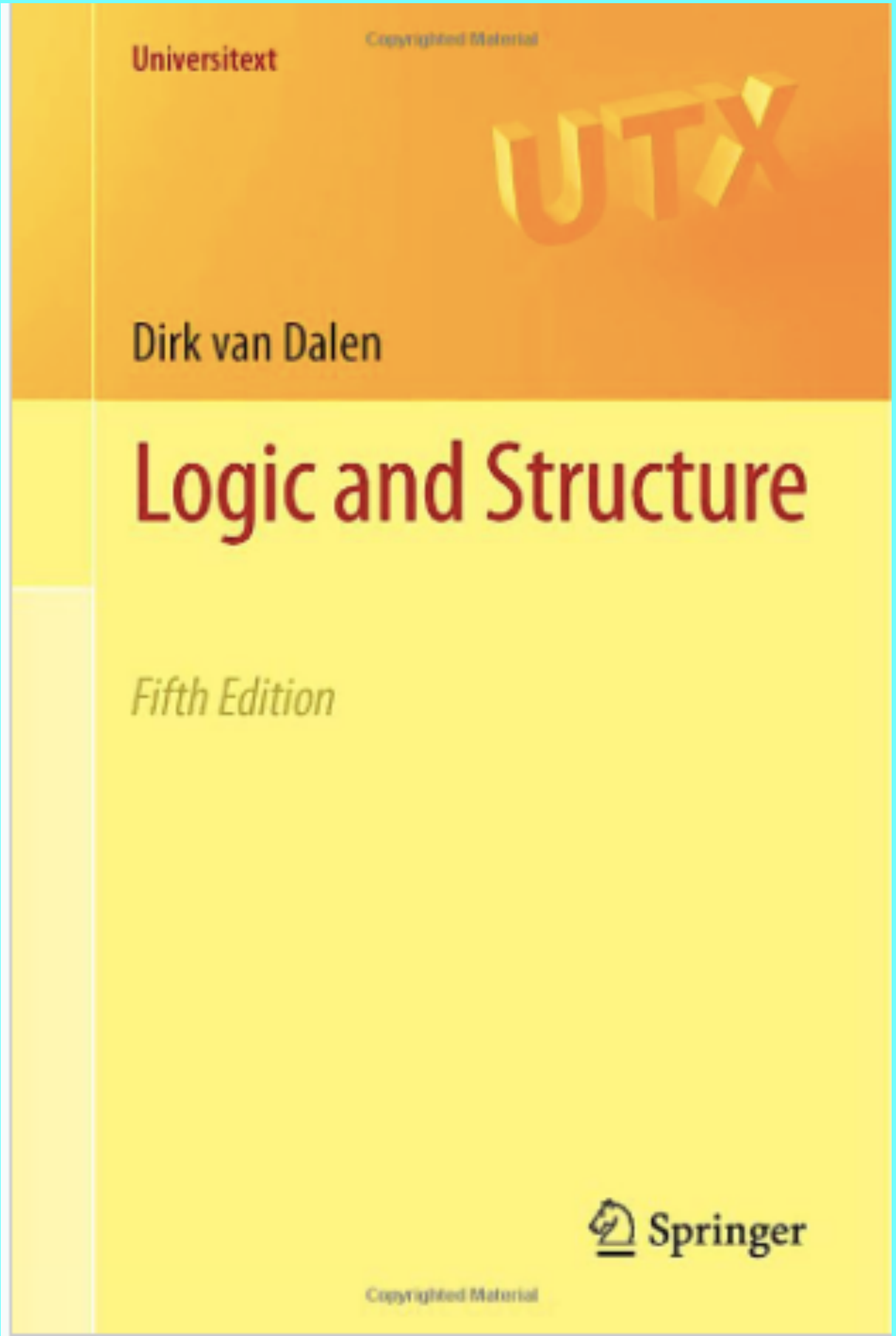
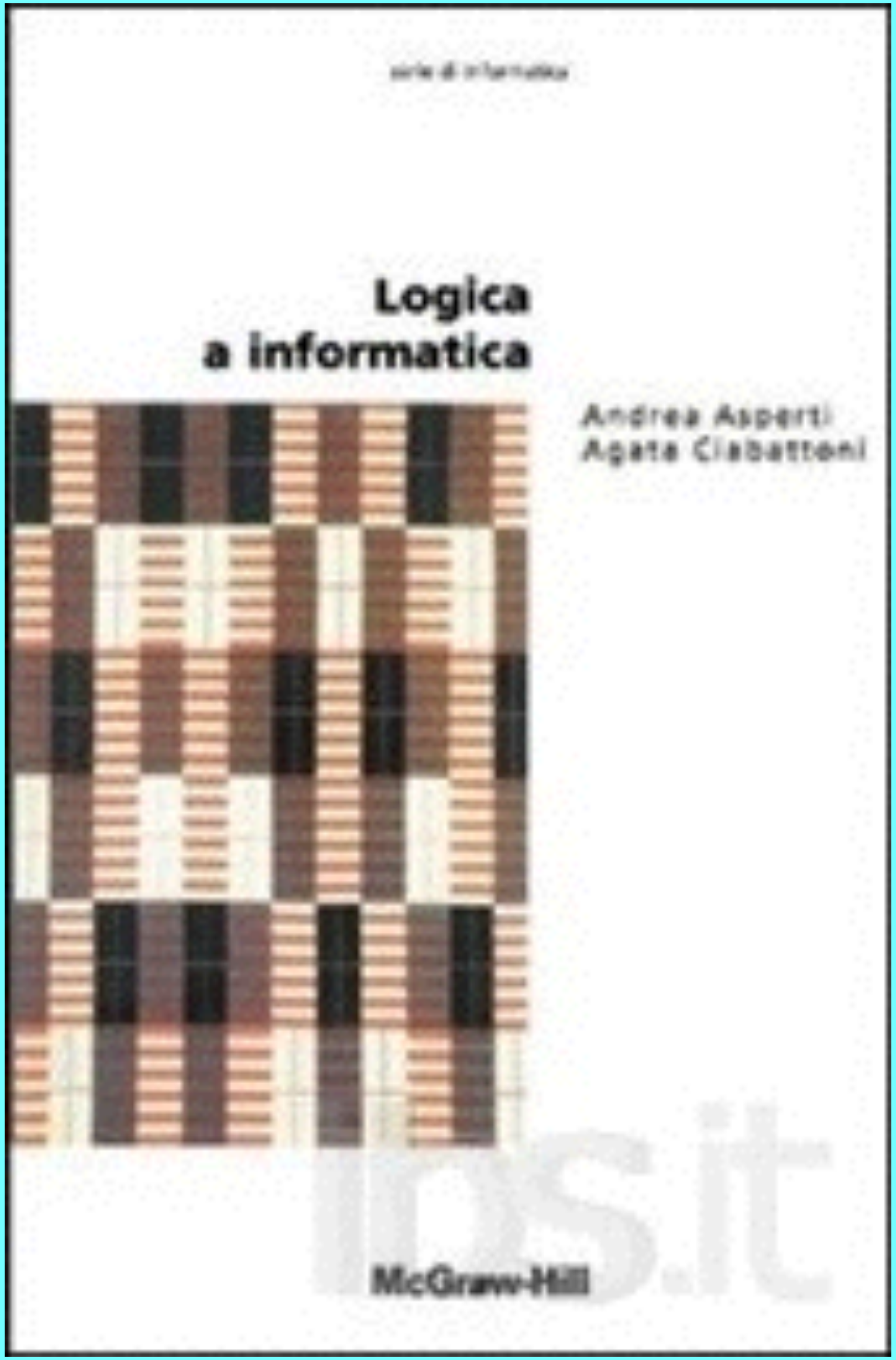
Propositional Logic

Libro di Testo



Lettura aggiuntiva



language of propositional logic

alphabet:

(i) proposition symbols : $p_0, p_1, p_2, \ldots,$

(ii) connectives : $\wedge, \vee, \rightarrow, \neg, \leftrightarrow, \perp,$

(iii) auxiliary symbols : (,).

$AT = \{p_0, p_1, p_2, \ldots,\} \cup \{\perp\}$

$\wedge$	**and**
$\vee$	**or**
$\rightarrow$	**if ..., then ...**
$\neg$	**not**
$\leftrightarrow$	**iff**
$\perp$	**falsity**

The set PROP of propositions is the smallest set X with the properties

(i) $p_i \in X (i \in N), \perp \in X,$

(ii) $\phi, \psi \in X \Rightarrow (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi) \in X,$

(iii) $\phi \in X \Rightarrow (\neg \phi) \in X.$

PROP is well defined? (PROP $\neq \varnothing$?)

$\neg \rightarrow \perp \notin$ PROP

The set PROP of propositions is the smallest set X with the properties

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(ii) $\phi, \psi \in X \Rightarrow (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi),$

$(\phi \leftrightarrow \psi) \in X$,

(iii) $\phi \in X \Rightarrow (\neg \phi) \in X$.

Suppose $\neg \rightarrow \perp \in$ PROP.

Y = PROP – $\{\neg \rightarrow \perp\}$ also satisfies (i), (ii) and (iii).

- $\perp, p_i \in$ **Y**.
- $\phi, \psi \in$ **Y** $\Rightarrow \phi, \psi \in$ PROP $\Rightarrow (\phi \circ \psi) \in$ PROP.

  $(\phi \circ \psi) \neq \neg \rightarrow \perp \Rightarrow (\phi \circ \psi) \in$ **Y** .
- $\phi \in$ **Y** $\Rightarrow \phi \in$ PROP $\Rightarrow (\neg \phi) \in$ PROP.

  $(\neg \phi) \neq \neg \rightarrow \perp \Rightarrow (\neg \phi) \in$ **Y** .
- PROP is not the smallest set satisfying (i), (ii) and (iii)!!! **impossible**

Theorem

Let h: $\mathbb{N} \times A \to A$ and $c \in A$.

There exist one and only one function $f : \mathbb{N} \to A$ t.c.:

1. $f(0)=c$
2. $\forall n \in \mathbb{N}$, $f(n+1)=h(n,f(n))$

the proof is difficult

$\square \in \{\wedge, \vee, \to\}$

Theorem 1.1.6 (Definition by Recursion) *Let mappings $H_\square : A^2 \to A$ and $H_\neg : A \to A$ be given and let H_{at} be a mapping from the set of atoms into A, then there exists exactly one mapping $F : PROP \to A$ such that*

$$\begin{cases} F(\varphi) & = H_{at}(\varphi) \text{ for } \varphi \text{ atomic,} \\ F((\varphi \square \psi)) & = H_\square(F(\varphi), F(\psi)), \\ F((\neg\varphi)) & = H_\neg(F(\varphi)). \end{cases}$$

exercise

Theorem 1.1.3 (Induction Principle)
Let A be a property, then $A(\phi)$ holds for all $\phi \in PROP$ if

(i) $A(p_i)$, for all i,and $A(\bot)$,

(ii) $A(\phi), A(\psi) \Rightarrow A((\phi \to \psi))$,

(iii) $A(\phi), A(\psi) \Rightarrow A((\phi \wedge \psi))$,

(iv) $A(\phi), A(\psi) \Rightarrow A((\phi \vee \psi))$,

(v) $A(\phi) \Rightarrow A((\neg \phi))$.

$$T(\varphi) \quad = \quad \bullet\,\varphi \qquad \text{for atomic } \varphi$$

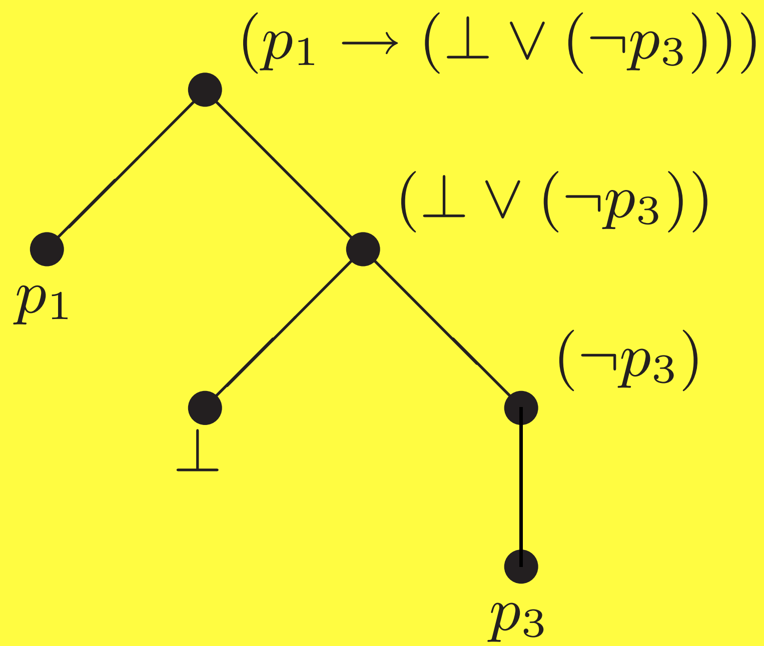
$$T((\varphi\Box\psi)) \quad = \quad \begin{tikzcd}[column sep=small] & \bullet\,(\varphi\Box\psi) \arrow[dl, dash] \arrow[dr, dash] & \\ T(\varphi) & & T(\psi) \end{tikzcd}$$

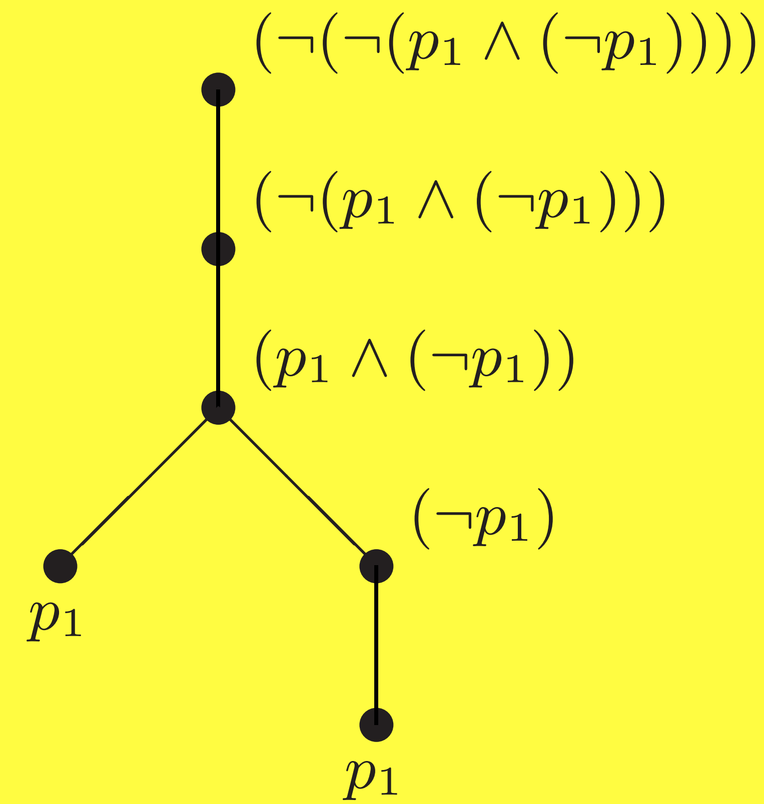
$$T((\neg\varphi)) \quad = \quad \begin{tikzcd} \bullet\,(\neg\varphi) \arrow[d, dash] \\ T(\varphi) \end{tikzcd}$$

Examples. $T\big((p_1 \to (\bot \vee (\neg p_3)))\big)$; $T\big(\neg(\neg(p_1 \wedge (\neg p_1)))\big)$

?

Examples. $T\big((p_1 \to (\bot \vee (\neg p_3)))\big)$; $T\big(\neg(\neg(p_1 \wedge (\neg p_1)))\big)$





SEMANTICS

truth table

$\wedge$	0	1
0	0	0
1	0	1

Definition 1

A mapping $v : PROP \to \{0, 1\}$ is a valuation if

$v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$,

$v(\phi \vee \psi) = \max(v(\phi), v(\psi))$,

$v(\phi \to \psi) = 0 \Leftrightarrow v(\phi) = 1$ and $v(\psi) = 0$,

$v(\phi \leftrightarrow \psi) = 1 \Leftrightarrow v(\phi) = v(\psi)$,

$v(\neg\phi) = 1 - v(\phi)$

$v(\perp) = 0$.

the two definitions are equivalent

Definition 2

A mapping $v : PROP \to \{0, 1\}$ is a valuation if

$v(\phi \wedge \psi) = 1 \Leftrightarrow v(\phi) = 1$ and $v(\psi) = 1$

$v(\phi \vee \psi) = 1 \Leftrightarrow v(\phi) = 1$ or $v(\psi) = 1$

$v(\phi \to \psi) = 1 \Leftrightarrow v(\phi) = 0$ or $v(\psi) = 1$,

$v(\phi \leftrightarrow \psi) = 1 \Leftrightarrow v(\phi) = v(\psi)$,

$v(\neg\phi) = 1 \Leftrightarrow v(\phi) = 0$

$v(\perp) = 0$.

Theorem

v: **AT**$\rightarrow$ {0, 1} s.t. v($\perp$) = 0 (assignment for atoms)

$\Rightarrow$

there exists a unique valuation **$[\cdot]_v$:PROP$\rightarrow$\{0,1\}**

such that $[\phi]_v = v(\phi)$ for each $\phi \in$**AT**

Lemma If v, w are two assignments for atoms s.t. for all p_i occurring in ϕ, $v(p_i) = w(p_i)$, then $[\phi]_v = [\phi]_w$.

Definition

- ϕ is a **tautology** if $[\phi]_v = 1$ for all valuations v,
- $\models \phi$ stands for 'ϕ is a tautology',
- let Γ be a set of propositions,

  $\Gamma \models \phi$ iff for all v: $([\psi]_v = 1 \text{ for all } \psi \in \Gamma) \Rightarrow [\phi]_v = 1$.

SUBSTITUTION

$$\varphi[\psi/p] = \begin{cases} \psi \text{ if } \varphi = p \\ \varphi \text{ if } \varphi =/= p \text{ if } \varphi \text{ atomic} \end{cases}$$

$(\phi_1 \Box \phi_2)[\psi/p] = (\phi_1[\psi/p] \Box \phi_2[\psi/p])$

$(\neg\phi)[\psi/p] = (\neg\phi[\psi/p])$

Substitution Theorem

- If $\models \phi_1 \leftrightarrow \phi_2$, then $\models \psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]$, where p is an atom.
- $[\phi_1 \leftrightarrow \phi_2]_v \leq [\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]]_v$
- $\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p])$

tautologies

- $(\phi \vee \psi) \vee \sigma \leftrightarrow \phi \vee (\psi \vee \sigma)$ $\quad$ $(\phi \wedge \psi) \wedge \sigma \leftrightarrow \phi \wedge (\psi \wedge \sigma)$
associativity
- $\phi \vee \psi \leftrightarrow \psi \vee \phi$ $\quad$ $\phi \wedge \psi \leftrightarrow \psi \wedge \phi$
commutativity
- $\phi \vee (\psi \wedge \sigma) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \sigma)$ $\quad$ $\phi \wedge (\psi \vee \sigma) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \sigma)$
distributivity
- $\neg(\phi \vee \psi) \leftrightarrow \neg\phi \wedge \neg\psi$ $\quad$ $\neg(\phi \wedge \psi) \leftrightarrow \neg\phi \vee \neg\psi$
De Morgan's laws
- $\phi \vee \phi \leftrightarrow \phi$ $\quad$ $\phi \wedge \phi \leftrightarrow \phi$
idempotency
- $\neg\neg\phi \leftrightarrow \phi$
double negation law

De Morgan's law: $[\neg(\phi \vee \psi)]=1 \Leftrightarrow [\phi \vee \psi]=0 \Leftrightarrow [\phi]=[\psi]=0 \Leftrightarrow [\neg\phi]=[\neg\psi]=1 \Leftrightarrow [\neg\phi \wedge \neg\psi] = 1$.

So $[\neg(\phi \vee \psi)] = [\neg\phi \wedge \neg\psi]$ for all valuations, i.e $\vDash \neg(\phi \vee \psi) \leftrightarrow \neg\phi \wedge \neg\psi$.

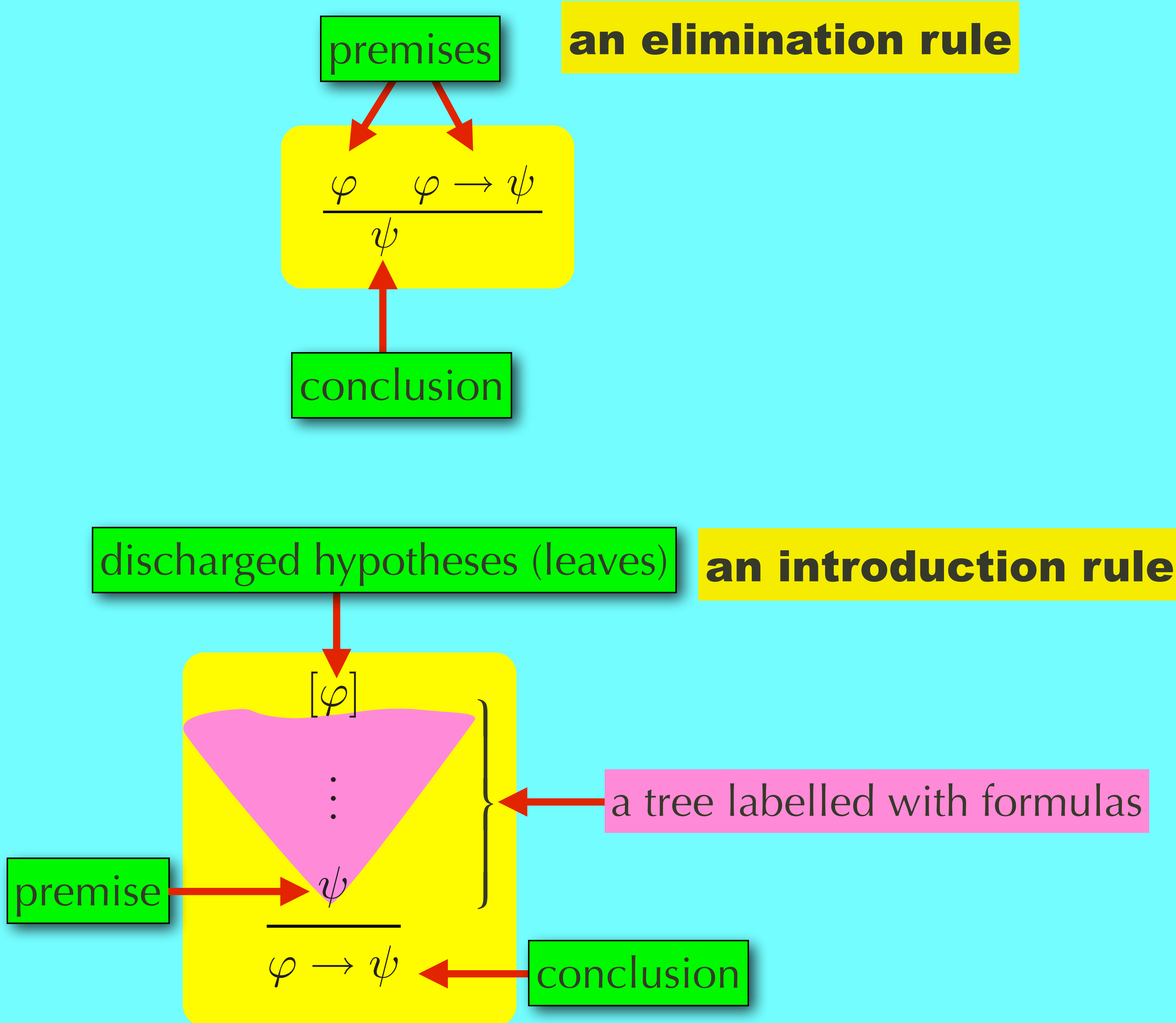
$\models (\varphi \leftrightarrow \psi) \leftrightarrow (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
$\models (\varphi \rightarrow \psi) \leftrightarrow (\neg\varphi \vee \psi)$
$\models \varphi \vee \psi \leftrightarrow (\neg\varphi \rightarrow \psi)$
$\models \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$
$\models \varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
$\models \neg\varphi \leftrightarrow (\varphi \rightarrow \bot)$,
$\models \bot \leftrightarrow \varphi \wedge \neg\varphi$.

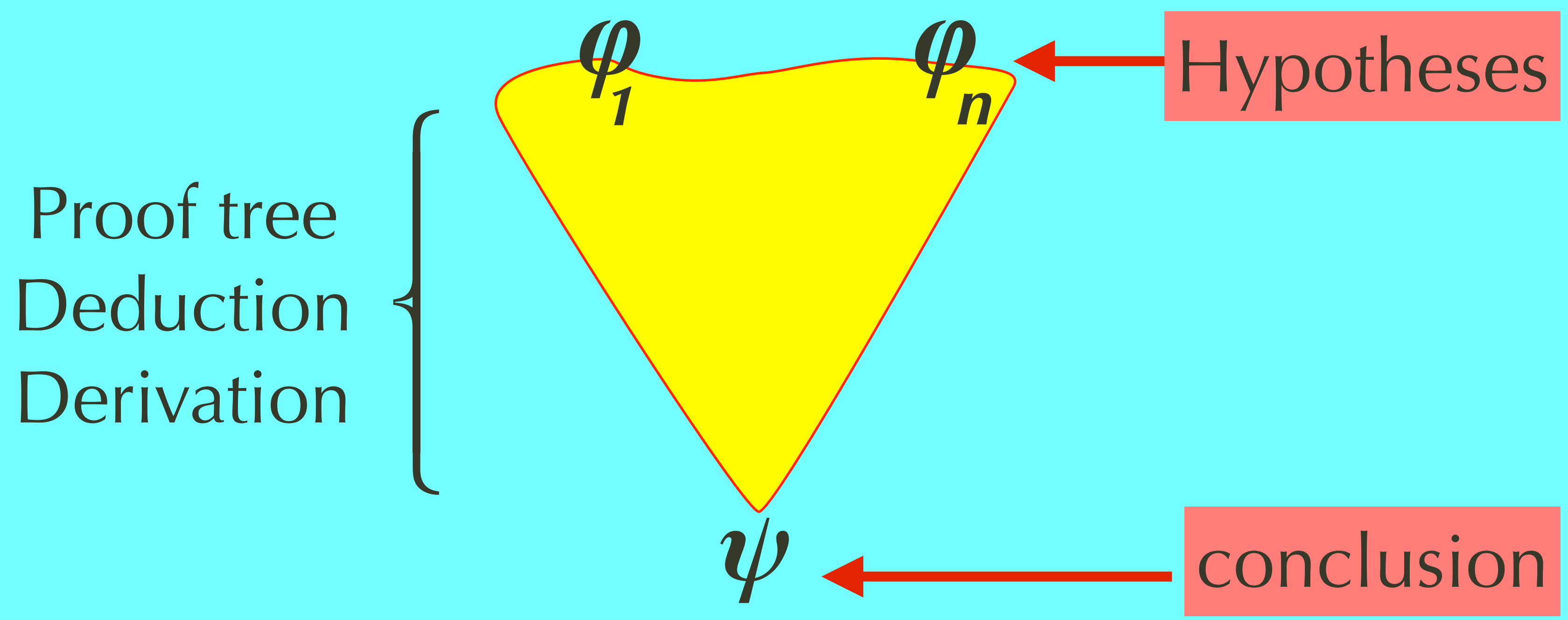
$\approx \subseteq \mathrm{PROP} \times \mathrm{PROP}$: $\phi \approx \psi$ iff $\models \phi \leftrightarrow \psi$.
exercise $\approx$ is an equivalence relation on PROP

Natural Deduction

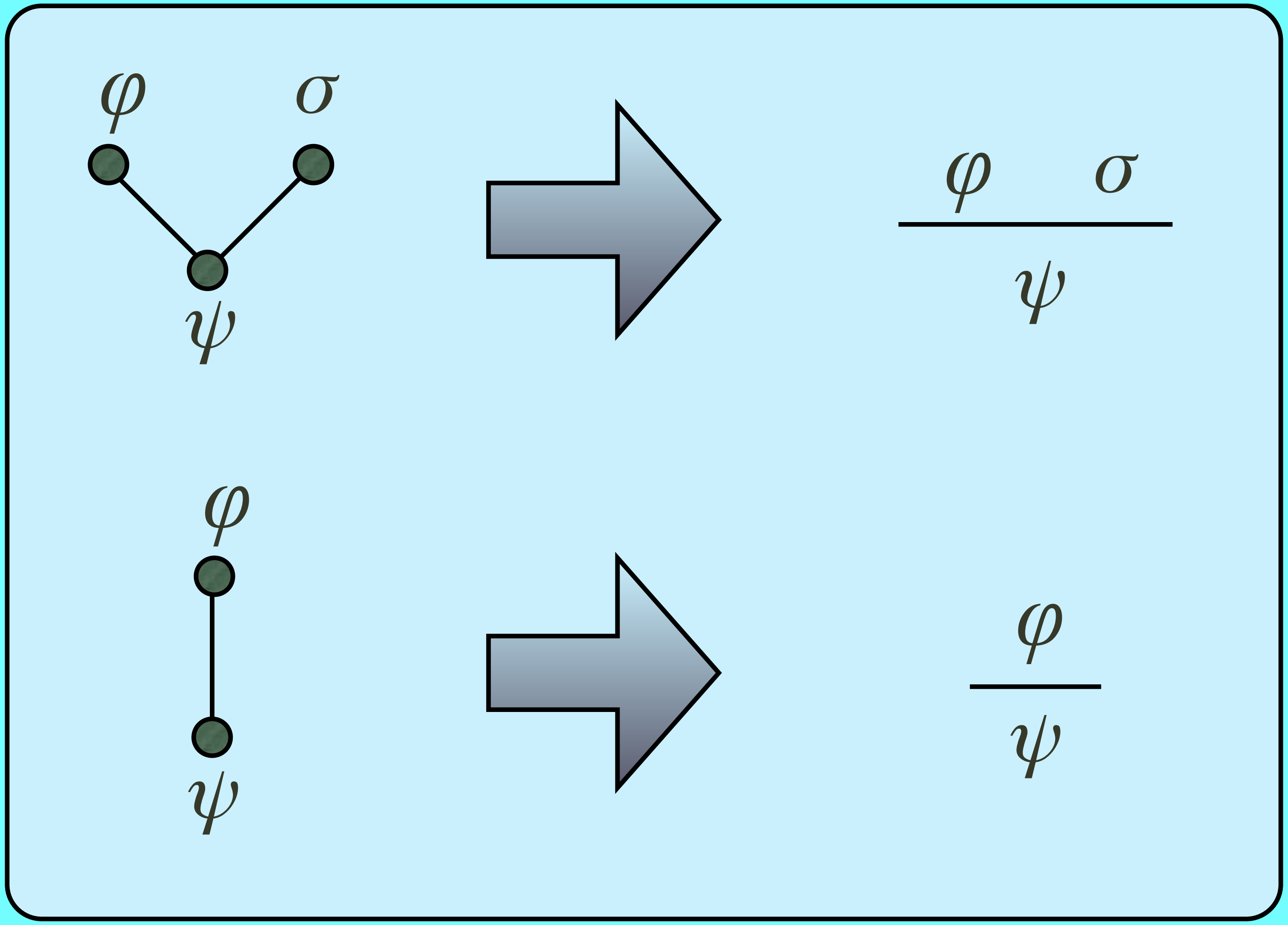
$$\neg\alpha \overset{\text{def}}{=} \alpha \rightarrow \bot$$

$$\neg\alpha \stackrel{\text{def}}{=} \alpha \rightarrow \bot$$


an elimination rule
premises
$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$
conclusion
an introduction rule
discharged hypotheses (leaves)
$[\varphi]$
$\vdots$
ψ
$\varphi \to \psi$
a tree labelled with formulas
premise
conclusion



φ_1
φ_n
Hypotheses
Proof tree
Deduction
Derivation
ψ
conclusion



φ
σ
ψ
$\dfrac{\varphi \quad \sigma}{\psi}$
φ
ψ
$\dfrac{\varphi}{\psi}$

The Elimination Rule for Implication

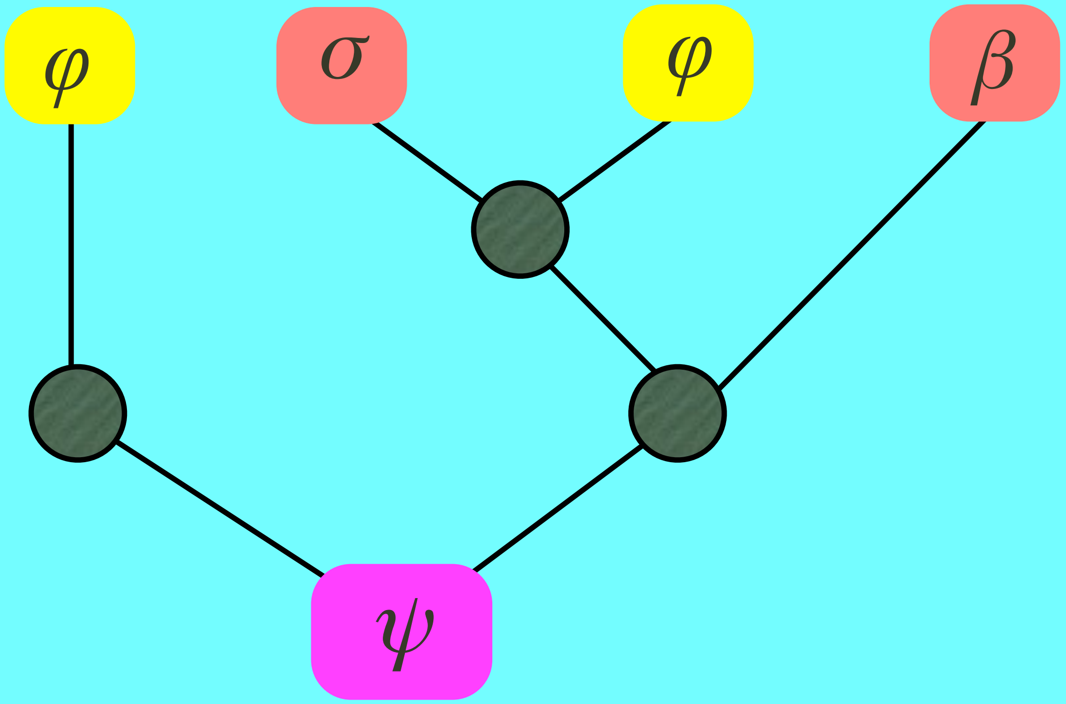
$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

The Introduction Rule for Implication

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}$$

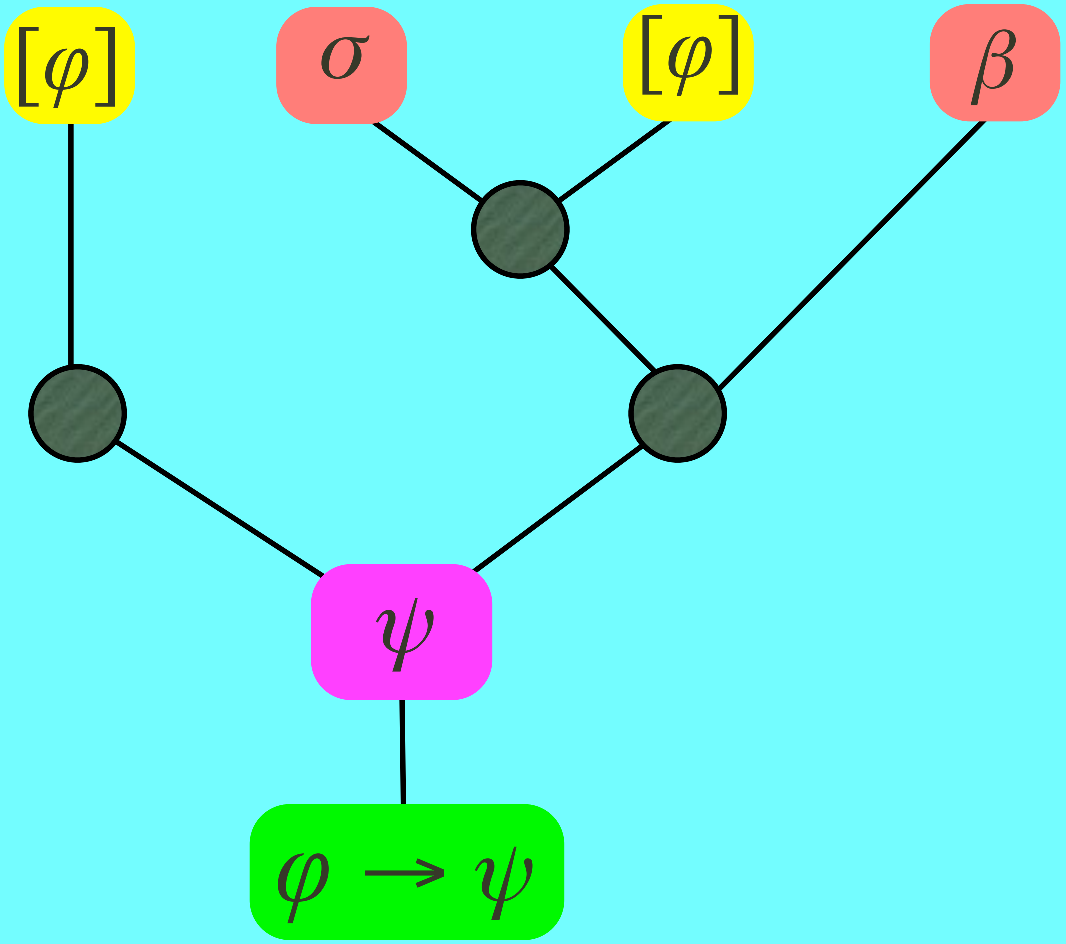
The Introduction Rule

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}$$



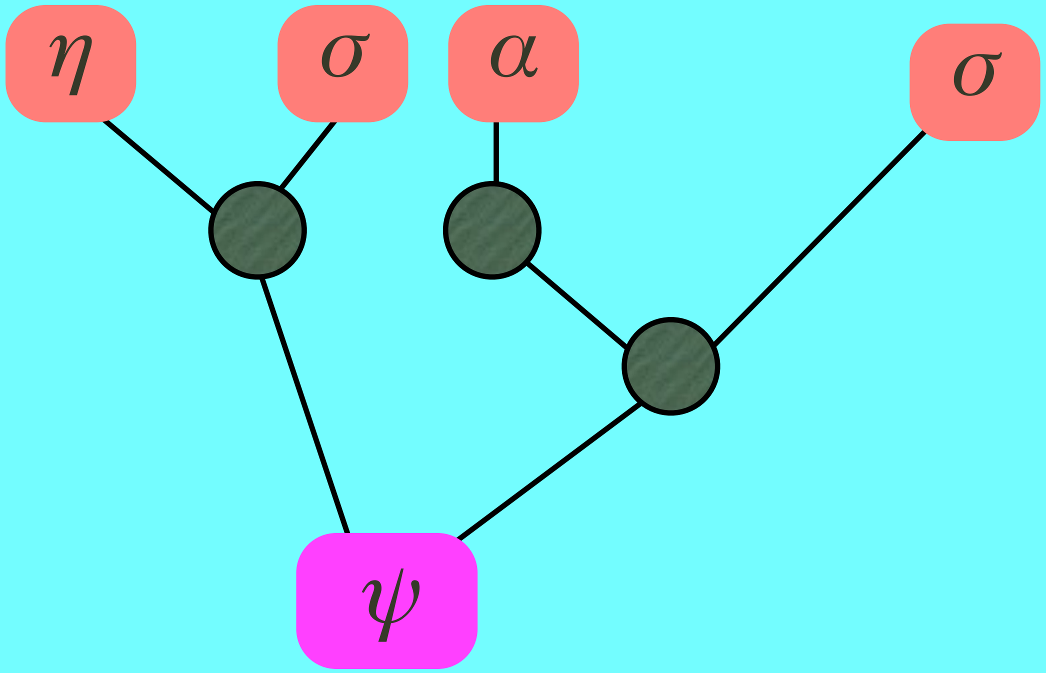
The Introduction Rule

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}$$



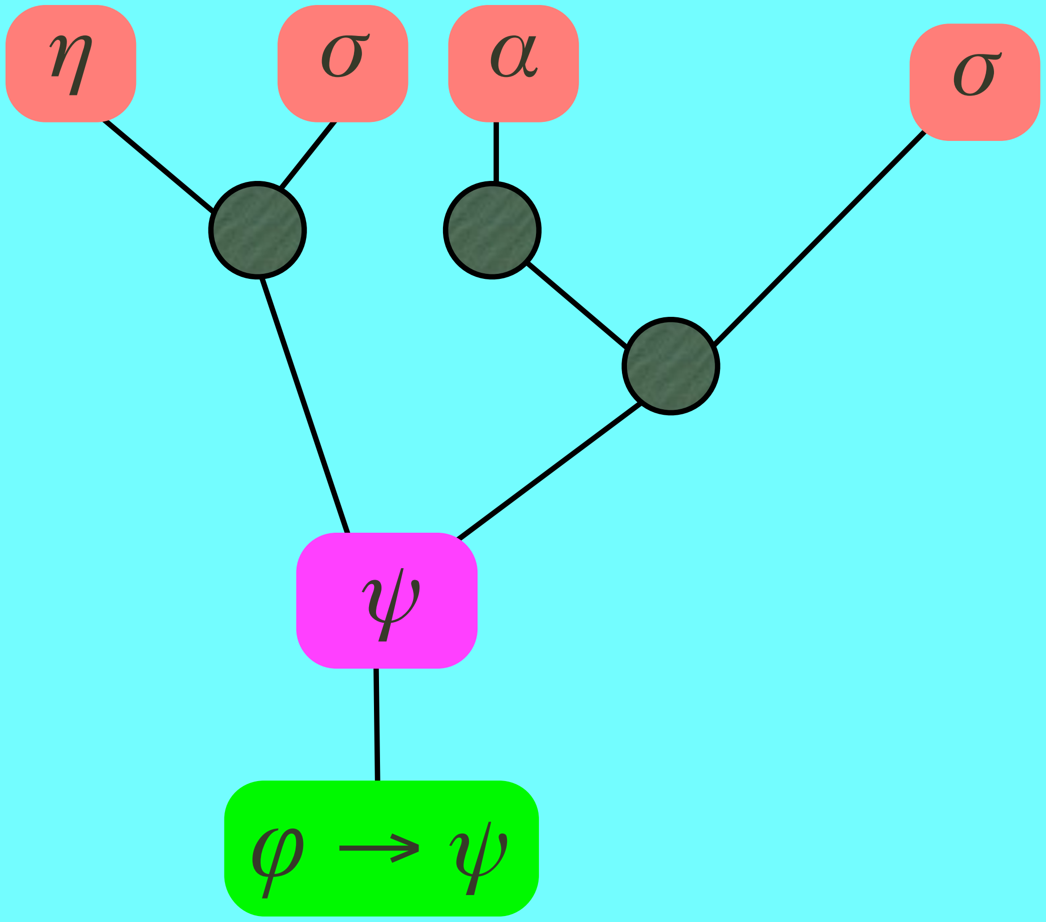
The Introduction Rule

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \longrightarrow \psi}$$



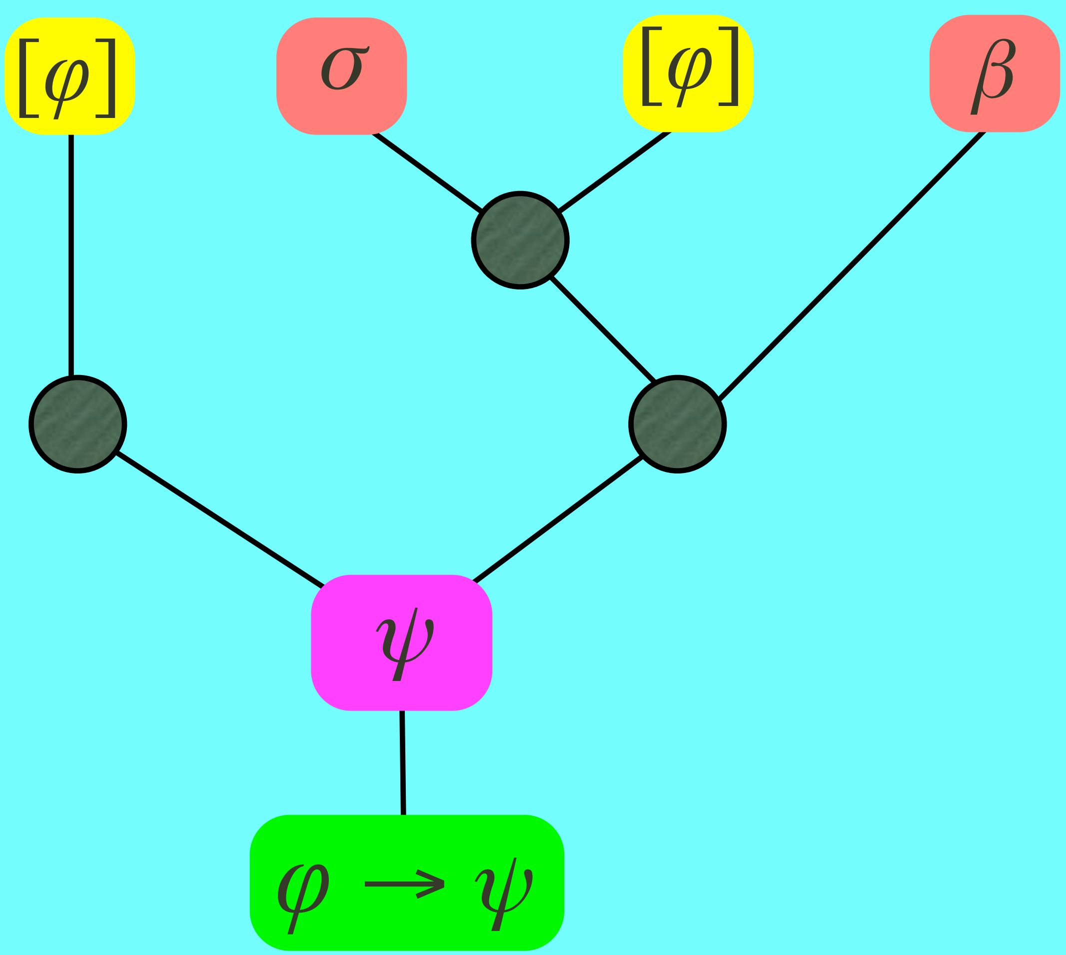
The Introduction Rule

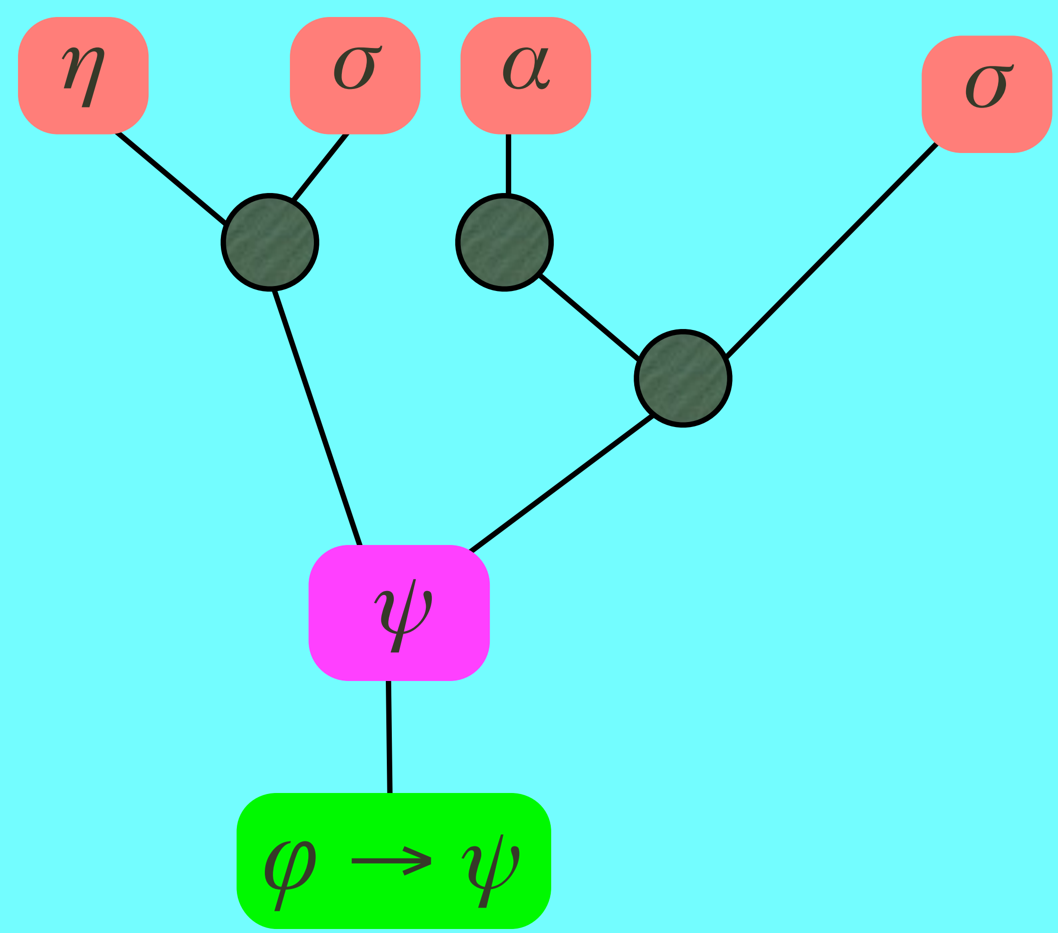
$$\dfrac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}$$



The Introduction Rule

$$\frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi}$$





Introduction rules **Elimination rules**

$$(\wedge I) \quad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I$$

$$(\rightarrow I) \quad \begin{array}{c} [\varphi] \\ \vdots \\ \dfrac{\psi}{\varphi \rightarrow \psi} \rightarrow I \end{array}$$

$$(\wedge E) \quad \frac{\varphi \wedge \psi}{\varphi} \wedge E_1 \quad \frac{\varphi \wedge \psi}{\psi} \wedge E_2$$

$$(\rightarrow E) \quad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \rightarrow E$$

(3 ore) fine lezione 5 marzo 2014

$$\frac{\varphi \wedge \psi}{\psi} \wedge E$$

$$\frac{\varphi \land \psi}{\psi}\ \land E \qquad \frac{\varphi \land \psi}{\varphi}\ \land E$$

$$\dfrac{\dfrac{\varphi \wedge \psi}{\psi}\,\wedge E \qquad \dfrac{\varphi \wedge \psi}{\varphi}\,\wedge E}{\psi \wedge \varphi}\,\wedge I$$

$$
\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\,\wedge E \qquad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\,\wedge E}{\psi \wedge \varphi}\,\wedge I}{\varphi \wedge \psi \to \psi \wedge \varphi}\,\to I_1
$$

$$\frac{\varphi \qquad \varphi \to \bot}{\bot} \to E$$

$$\dfrac{\dfrac{\varphi \qquad [\varphi \to \bot]^1}{\bot} \to E}{(\varphi \to \bot) \to \bot} \to I_1$$

$$\cfrac{\cfrac{\cfrac{[\varphi]^2 \qquad [\varphi \to \bot]^1}{\bot}\, \to E}{(\varphi \to \bot) \to \bot}\, \to I_1}{\varphi \to ((\varphi \to \bot) \to \bot)}\, \to I_2$$

$$\frac{\varphi \wedge \psi}{\psi} \wedge E$$

$$\frac{\varphi \wedge \psi}{\varphi} \wedge E$$

$$\frac{\varphi \wedge \psi}{\psi} \wedge E$$

$$\dfrac{\varphi \wedge \psi}{\psi}\ \wedge E$$

$$\dfrac{\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge E \qquad \varphi \to (\psi \to \sigma)}{\psi \to \sigma}\ \to E$$

$$
\dfrac{\dfrac{\varphi \wedge \psi}{\psi}\,\wedge E \qquad \dfrac{\dfrac{\varphi \wedge \psi}{\varphi}\,\wedge E \qquad \varphi \to (\psi \to \sigma)}{\psi \to \sigma}\,\to E}{\sigma}\,\to E
$$

$$
\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E \qquad \varphi \to (\psi \to \sigma)}{\psi \to \sigma} \to E}{\dfrac{\sigma}{\varphi \wedge \psi \to \sigma} \to I_1} \to E
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E \qquad [\varphi \to (\psi \to \sigma)]^2}{\psi \to \sigma} \to E}{\sigma} \to E}{\varphi \wedge \psi \to \sigma} \to I_1}{(\varphi \to (\psi \to \sigma)) \to (\varphi \wedge \psi \to \sigma)} \to I_2
$$

$\neg\alpha \overset{\text{def}}{=\!=} \alpha\rightarrow\perp$

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\neg\varphi]^1}{\perp} \longrightarrow E}{\neg\neg\varphi} \longrightarrow I_1}{\varphi \longrightarrow \neg\neg\varphi} \longrightarrow I_2$$

Derivations

$$\begin{array}{c}\mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array}$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\end{array}}{\psi} \qquad\qquad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\psi}$$

new derivations
obtained by:
i) unary rule
ii) binary rule

Derivation with
hypothesis ψ

$$\begin{matrix}\psi\\ \mathcal{D}\\ \varphi\end{matrix}$$

denotes the set
(possibly empty)
of all the leaves
labelled with the
formula ψ

A derivation with
hypothesis ψ cancelled

$$\dfrac{\begin{matrix}[\psi]\\ \mathcal{D}\\ \varphi\end{matrix}}{\sigma}$$

denotes the set
of all the leaves labelled with
the formula ψ
marked
as *"cancelled"* / *"discharged"*

the set of derivations is the smallest set X s.t.

(1) The one element tree φ *belongs to* X *for all* $\varphi \in PROP$.

$$(2\wedge)\ \textit{If}\ \begin{array}{c}\mathcal{D}\\ \varphi\end{array},\ \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array} \in X,\ \textit{then}\quad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \varphi & \varphi'\end{array}}{\varphi \wedge \varphi'} \in X.$$

$$\textit{If}\ \begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array} \in X,\ \textit{then}\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi},\ \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi} \in X.$$

$$(2{\to})\ \textit{If}\ \begin{array}{c}\varphi\\ \mathcal{D}\\ \psi\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi} \in X.$$

$$\textit{If}\ \begin{array}{c}\mathcal{D}\\ \varphi\end{array},\ \begin{array}{c}\mathcal{D}'\\ \varphi \to \psi\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \varphi \to \psi\end{array}}{\psi} \in X.$$

$$(2\bot)\ \textit{If}\ \begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi} \in X.$$

$$\textit{If}\ \begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi} \in X.$$

$$\Gamma \vdash \varphi$$

$$\Updownarrow$$

there is a derivation with conclusion φ and with all (uncancelled) hypotheses in Γ

$\vdash \varphi \quad \stackrel{\text{def}}{\equiv} \quad \varnothing \vdash \varphi$ $\Longleftrightarrow$ there is a derivation with conclusion φ and with all hypotheses cancelled

$\Gamma \vdash \varphi$ if $\varphi \in \Gamma$

$\Gamma \vdash \varphi, \Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \varphi \wedge \psi$

$\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$ and $\Gamma \vdash \psi$

$\Gamma \cup \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \rightarrow \psi$

$\Gamma \vdash \varphi, \Gamma' \vdash \varphi \rightarrow \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi$

$\Gamma \vdash \perp \Rightarrow \Gamma \vdash \varphi$

$\Gamma \cup \{\neg\varphi\} \vdash \perp \Rightarrow \Gamma \vdash \varphi$

(1) $\vdash \phi \to (\psi \to \phi)$

(2) $\vdash \phi \to (\neg\phi \to \psi)$

(3) $\vdash (\phi \to \psi) \to [(\psi \to \sigma) \to (\phi \to \sigma)]$

(4) $\vdash (\phi \to \psi) \leftrightarrow (\neg\psi \to \neg\phi)$

(5) $\vdash \neg\neg\phi \leftrightarrow \phi$

(6) $\vdash [\phi \to (\psi \to \sigma)] \leftrightarrow [\phi \wedge \psi \to \sigma]$

(7) $\vdash \bot \leftrightarrow (\phi \wedge \neg\phi)$

1. $$\dfrac{\dfrac{[\varphi]^1}{\psi \to \varphi} \to I}{\varphi \to (\psi \to \varphi)} \to I_1$$

$$2.\qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\neg\varphi]^1}{\perp}\rightarrow E}{\psi}\perp}{\neg\varphi\rightarrow\psi}\rightarrow I_1}{\varphi\rightarrow(\neg\varphi\rightarrow\psi)}\rightarrow I_2$$

3.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1 \quad [\varphi \to \psi]^3}{\psi}\to E \qquad [\psi \to \sigma]^2}{\sigma}\to E}{\varphi \to \sigma}\to I_1}{(\psi \to \sigma) \to (\varphi \to \sigma)}\to I_2}{(\varphi \to \psi) \to ((\psi \to \sigma) \to (\varphi \to \sigma))}\to I_3
$$

Soundness

$$\Gamma \vdash \phi \Rightarrow \Gamma \models \phi.$$

Towards Soundness

Notation:

$\Gamma, \Gamma' \overset{\text{def}}{=} \Gamma \cup \Gamma'$

$\Gamma, \phi \overset{\text{def}}{=} \Gamma, \{\phi\}$

- $\Gamma \models \phi \;\&\; \Gamma \subseteq \Gamma' \Rightarrow \Gamma' \models \phi$
- $\phi \models \phi$
- $\Gamma, \phi \models \phi$
- $\Gamma \models \phi \;\&\; \Gamma' \models \phi' \Rightarrow \Gamma, \Gamma' \models \phi \wedge \phi'$
- $\Gamma \models \phi \wedge \phi' \Rightarrow \Gamma \models \phi \;\&\; \Gamma \models \phi'$
- $\perp \models \phi$
- $\Gamma, \neg\phi \models \perp \Rightarrow \Gamma \models \phi$
- $\Gamma \models \perp \Rightarrow \Gamma - \{\neg\phi\} \models \phi$
- $\Gamma \models \perp \Rightarrow \Gamma \models \phi$
- $\Gamma \models \phi \rightarrow \sigma \;\&\; \Gamma' \models \phi \Rightarrow \Gamma, \Gamma' \models \sigma$
- $\Gamma, \phi \models \sigma \Rightarrow \Gamma \models \phi \rightarrow \sigma$
- $\Gamma \models \sigma \Rightarrow \Gamma - \{\phi\} \models \phi \rightarrow \sigma$
- $\Gamma \models \sigma \;\&\; \Gamma', \sigma \models \phi \Rightarrow \Gamma, \Gamma' \models \phi$

$\Gamma, \phi \vDash \sigma \Rightarrow \Gamma \vDash \phi \to \sigma$

$$\Gamma, \phi \vDash \sigma$$

$$\Rightarrow$$

$$\forall v.\ \{([\Gamma]_v=1 \,\&\, [\phi]_v=1) \Rightarrow [\sigma]_v=1\}$$

$$\Rightarrow$$

$$\forall v.\ \{\text{NOT}([\Gamma]_v=1 \,\&\, [\phi]_v=1)\ \text{OR}\ [\sigma]_v=1\}$$

$$\Rightarrow$$

$$\forall v.\ \{([\Gamma]_v \neq 1\ \text{OR}\ [\phi]_v=0)\ \text{OR}\ [\sigma]_v=1\}$$

$$\Rightarrow$$

$$\forall v.\ \{[\Gamma]_v \neq 1\ \text{OR}\ ([\phi]_v=0\ \text{OR}\ [\sigma]_v=1)\}$$

$$\Rightarrow$$

$$\forall v.\ \{[\Gamma]_v \neq 1\ \text{OR}\ ([\phi \to \sigma]_v=1)\}$$

$$\Rightarrow$$

$$\forall v.\ \{[\Gamma]_v=1 \Rightarrow [\phi \to \sigma]_v=1\}$$

$$\Rightarrow$$

$$\Gamma \vDash \phi \to \sigma$$

Soundness

$$\Gamma \vdash \phi \Rightarrow \Gamma \models \phi.$$

Notation: **hp**$\mathcal{D}$ is the set of uncancelled hypoteses of $\mathcal{D}$

We prove, by induction on the lenght of derivations, that

for each derivation $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and Γ, with **hp**$\mathcal{D} \subseteq \Gamma$

we have $\Gamma \models \varphi$

Basis: $\boxed{\mathcal{D} = \varphi}$

$$\mathcal{D} = \varphi \Rightarrow \varphi \in \Gamma \Rightarrow \Gamma \models \varphi$$

Inductive cas

We prove, by induction on the lenght of derivations, that

for each derivation $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}$ and Γ, with **hp**$\mathcal{D}$⊆Γ

we have $\Gamma \vDash \varphi$

1: ∧ I

$$\mathcal{D}'' = \left\{ \frac{\begin{array}{c}\mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \varphi'\end{array}}{\varphi \wedge \varphi'} \right.$$

hp$\mathcal{D}''\subseteq \Gamma''$

Inductive Hypothesis (IH)

$\Longrightarrow$

hp$\mathcal{D} \vDash \varphi$ & **hp**$\mathcal{D}' \vDash \varphi'$

$\Longrightarrow$

hp$\mathcal{D}$ ∪ **hp**$\mathcal{D}' \vDash \varphi \wedge \varphi'$

$\Longrightarrow$

$\Gamma'' \vDash \varphi \wedge \varphi'$

2: $\wedge E_1$

$$\mathcal{D}' = \left\{ \begin{array}{c} \mathcal{D} \\ \dfrac{\varphi \wedge \psi}{\varphi} \end{array} \right.$$

hp$\mathcal{D}' \subseteq \Gamma'$

Inductive Hypothesis (IH)

$\Rightarrow$

hp$\mathcal{D} \vDash \varphi \wedge \psi$

$\Rightarrow$

hp$\mathcal{D} \vDash \varphi$

$\Rightarrow$

$\Gamma' \vDash \varphi$

3: $\wedge E_2$ as the previous one

2: $\rightarrow$ I

$$\mathcal{D}' = \left\{ \begin{array}{c} [\varphi] \\ \mathcal{D} \\ \dfrac{\psi}{\varphi \rightarrow \psi} \end{array} \right.$$

hp$\mathcal{D}' \subseteq \Gamma'$

Inductive Hypothesis (IH)

$\Rightarrow$

hp$\mathcal{D} \vDash \psi$

$\Rightarrow$

hp$\mathcal{D} - \{\varphi\} \vDash \varphi \rightarrow \psi$

$\Rightarrow$ *(since* **hp**$\mathcal{D}'$ = **hp**$\mathcal{D} - \{\varphi\}$*)*

$\Gamma' \vDash \varphi \rightarrow \psi$

4: $\rightarrow$E

$$\mathcal{D}'' = \left\{ \begin{array}{c} \mathcal{D} \qquad\qquad \mathcal{D}' \\ \\ \dfrac{\varphi \qquad\qquad \varphi \rightarrow \psi}{\psi} \end{array} \right.$$

hp$\mathcal{D}'' \subseteq \Gamma''$

Inductive Hypothesis (IH)

$\Longrightarrow$

hp$\mathcal{D} \vDash \varphi$ & **hp**$\mathcal{D}' \vDash \varphi \rightarrow \psi$

$\Longrightarrow$

hp$\mathcal{D}$ ∪ **hp**$\mathcal{D}' \vDash \psi$

$\Longrightarrow$

$\Gamma'' \vDash \psi$

4: RAA

$$\mathcal{D}' = \left\{ \begin{array}{c} [\neg\varphi] \\ \mathcal{D} \\ \perp \\ \hline \varphi \end{array} \right.$$

hp$\mathcal{D}' \subseteq \Gamma'$

Inductive Hypothesis (IH)

$\Longrightarrow$

hp$\mathcal{D} \vDash \perp$

$\Longrightarrow$

hp$\mathcal{D} - \{\neg\varphi\} \vDash \varphi$

$\Longrightarrow$ *(since* **hp**$\mathcal{D}'$ = **hp**$\mathcal{D} - \{\neg\varphi\}$*)*

$\Gamma' \vDash \varphi$

An application of **soundness**

$$\Gamma \not\models \phi \Rightarrow \Gamma \nvdash \phi$$

$$\nvdash (\varphi \vee \sigma) \longrightarrow \varphi$$

1. let $\varphi = p_0$ and $\sigma = p_1$
2. let $v(p_0)=0$ and $v(p_1)=1$
3. $v((p_0 \vee p_1) \rightarrow p_0)=0$
4. $\not\models (p_0 \vee p_1) \rightarrow p_0$
5. $\nvdash (p_0 \vee p_1) \rightarrow p_0$

Completeness

$$\Gamma \vDash \phi \Rightarrow \Gamma \vdash \phi$$

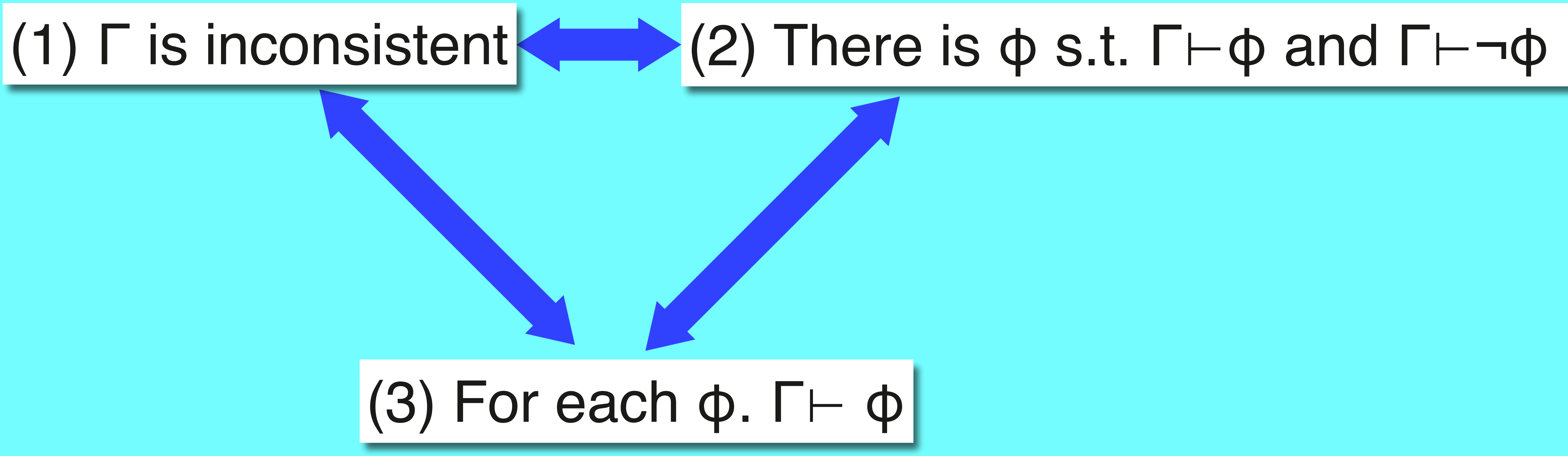
A set Γ of propositions is **consistent** if $\Gamma \nvdash \bot$.

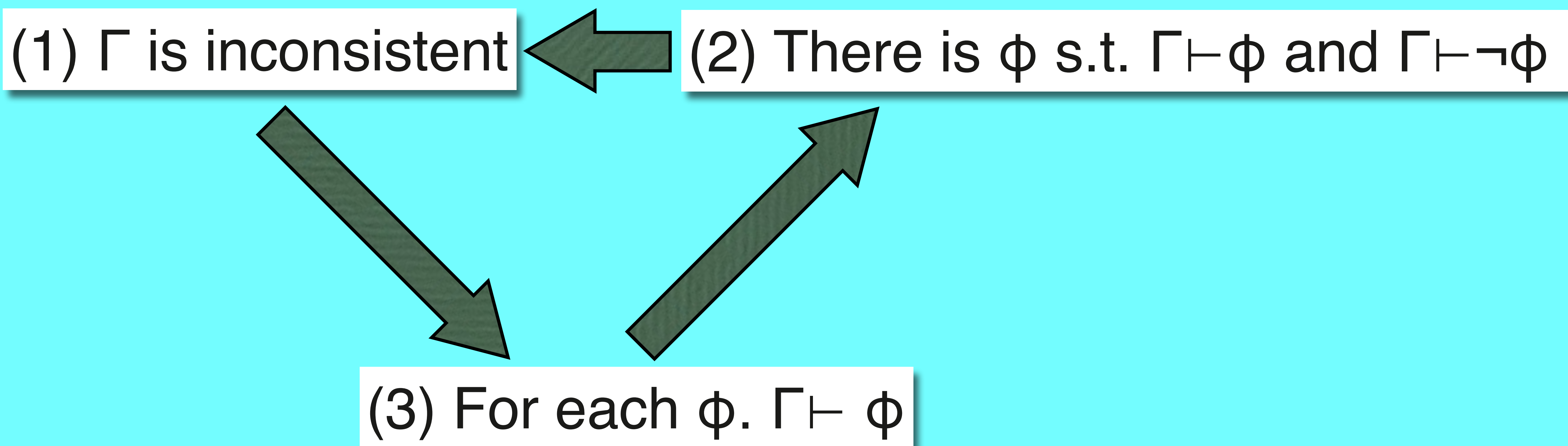
A set Γ of propositions is **inconsistent** if $\Gamma \vdash \bot$.

(1) Γ is consistent

(2) For no ϕ, $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

(3) There is at least one ϕ such that $\Gamma \nvdash \phi$


(1) Γ is inconsistent
(2) There is ϕ s.t. Γ⊢ϕ and Γ⊢¬ϕ
(3) For each ϕ. Γ⊢ ϕ


(1) Γ is inconsistent
(2) There is ϕ s.t. Γ⊢ϕ and Γ⊢¬ϕ
(3) For each ϕ. Γ⊢ ϕ

(1) Γ is inconsistent

(2) There is ϕ s.t. $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

(3) For each φ. $\Gamma \vdash \varphi$

$$\Gamma \vdash \bot \Rightarrow \exists \mathcal{D} \text{ s.t. } \begin{array}{c}\mathcal{D}\\ \underline{\bot}\end{array} \text{ with } \mathbf{hp}\mathcal{D} \subseteq \Gamma$$

$$\Rightarrow \dfrac{\begin{array}{c}\mathcal{D}\\ \underline{\bot}\end{array}}{\varphi} \Rightarrow \Gamma \vdash \varphi$$

(1) Γ is inconsistent

(2) There is ϕ s.t. $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

(3) For each ϕ. $\Gamma \vdash \phi$

immediate

(1) Γ is inconsistent

(2) There is φ s.t. $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$

(3) For each φ. $\Gamma \vdash \phi$

$$\Gamma \vdash \phi \Rightarrow \exists\, \mathcal{D}' \text{ s.t. } \begin{array}{c} \mathcal{D}' \\ \phi \end{array} \text{ with } \mathbf{hp}\mathcal{D}' \subseteq \Gamma$$

$$\Gamma \vdash \neg\phi \Rightarrow \exists\, \mathcal{D}'' \text{ s.t. } \begin{array}{c} \mathcal{D}' \\ \neg\phi \end{array} \text{ with } \mathbf{hp}\mathcal{D}'' \subseteq \Gamma$$

$$\Rightarrow$$

$$\dfrac{\begin{array}{cc} \mathcal{D}' & \mathcal{D}'' \\ \phi & \neg\phi \end{array}}{\bot} \quad \Rightarrow \quad \Gamma \vdash \bot$$

Proposition:
If there is a valuation such that $[\psi]_v = 1$ for all $\psi \in \Gamma$, then Γ is consistent.

Proof:
Suppose $\Gamma \vdash \perp$, then $\Gamma \vDash \perp$, so for any valuation v

$$[(\psi)]_v = 1 \text{ for all } \psi \in \Gamma \Rightarrow [\perp]_v = 1$$

Since $[\perp]_v = 0$ for all valuations, there is no valuation with $[\psi]_v = 1$ for all $\psi \in \Gamma$. ***Contradiction***.
Hence Γ is consistent.

$\Gamma \cup \{\neg\phi\}$ is inconsistent $\Rightarrow \Gamma \vdash \phi$,

$\Gamma \cup \{\phi\}$ is inconsistent $\Rightarrow \Gamma \vdash \neg\phi$.

$\Gamma \cup \{\neg\phi\}$ is inconsistent $\Rightarrow \exists\, \mathcal{D}'$ s.t. $\begin{array}{c}\mathcal{D}'\\ \perp\end{array}$ with $\mathbf{hp}\mathcal{D}' \subseteq \Gamma \cup \{\neg\phi\}$

$$\Rightarrow \quad \dfrac{\begin{array}{c}[\neg\phi]\\ \mathcal{D}'\\ \perp\end{array}}{\phi}\ \mathrm{RAA}$$

$\Gamma \cup \{\phi\}$ is inconsistent $\Rightarrow \exists\, \mathcal{D}'$ s.t. $\begin{array}{c}\mathcal{D}'\\ \perp\end{array}$ with $\mathbf{hp}\mathcal{D}' \subseteq \Gamma \cup \{\phi\}$

$$\Rightarrow \quad \dfrac{\begin{array}{c}[\phi]\\ \mathcal{D}'\\ \perp\end{array}}{\neg\phi}\ \rightarrow\mathrm{I}$$

A set Γ is maximally consistent iff
(a) Γ is consistent,
(b) $\Gamma \subseteq \Gamma'$ and Γ' consistent $\Rightarrow \Gamma = \Gamma'$.

example: Let v a valuation, $\Gamma = \{\phi : [\phi]_v = 1\}$. Γ is consistent.
Let Γ' such that $\Gamma \subseteq \Gamma'$.
Let $\psi \in \Gamma'$ s.t. $\psi \notin \Gamma$ i.e. $[\psi]_v = 0$, then $[\neg\psi]_v = 1$, and so $\neg\psi \in \Gamma$.
But since $\Gamma \subseteq \Gamma'$ this implies that Γ' is inconsistent.
Contradiction.

Theorem:
Each consistent set Γ is contained in a maximally consistent set Γ^*

1) enumerate all the formulas

$\varphi_0, \varphi_1, \varphi_2, \ldots..$

2) define the non decreasing sequence:

$\Gamma_0 = \Gamma$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\phi_n\} \text{ if } \Gamma_n \cup \{\phi_n\} \text{ is consistent,} \\ \Gamma_n \text{ otherwise} \end{cases}$$

3) define

$$\Gamma^* = \bigcup_{n \geq 0} \Gamma_n \, .$$

(a) Γ_n is consistent for all n (a trivial induction on n)

(b) Γ^* is consistent

suppose $\Gamma^* \vdash \bot$

we have $\exists \begin{array}{c}\mathcal{D}\\ \bot\end{array}$ with hp$\mathcal{D}=\{\psi_0,\ldots,\psi_k\} \subseteq \Gamma^*$;

$\Gamma^* = \bigcup_{n\geq 0} \Gamma_n \Rightarrow \forall i \leq k\ \exists n_i : \psi_i \in \Gamma_{n_i}$.

Let n=max$\{n_i : i \leq k\}$, then $\psi_0,\ldots,\psi_k \in \Gamma_n$ and hence $\Gamma_n \vdash \bot$.

But Γ_n is consistent. Contradiction.

(c) Γ^* is maximally consistent

Let $\Gamma^* \subseteq \Delta$ and Δ consistent. If $\psi \in \Delta$, then $\exists m.\ \psi = \phi_m$;

$\Gamma_m \subseteq \Gamma^* \subseteq \Delta$ and Δ is consistent, $\Gamma_m \cup \{\phi_m\}$ is consistent.

Therefore $\Gamma_{m+1} = \Gamma_m \cup \{\phi_m\}$, i.e. $\phi_m \in \Gamma_{m+1} \subseteq \Gamma^*$.

$\Gamma^* = \Delta$.

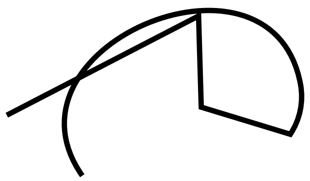
If Γ is maximally consistent, then Γ is closed under derivability (i.e. $\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma$).

Let $\Gamma \vdash \phi$ and suppose $\phi \notin \Gamma$. Then $\Gamma \cup \{\phi\}$ must be inconsistent. Hence $\Gamma \vdash \neg\phi$, so Γ is inconsistent. Contradiction.

Let Γ be maximally consistent;

a)$\forall\phi$ either $\phi\in\Gamma$, or $\neg\phi\in\Gamma$,

b)$\forall\phi,\psi.\ \phi\rightarrow\psi\in\Gamma\Leftrightarrow(\phi\in\Gamma\Rightarrow\psi\in\Gamma)$.



(a) We know that not both ϕ and $\neg\phi$ can belong to Γ. Consider $\Gamma' = \Gamma \cup \{\phi\}$. If Γ' is inconsistent, then, $\neg\phi \in \Gamma$. If Γ' is consistent, then $\phi \in \Gamma$ by the maximality of Γ.

(b) **b1) Let $\phi\rightarrow\psi\in\Gamma$ and $\phi\in\Gamma$.**

Since $\phi,\phi\rightarrow\psi\in\Gamma$ and since Γ is closed under derivability we get $\psi\in\Gamma$ by $\rightarrow$ E.

b2) Let $\phi \in \Gamma \Rightarrow \psi \in \Gamma$.

If $\phi \in \Gamma$ then obviously $\Gamma\vdash\psi$, so $\Gamma\vdash\phi\rightarrow\psi$.

If $\phi\notin \Gamma$,then $\neg\phi\in\Gamma$, and then $\Gamma \vdash\neg\phi$.

Therefore $\Gamma \vdash\phi\rightarrow\psi$.

Corollary

If Γ is maximally consistent, then $\phi \in \Gamma \Leftrightarrow \neg\phi \notin \Gamma$, and $\neg\phi \in \Gamma \Leftrightarrow \phi \notin \Gamma$.

If Γ is consistent, then there exists a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$.

Proof. (a) Γ is contained in a maximally consistent Γ^*

(b) Define $v(p_i) = \begin{cases} 1 \text{ if } p_i \in \Gamma^* \\ 0 \text{ else} \end{cases}$

and extend v to the valuation $[\![\ ]\!]_v$.

Claim: $[\![\varphi]\!] = 1 \Leftrightarrow \varphi \in \Gamma^*$. Use induction on φ.

1. For atomic φ the claim holds by definition.
2. $\varphi = \psi \wedge \sigma$. $[\![\varphi]\!]_v = 1 \Leftrightarrow [\![\psi]\!]_v = [\![\sigma]\!]_v = 1 \Leftrightarrow$ (induction hypothesis) $\psi, \sigma \in \Gamma^*$ and so $\varphi \in \Gamma^*$. Conversely $\psi \wedge \sigma \in \Gamma^* \Longrightarrow \psi, \sigma \in \Gamma^*$ The rest follows from the induction hypothesis.
3. $\varphi = \psi \to \sigma$. $[\![\psi \to \sigma]\!]_v = 0 \Leftrightarrow [\![\psi]\!]_v = 1$ and $[\![\sigma]\!]_v = 0 \Leftrightarrow$ (induction hypothesis) $\psi \in \Gamma^*$ and $\sigma \notin \Gamma^* \Leftrightarrow \psi \to \sigma \notin \Gamma^*$

(c) Since $\Gamma \subseteq \Gamma^*$ we have $[\![\psi]\!]_v = 1$ for all $\psi \in \Gamma$. $\square$

Corollary

$\Gamma \nvdash \phi \Leftrightarrow$ there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$ and $[\phi]=0$.

$\Gamma \nvdash \phi \Leftrightarrow \Gamma \cup \{\neg\phi\}$ consistent $\Leftrightarrow$ there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma \cup \{\neg\phi\}$, namely, $[\psi]=1$forall $\psi \in \Gamma$ and$[\phi]=0$

Theorem (Completeness Theorem)

$$\Gamma \vDash \phi \Longrightarrow \Gamma \vdash \phi$$

Proof. $\Gamma \nvdash \phi \Rightarrow \Gamma \nvDash \phi$

$$\Gamma \vDash \phi \Longleftrightarrow \Gamma \vdash \phi$$

The connective $\vee$

$$\frac{\varphi}{\varphi \vee \psi}\,\vee I \qquad \frac{\psi}{\varphi \vee \psi}\,\vee I$$

$$\dfrac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \sigma\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \sigma\end{matrix}}{\sigma}\,\vee E$$

proof by cases

$$\vdash (\varphi \wedge \psi) \vee \sigma \leftrightarrow (\varphi \vee \sigma) \wedge (\psi \vee \sigma).$$

$$\vdash (\varphi \wedge \psi) \vee \sigma \leftrightarrow (\varphi \vee \sigma) \wedge (\psi \vee \sigma).$$

$$\dfrac{\dfrac{(\varphi \wedge \psi) \vee \sigma \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}}{\varphi \vee \sigma} \qquad \dfrac{[\sigma]^1}{\varphi \vee \sigma}}{\varphi \vee \sigma}\,1 \qquad \dfrac{(\varphi \wedge \psi) \vee \sigma \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^2}{\psi}}{\psi \vee \sigma} \qquad \dfrac{[\sigma]^2}{\psi \vee \sigma}}{\psi \vee \sigma}\,2}{(\varphi \vee \sigma) \wedge (\psi \vee \sigma)}$$

$$\vdash (\varphi \wedge \psi) \vee \sigma \leftrightarrow (\varphi \vee \sigma) \wedge (\psi \vee \sigma).$$

$$\dfrac{\dfrac{(\varphi \vee \sigma) \wedge (\psi \vee \sigma)}{\varphi \vee \sigma} \qquad \dfrac{\dfrac{(\varphi \vee \sigma) \wedge (\psi \vee \sigma)}{\psi \vee \sigma} \quad \dfrac{\dfrac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi}}{(\varphi \wedge \psi) \vee \sigma} \quad \dfrac{[\sigma]^1}{(\varphi \wedge \psi) \vee \sigma}}{(\varphi \wedge \psi) \vee \sigma}\,1 \qquad \dfrac{[\sigma]^2}{(\varphi \wedge \psi) \vee \sigma}}{(\varphi \wedge \psi) \vee \sigma}\,2$$

$$\vdash \varphi \vee \neg\varphi$$

$$\vdash \varphi \vee \neg\varphi$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \vee \neg\varphi}\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\to E}{\neg\varphi}\to I_1}{\varphi \vee \neg\varphi}\vee I \qquad [\neg(\varphi \vee \neg\varphi)]^2}{\bot}\to E}{\varphi \vee \neg\varphi}\mathrm{RAA}_2$$

$$\vdash (\varphi \to \psi) \lor (\psi \to \varphi)$$

$$\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\psi \to \varphi} \to I_1}{(\varphi \to \psi) \vee (\psi \to \varphi)} \vee I \qquad [\neg((\varphi \to \psi) \vee (\psi \to \varphi))]^2}{\bot} \to E}{\psi} \bot}{\varphi \to \psi} \to I_1}{(\varphi \to \psi) \vee (\psi \to \varphi)} \vee I \qquad [\neg((\varphi \to \psi) \vee (\psi \to \varphi))]^2}{\bot} \to E}{(\varphi \to \psi) \vee (\psi \to \varphi)} \mathrm{RAA}_2$$

$$\vdash \neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$$

$$\vdash \neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$$

$$\dfrac{\dfrac{\dfrac{[\neg(\varphi \wedge \psi)] \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg\varphi \vee \neg\psi)] \qquad \dfrac{[\neg\varphi]}{\neg\varphi \vee \neg\psi}}{\bot}}{\varphi} \qquad \dfrac{\dfrac{[\neg(\neg\varphi \vee \neg\psi)] \qquad \dfrac{[\neg\psi]}{\neg\varphi \vee \neg\psi}}{\bot}}{\psi}}{\varphi \wedge \psi}}{\bot}}{\neg\varphi \vee \neg\psi}}{\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi}$$

$$\vdash \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi).$$

exercise