Università degli Studi di Verona

Dipartimento di Informatica

Edificio Ca' Vignal 2
Strada Le Grazie, 15
37134 Verona - Italia
Tel. +39 045 802 7069
Fax. +39 045 802 7068

MASTER'S DEGREE IN MATHEMATICS

COURSE OF OPTIMIZATION

Antonio Marigonda

Team exercises

Exercise 1. Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex. Define

$$(f_1 \oplus f_2)(x) := \inf\{f_1(x_1) + f_2(x_2) : x_1 + x_2 = x\},$$

and suppose that $(f_1 \oplus f_2)(x) > -\infty$ for all $x \in \mathbb{R}^n$. Prove that $f_1 \oplus f_2$ is convex. Calculate $\partial(f_1 \oplus f_2)(x)$ for all $x \in \operatorname{dom}(f_1 \oplus f_2)$, and compute $(f_1 \oplus f_2)^*$.

Exercise 2. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and assume that

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\},$$

for all $x, y \in \mathbb{R}^n$, $\lambda \in]0,1[$. Prove that for every $\alpha \in \mathbb{R}$ the set $\{x : f(x) \le \alpha\}$ is either empty or convex. Can we conclude that f is convex? Give a proof or a counterexample.

Exercise 3. Let Ω_1, Ω_2 be convex, closed, and nonempty subsets of a Banach space X. Suppose that $\operatorname{int}\Omega_1 \neq \emptyset$ and $(\operatorname{int}\Omega_1) \cap \Omega_2 = \emptyset$. Then prove that there exist $v_1, v_2 \in X'$ such that at least one between v_1 and v_2 is nonzero, and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

(1) $\langle v_i, x\rangle \le \alpha_i$ for all $x \in \Omega_i$, $i = 1, 2$,
(2) $v_1 + v_2 = 0$, $\alpha_1 + \alpha_2 = 0$.

Exercise 4. Show that C is convex if and only if for $\alpha, \beta \ge 0$ we have $(\alpha+\beta)C = \alpha C + \beta C$, where the sum of sets is Minkowski sum: $A + B = \{a + b : a \in A, b \in B\}$.

Exercise 5. Let $h : [0,1] \to [0, +\infty[$ be a concave function. Define a map $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f(\rho, v) = \begin{cases} \dfrac{|v|^2}{h(\rho)}, & \text{if } \rho \in [0,1] \text{ and } h(\rho) \neq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Compute f^*, f^{**}, ∂f^{**}, ∂f^*, precising their domain.

Exercise 6. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a symmetric, positive definite $n \times n$ matrix with real entries, $a, b \in \mathbb{R}^n \setminus \{0\}$, $a \neq b$, $\alpha, \beta \in \mathbb{R}$. We consider the problem of minimizing the map $f(x) := \langle x, Ax\rangle$ on $C_1 \cap C_2$ where $C_1 := \{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\}$ and $C_2 := \{x \in \mathbb{R}^n : \langle b, x\rangle = \beta\}$. Prove that the problem admits a unique solution $\bar{x}$ and that there exists $\bar{\delta} \in \mathbb{R}$ such that $\bar{x}$ minimizes $f(x) + \bar{\delta}\langle a, x\rangle$ on C_2.

Teams
Team 1: VR394480, VR389636, VR386856
Team 2: VR388559, VR388232, VR388607, VR388555
Team 3: VR388560, VR391980, VR390313, VR393747, F.V.*
Team 4: VR388432, VR394076, VR394042
Team 5: VR387938, VR393491, VR387762
Team 6: VR389710, VR392130, VR390987
*missing ID number, please communicate it as soon as possible.

Remarks
Each group must work **strictly independently** on the others.
Correctness, quality, rigor, and cleariness of the answers are **more important** than quantity.
Please return neatly written solutions before **20th November 2015**.