

8.7 Exercises - Part 5

(Published on December 13, **solutions to be submitted on January 10, 2017.**)

Exercise 17. Let K be a field and Q be the Kronecker quiver $1 \xrightarrow{\quad} 2$.

(a) Let $\lambda \in K \setminus \{0\}$ and let M_λ be the representation $K \xrightarrow[\quad 1]{\lambda} K$. Show that there is a short exact sequence $\varepsilon_\lambda: 0 \rightarrow S(2) \rightarrow M_\lambda \rightarrow S(1) \rightarrow 0$.

(b) Let $\lambda, \mu \in K \setminus \{0\}$. Show that ε_λ and ε_μ are equivalent if and only if $\lambda = \mu$.

Exercise 18. Let A, B and B' be R -modules. Let $\beta \in \text{Hom}_R(B, B')$. Show that the map $\text{Ext}_R^1(A, \beta): \text{Ext}_R^1(A, B) \rightarrow \text{Ext}_R^1(A, B')$, where the assignment $[\varepsilon] \mapsto [\beta\varepsilon]$ is given by sending the short exact sequence $\varepsilon: 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ to the short exact sequence $\beta\varepsilon$ via the pushout,

$$\begin{array}{ccccccc} \varepsilon: 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\ & & \beta \downarrow & & \downarrow & & \parallel \\ \beta\varepsilon: 0 & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & A \longrightarrow 0 \end{array}$$

is well defined.

Exercise 19. (a) Show that an R -module P is projective if and only if $\text{Ext}_R^n(P, B) = 0$ for all R -modules B and for all $n > 0$.

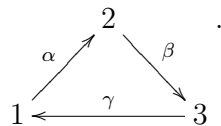
(b) Let $P_\bullet: \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \rightarrow 0$ be a projective resolution of an R -module A and $K_n = \ker p_n$ for each $n \geq 0$. Show that $\text{Ext}_R^1(K_n, B) \cong \text{Ext}_R^{n+2}(A, B)$ for all $n \geq 0$.

(c) Given $A \in R \text{ Mod}$, show that if $\text{Ext}_R^{n+1}(A, B) = 0$ for all R -modules B , then there is a projective resolution of A of the form $0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A$.

(d) Given $B \in R \text{ Mod}$, show that if $\text{Ext}_R^{n+1}(A, B) = 0$ for all R -modules A , then there is an injective (co)resolution of B of the form $0 \rightarrow B \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{n-1} \rightarrow E^n \rightarrow 0$.

(e) Conclude that $\sup\{\text{proj dim } A \mid A \in R \text{ Mod}\} = \sup\{\text{inj dim } B \mid B \in R \text{ Mod}\}$.

Exercise 20. Let K be a field and Q the quiver



(a) Let $\Lambda_1 = KQ/\mathcal{I}_1$, where $\mathcal{I}_1 = (\alpha\gamma)$.

- (i) Determine all indecomposable projective representations of Λ_1 .
- (ii) Compute the global dimension of Λ_1 .

(b) Let $\Lambda_2 = KQ/\mathcal{I}_2$, where $\mathcal{I}_2 = (\alpha\gamma, \gamma\beta)$. Compute the global dimension of Λ_2 .

(c) Let $\Lambda_3 = KQ/\mathcal{I}_3$, where $\mathcal{I}_3 = (\alpha\gamma, \gamma\beta, \beta\alpha)$. Compute the global dimension of Λ_3 .