hodge star operators $*$,
2. the codifferential operators δ, and
3. the Laplace-Beltrami operators $\triangle$

with respect to g and Ω.

9 Poincarè duals, Intersection numbers, Euler characteristic

Exercise 26. Let T^2 be the torus, and M_1 and M_2 its two meridians. Compute $I(M_1, M_1)$ and $I(M_1, M_2)$. Conclude that M_1 and M_2 are not isotopic to each other.

Exercise 27. Let S be the curve in T^2 depicted in red below. Choose an orientation for S and compute $I(S,S)$. Let M_1 and M_2 be the two meridians of T^2. Compute $I(S, M_1)$ and $I(S, M_2)$ with respect to any orientation of M_1 and M^2. Conclude that S is not isotopic to either M_1 or M_2.



Exercise 28. Let S_1 and S_2 be two compact oriented submanifolds of $\mathbb{R}^n$ of complementary dimension, with Poincaré duals η_{S_1} and η_{S_2}. Show that $\int_M \eta_{S_1} \wedge \eta_{S_2} = 0$.

Exercise 29. Prove that the Euler characteristic of the product of two compact, oriented manifolds is the product of their Euler characteristics.

Exercise 30. Let $\triangle \subset S^2 \times S^2$ be the diagonal, which itself is isomorphic to a sphere. Show that there is no isotopy $\Phi : S^2 \times S^2 \to S^2 \times S^2$ such that $\Phi(\triangle) \cap \triangle = \emptyset$.

Exercise 31. More generally, for $k > 0$ even, let S^k be the k-dimensional sphere. Show that there is no isotopy $\Phi : S^k \times S^k$ such that $\Phi(\triangle) \cap \triangle = \emptyset$.