8.7 Exercises - Part 5

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Exercise 17. Let K be a field and Q be the Kronecker quiver $1 \rightrightarrows 2$.

(a) Let $\lambda \in K \setminus \{0\}$ and let M_λ be the representation $K \overset{\lambda}{\underset{1}{\rightrightarrows}} K$. Show that there is a short exact sequence $\varepsilon_\lambda \colon 0 \to S(2) \to M_\lambda \to S(1) \to 0$.

(b) Let $\lambda, \mu \in K \setminus \{0\}$. Show that ε_λ and ε_μ are equivalent if and only if $\lambda = \mu$.

Exercise 18. Let A, B and B' be R-modules. Let $\beta \in \operatorname{Hom}_R(B, B')$. Show that the map $\operatorname{Ext}^1_R(A, \beta) \colon \operatorname{Ext}^1_R(A, B) \to \operatorname{Ext}^1_R(A, B')$, where the assignment $[\varepsilon] \mapsto [\beta\varepsilon]$ is given by sending the short exact sequence $\varepsilon \colon 0 \to B \to E \to A \to 0$ to the short exact sequence $\beta\varepsilon$ via the pushout,

$$\begin{tikzcd}
\varepsilon\colon 0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\beta\varepsilon\colon 0 \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is well defined.

Exercise 19. (a) Show that an R-module P is projective if and only if $\operatorname{Ext}^n_R(P, B) = 0$ for all R-modules B and for all $n > 0$.

(b) Let $P_\bullet \colon \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \longrightarrow 0$ be a projective resolution of an R-module A and $K_n = \ker p_n$ for each $n \geq 0$. Show that $\operatorname{Ext}^1_R(K_n, B) \cong \operatorname{Ext}^{n+2}_R(A, B)$ for all $n \geq 0$.

(c) Given $A \in R$ Mod, show that if $\operatorname{Ext}^{n+1}_R(A, B) = 0$ for all R-modules B, then there is a projective resolution of A of the form $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A$.

(d) Given $B \in R$ Mod, show that if $\operatorname{Ext}^{n+1}_R(A, B) = 0$ for all R-modules A, then there is an injective (co)resolution of B of the form $0 \to B \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$.

(e) Conclude that $\sup\{\operatorname{proj\,dim} A \mid A \in R \text{ Mod}\} = \sup\{\operatorname{inj\,dim} B \mid B \in R \text{ Mod}\}$.

Exercise 20. Let K be a field and Q the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"]
\end{tikzcd}.$$

(a) Let $\Lambda_1 = KQ/\mathcal{I}_1$, where $\mathcal{I}_1 = (\alpha\gamma)$.

  (i) Determine all indecomposable projective representations of Λ_1.

  (ii) Compute the global dimension of Λ_1.

(b) Let $\Lambda_2 = KQ/\mathcal{I}_2$, where $\mathcal{I}_2 = (\alpha\gamma, \gamma\beta)$. Compute the global dimension of Λ_2.

(c) Let $\Lambda_3 = KQ/\mathcal{I}_3$, where $\mathcal{I}_3 = (\alpha\gamma, \gamma\beta, \beta\alpha)$. Compute the global dimension of Λ_3.