Morphological Image Processing

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Introduction

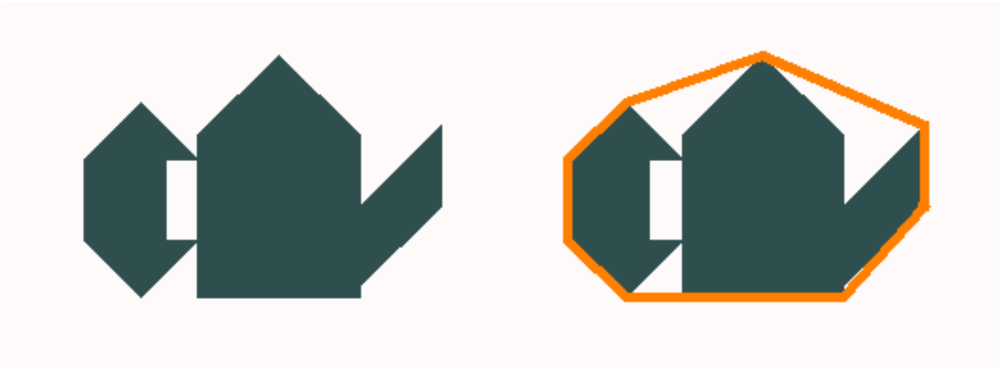
- In many areas of knowledge Morphology deals with form and structure (biology, linguistics, social studies, etc)

- Mathematical Morphology deals with set theory

- Sets in Mathematical Morphology represents objects in an Image

Mathematatic Morphology

- Used to extract image components that are useful in the representation and description of region shape, such as
  - boundaries extraction
  - skeletons
  - convex hull (italian: inviluppo convesso)
  - morphological filtering
  - thinning
  - pruning



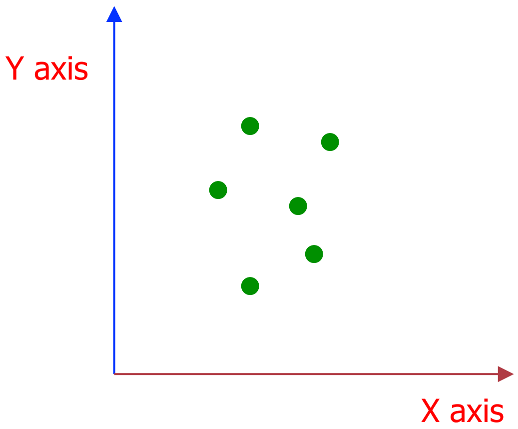
Mathematic Morphology

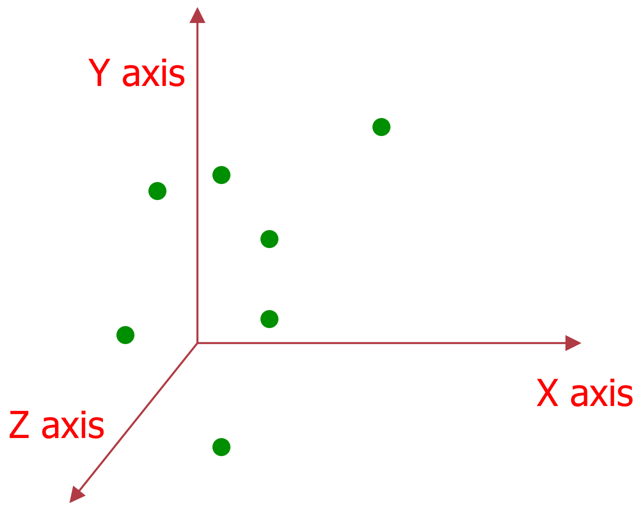
mathematical framework used for:

- pre-processing
  - noise filtering, shape simplification, ...
- enhancing object structure
  - skeletonization, convex hull...
- segmentation
  - watershed,…
- quantitative description
  - area, perimeter, ...

Z^2 and Z^3

- set in mathematic morphology represent objects in an image
  - binary image (0 = white, 1 = black) : the element of the set is the coordinates (x,y) of pixel belong to the object ⇨ Z^2
- gray-scaled image : the element of the set is the coordinates (x,y) of pixel belong to the object and the gray levels ⇨ Z^3

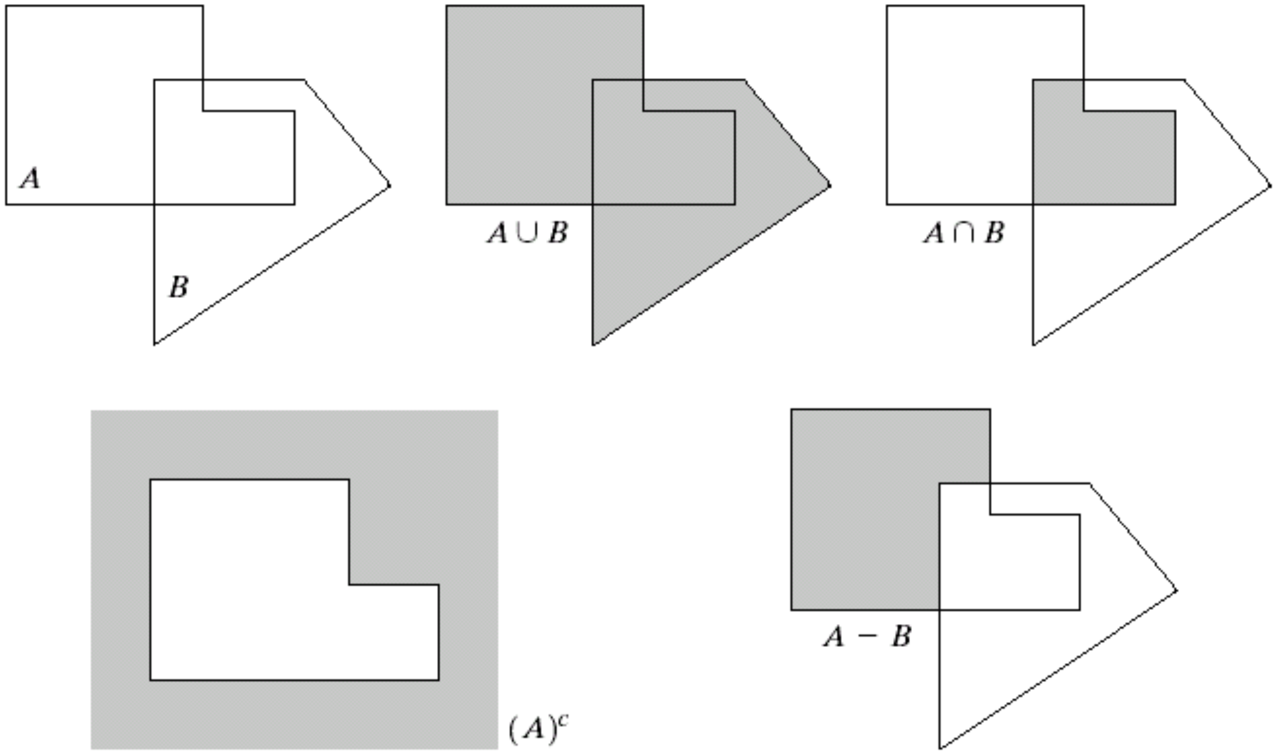




Basic Set Operators

Set operators	Denotations
A Subset B	$A \subseteq B$
Union of A and B	$C = A \cup B$
Intersection of A and B	$C = A \cap B$
Disjoint	$A \cap B = \varnothing$
Complement of A	$A^c = \{ w \mid w \notin A \}$
Difference of A and B	$A - B = \{ w \mid w \in A, w \notin B \}$
Reflection of A	$\hat{A} = \{ w \mid w = -a \text{ for } a \in A \}$
Translation of set A by point $z(z_1, z_2)$	$(A)_z = \{ c \mid c = a + z, \text{ for } a \in A \}$

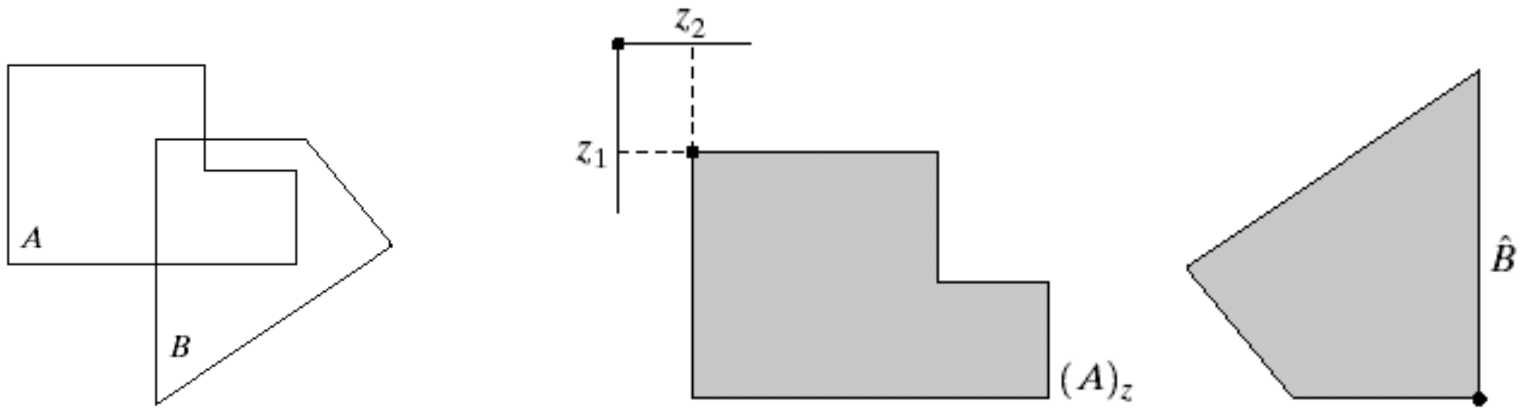
Basic Set Theory



Reflection and Translation

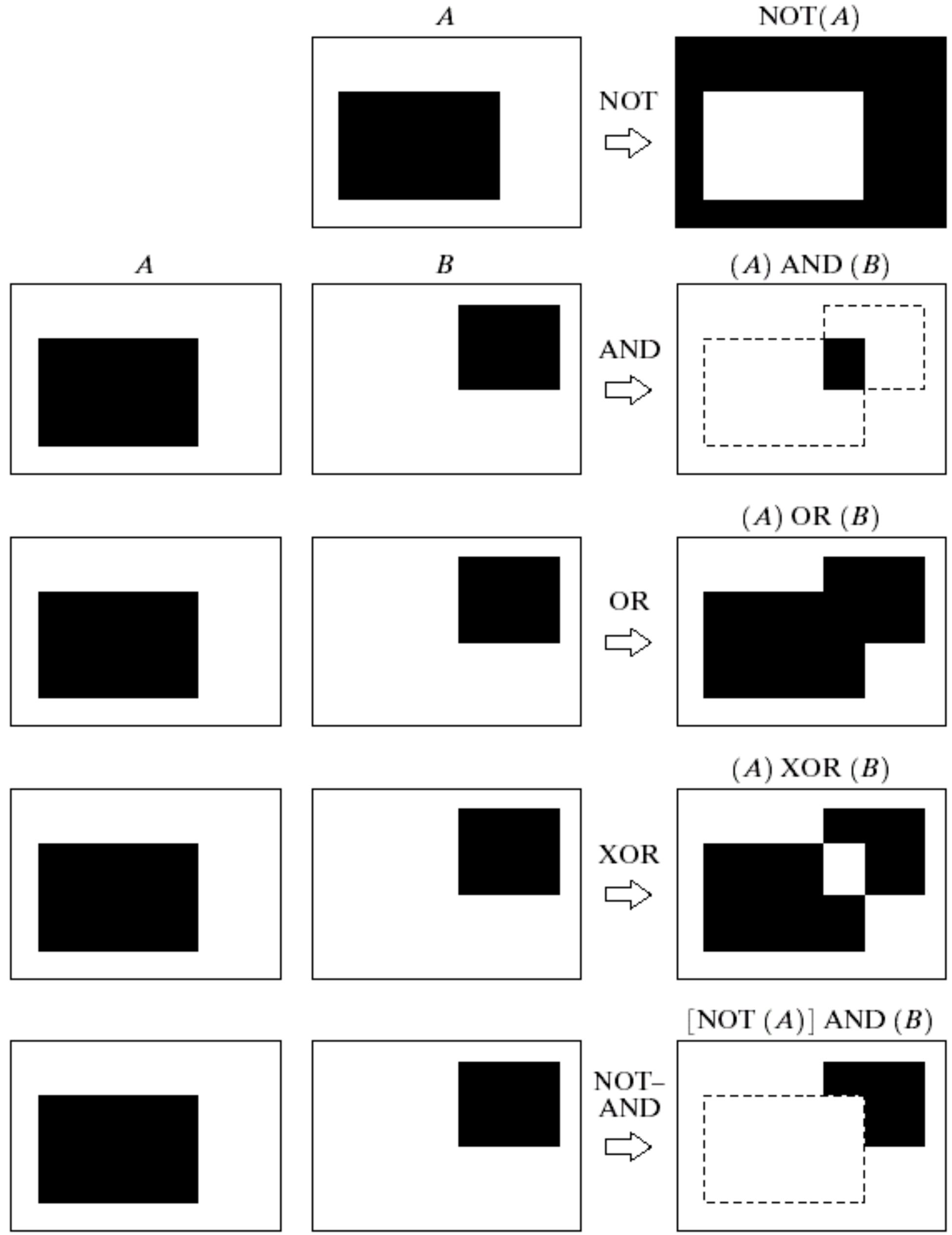
$$\hat{B} = \{w \in E^2 : w = -b, \quad for\ b \in B\}$$

$$(A)_z = \{c \in E^2 : c = a + z, \quad for\ a \in A\}$$



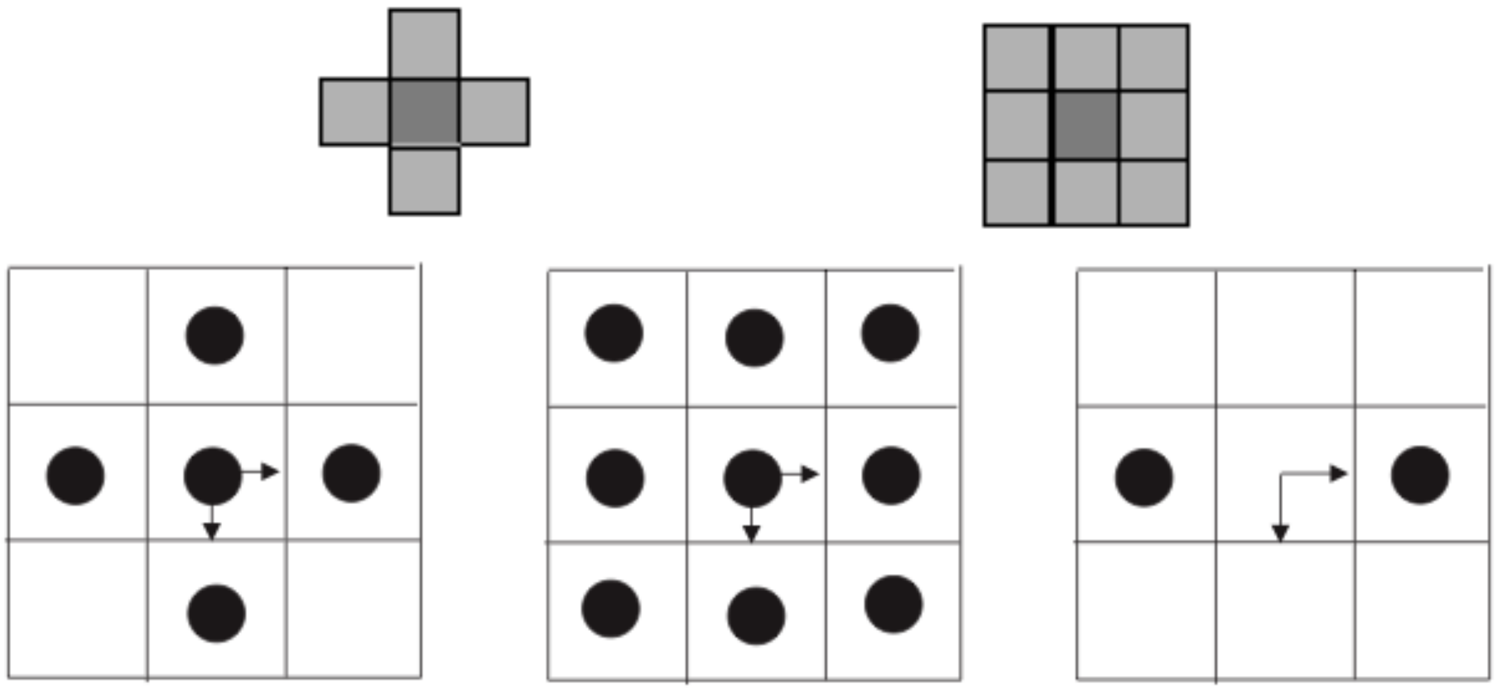
Logic Operations

p	q	p **AND** q **(also** $p \cdot q$**)**	p **OR** q **(also** $p + q$**)**	**NOT (**p**) (also** $\bar{p}$**)**
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0


A
NOT(A)
NOT
A
B
(A) AND (B)
AND
(A) OR (B)
OR
(A) XOR (B)
XOR
[NOT (A)] AND (B)
NOT–
AND

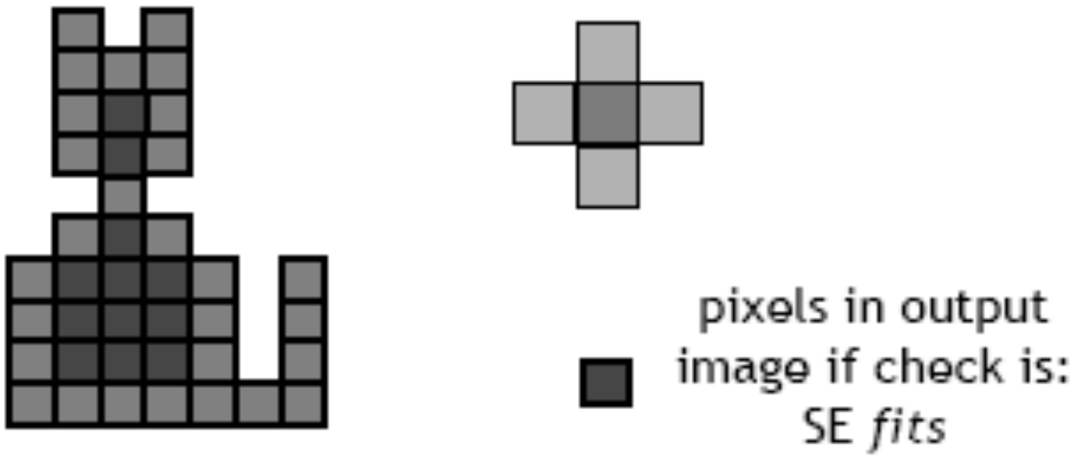
Structuring element (SE)

- small set to probe the image under study
- for each SE, define origin
- shape and size must be adapted to geometric properties for the objects



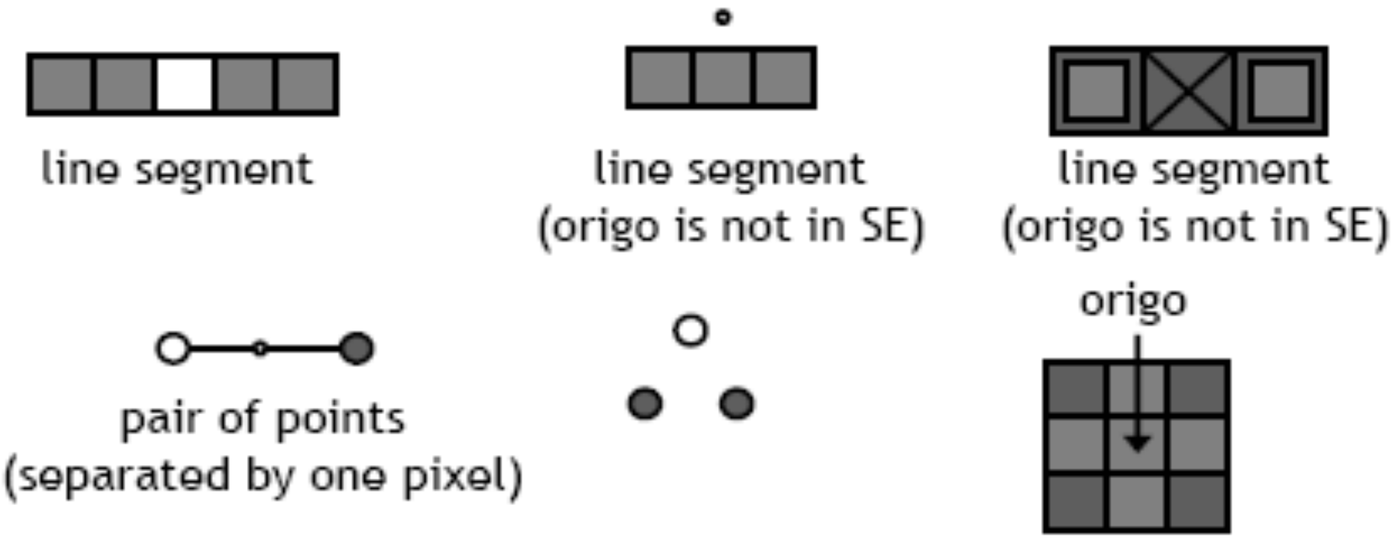
Basic idea

- in parallel for each pixel in binary image:
  - check if SE is ”satisfied”
  - output pixel is set to 0 or 1 depending on used operation



How to describe SE

- Can be described in many different ways
- information needed:
  - position of origin for SE
  - positions of elements belonging to SE



Five Binary Morphological Operations

- $\ominus$ Erosion
- $\oplus$ Dilation
- $\circ$ Opening
- $\bullet$ Closing
- $\otimes$ Hit-or-Miss transform

Basic morphological operations

- Erosion 

- Dilation 

- combine to keep general shape but smooth with respect to
  - Opening → object
  - Closening → background

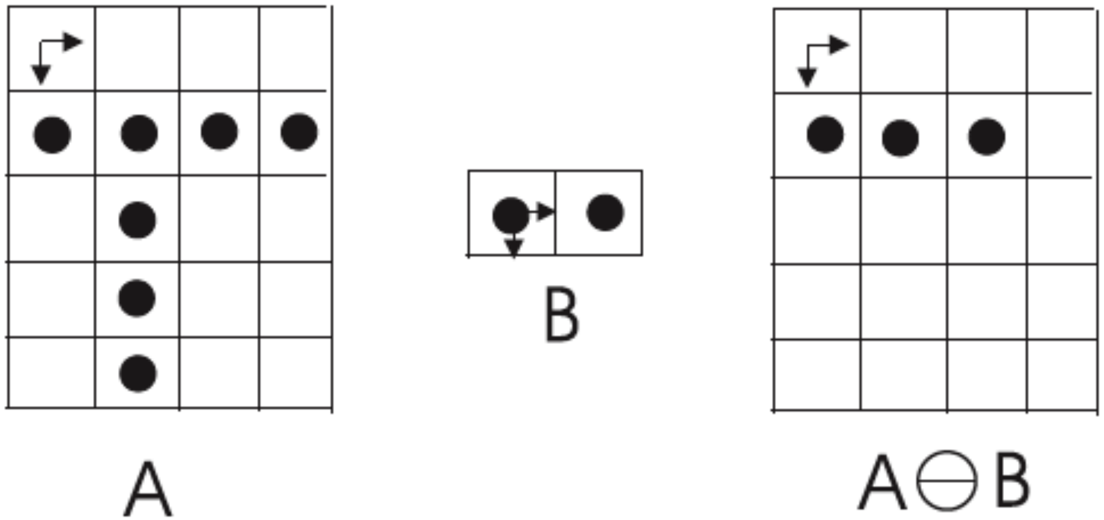
Erosion

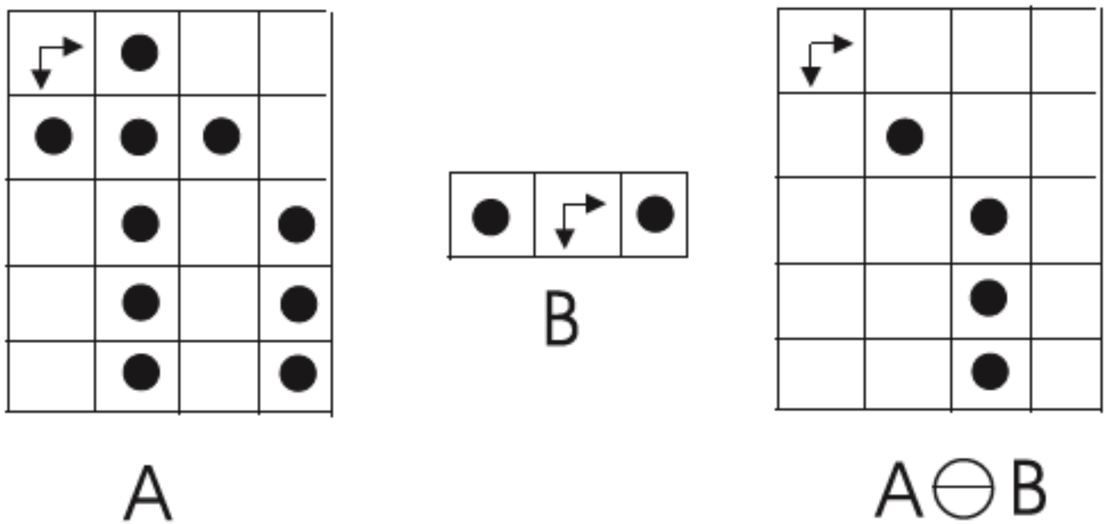
- Does the structuring element **fit the set?**

- Erosion of a set A by structuring element B: all z in A such that B is in A when origin of B=z

shrink the object

$$A \ominus B = \{z \in E^2 : (B)_z \subseteq A\}$$


A
B
A⊖B


A
B
A⊖B

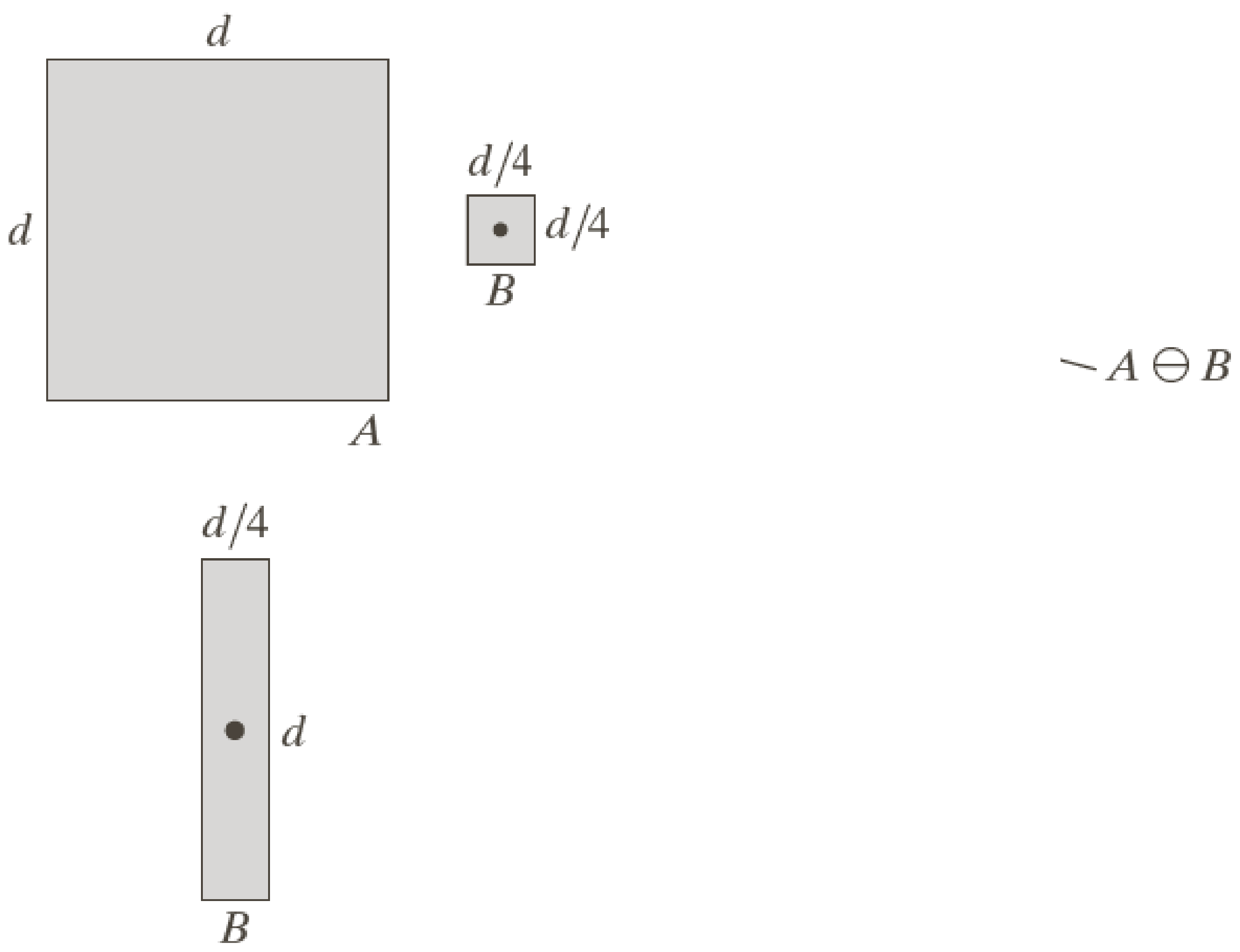
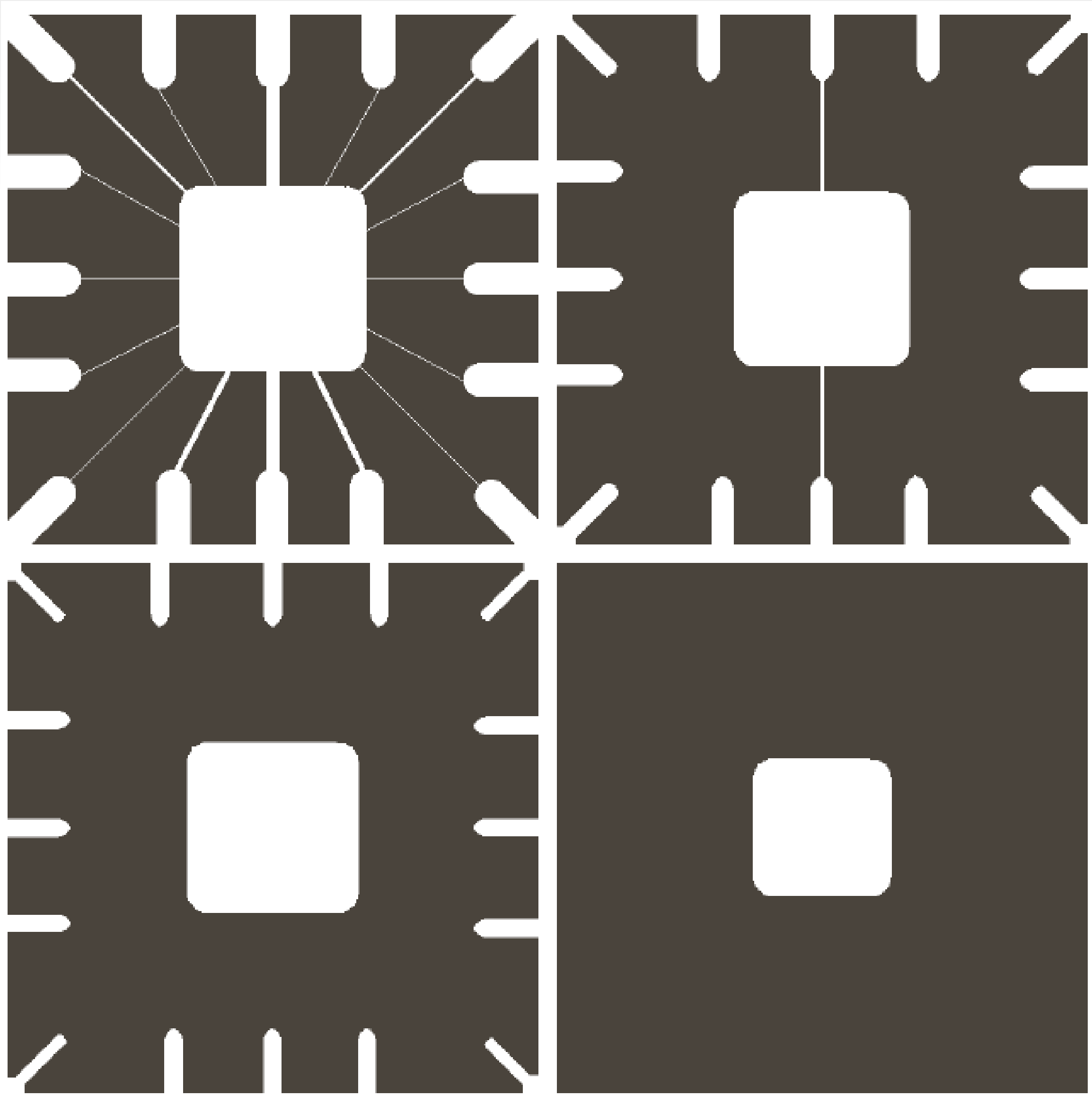


FIGURE 9.4 (a) Set A. (b) Square structuring element, B. (c) Erosion of A by B, shown shaded. (d) Elongated structuring element. (e) Erosion of A by B using this element. The dotted border in (c) and (e) is the boundary of set A, shown only for reference.



a b
c d

FIGURE 9.5 Using erosion to remove image components. (a) A 486×486 binary image of a wire-bond mask. (b)–(d) Image eroded using square structuring elements of sizes 11×11, 15×15, and 45×45, respectively. The elements of the SEs were all 1s.

Erosion

- Properties
  - L'erosione non è commutativa

$$A \ominus B \neq B \ominus A$$

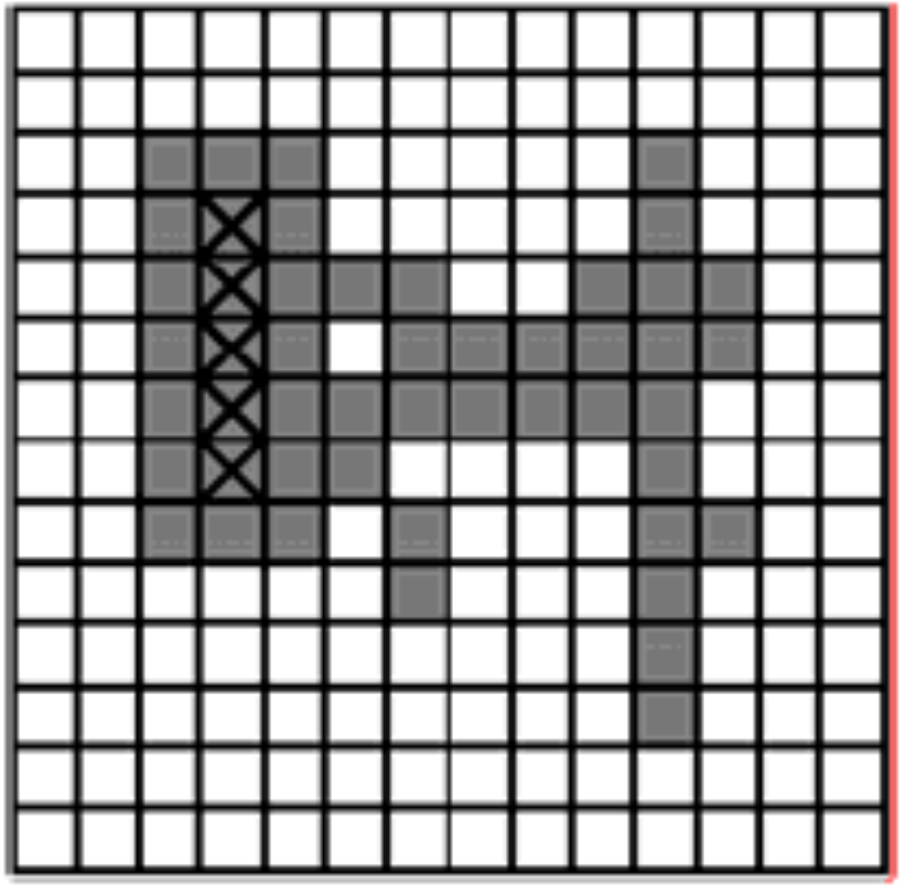
  - L'erosione è associativa quando l'elemento strutturante è decomponibile intermini di dilatazioni:

$$A \ominus (B \oplus C) = (A \ominus B) \ominus C$$

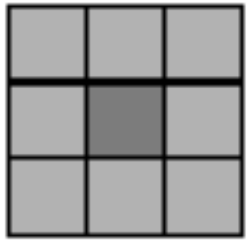
  - Se l'elemento strutturante contiene l'origine (O ∈ B) l'erosione è una trasformazione antiestensiva: l'insieme eroso è contenuto nell'insieme
  - L'erosione è una trasformazione crescente

$$A \subseteq C \Rightarrow A \ominus B \subseteq C \ominus B$$

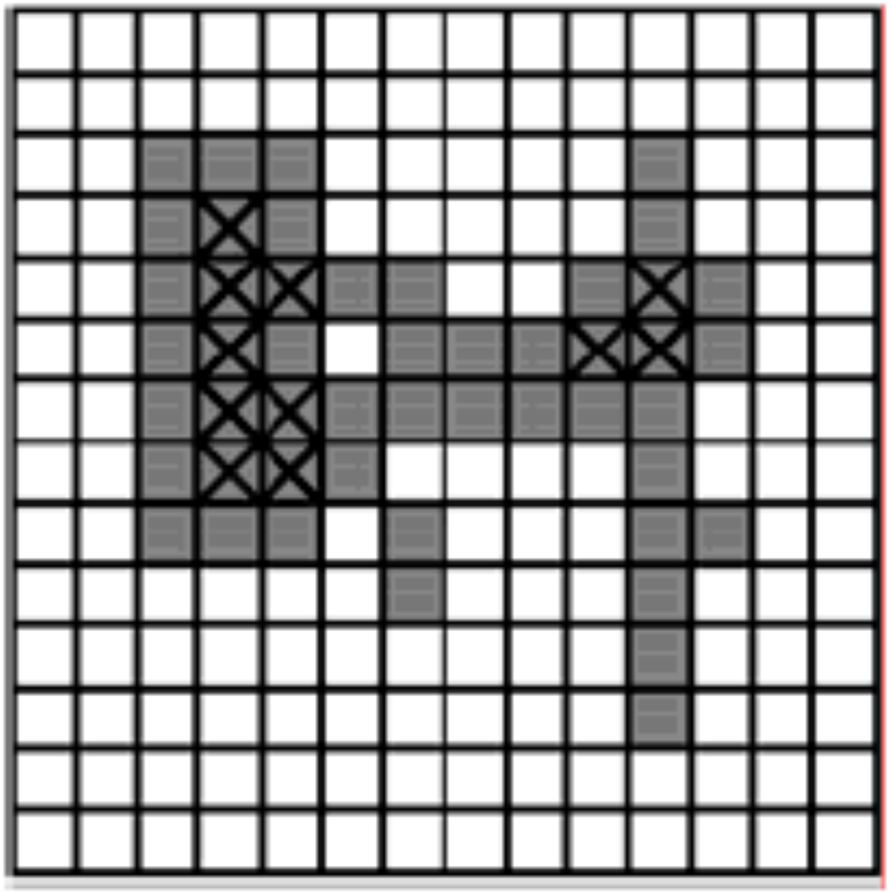
Erosion

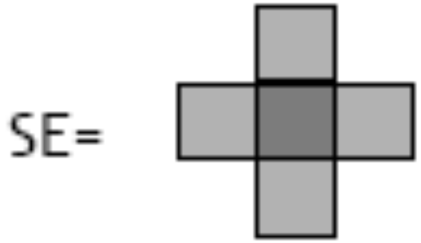


SE=



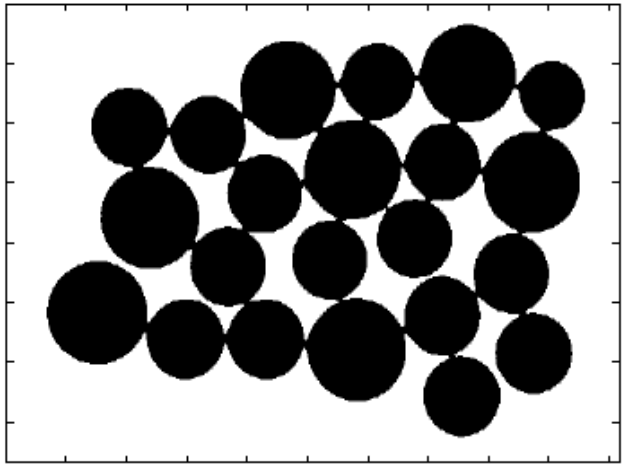
Erosion





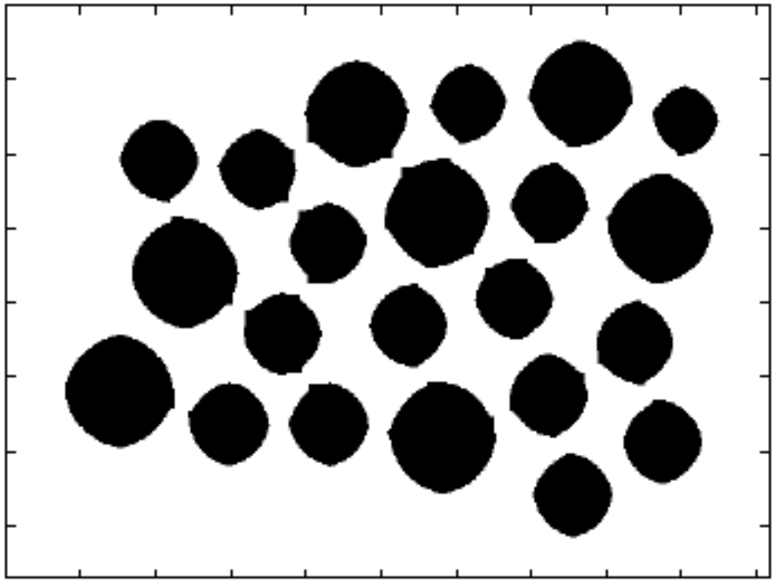
Erosion

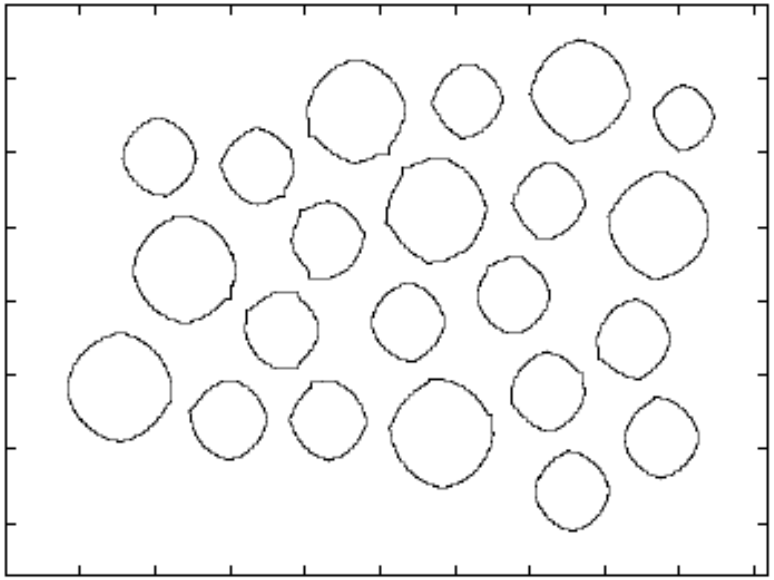
- Consideriamo ora l'immagine binaria seguente:



- A causa del valore troppo elevato della soglia alcuni oggetti che dovrebbero essere separati risultano connessi. Ciò può introdurre degli errori nelle elaborazioni successive (ad esempio, nel conteggio del numero di oggetti presenti nell'immagine).

Erosion





Dilation

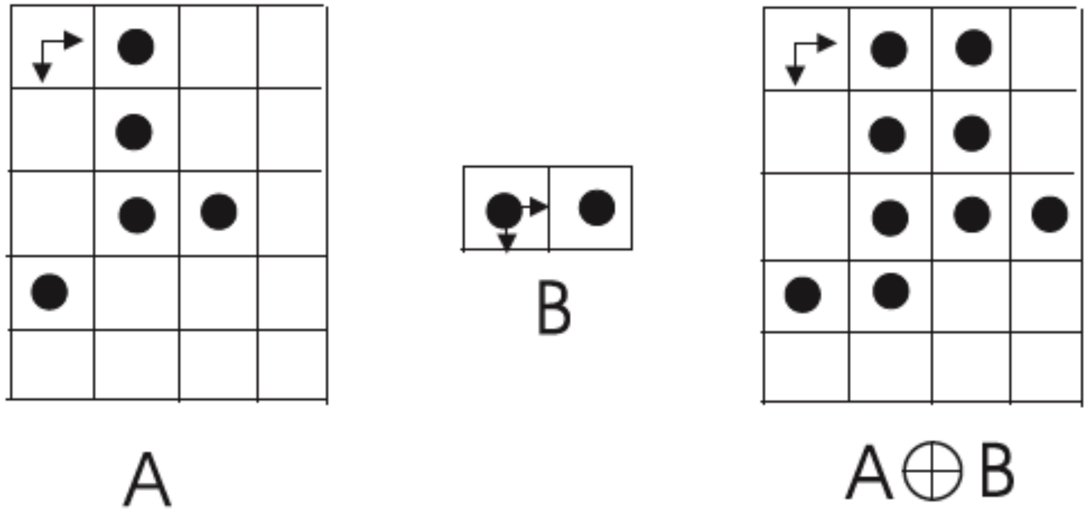
- Does the structuring element **hit the set?**

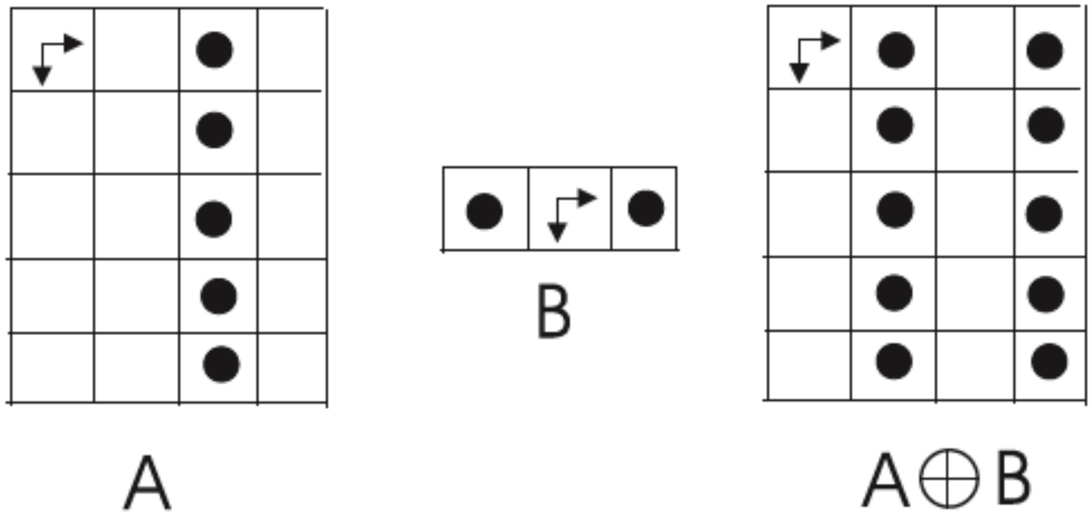
- Dilation of a set A by structuring element B: all z in A such that B hits A when origin of B=z

$$A \oplus B = \bigcup_{b \in B} A_b$$

grow the object

$$A \oplus B = \{c \in E^2 : \; c = a + b, \;\; a \in A \;\; \mathrm{e} \;\; b \in B\}$$


A
B
A⊕B


A
B
A⊕B

Dilation

- Properties

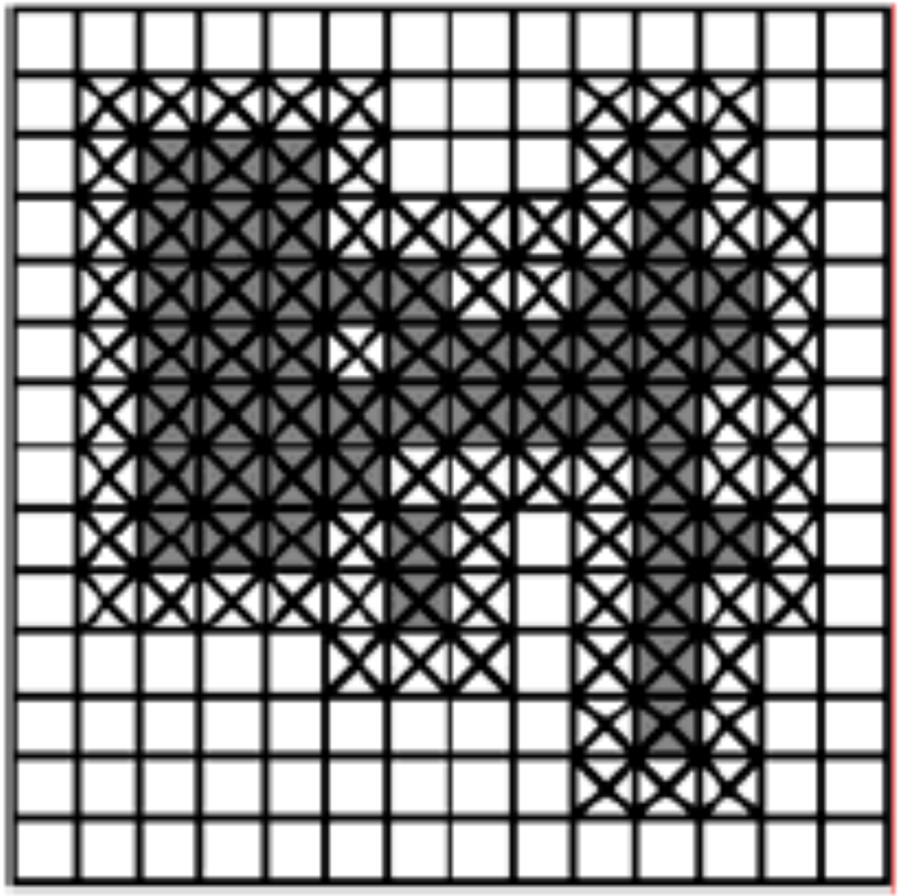
- La dilatazione è commutativa
  - $A \oplus B = B \oplus A$

- La dilatazione è associativa
  - $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

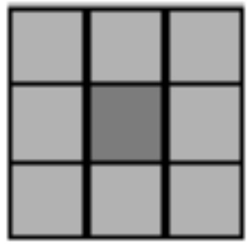
- Se l'elemento strutturante contiene l'origine ($O \in B$) la dilatazione è una trasformazione estensiva: l'insieme originario è contenuto nell'insieme dilatato ($A \subseteq A \oplus B$)

- La dilatazione è una trasformazione crescente
  - $A \subseteq C \Rightarrow A \oplus B \subseteq C \oplus B$

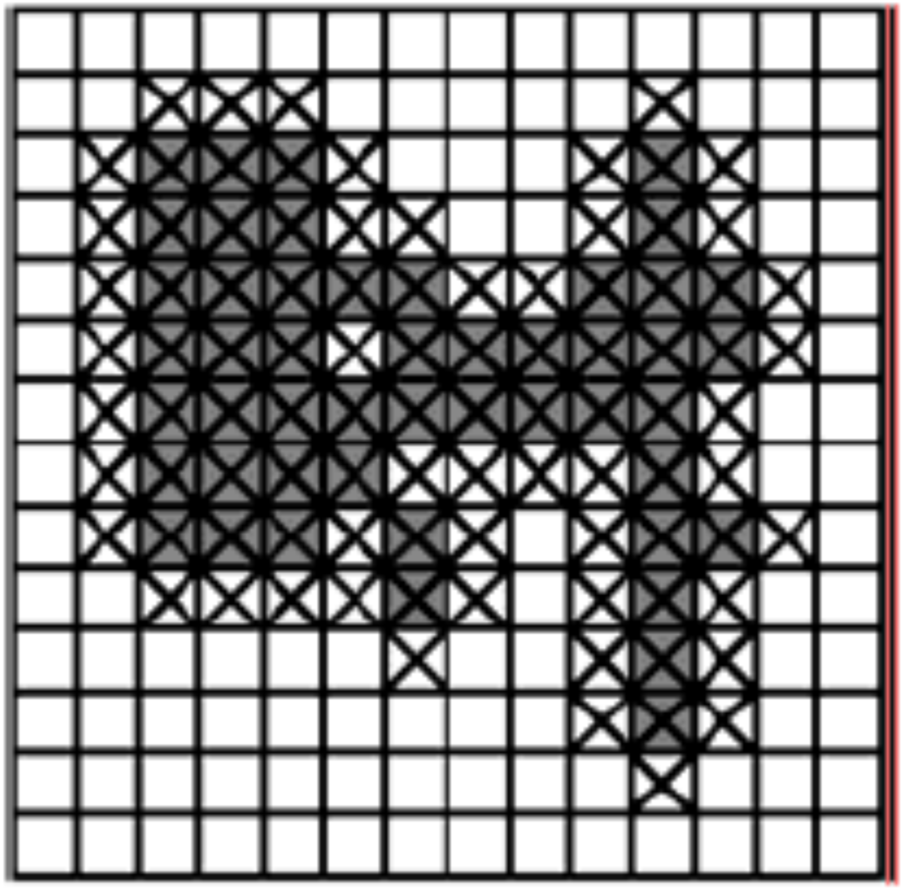
Dilation



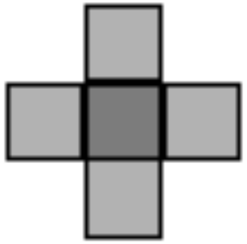
SE=



Dilation

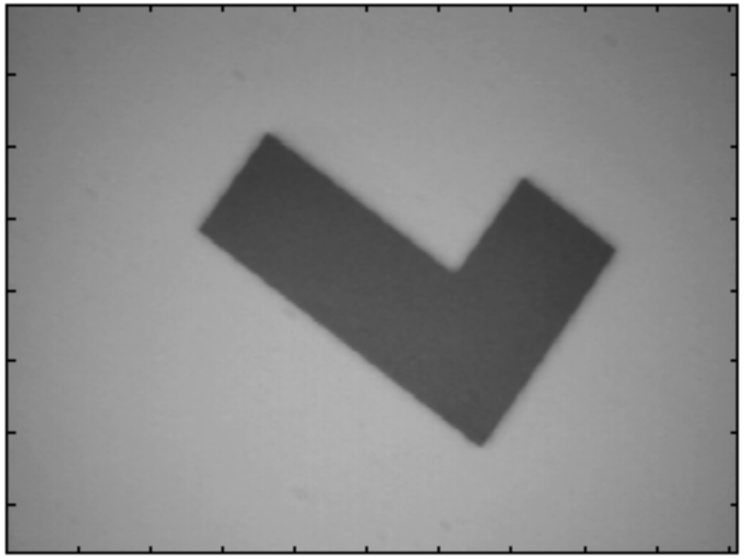


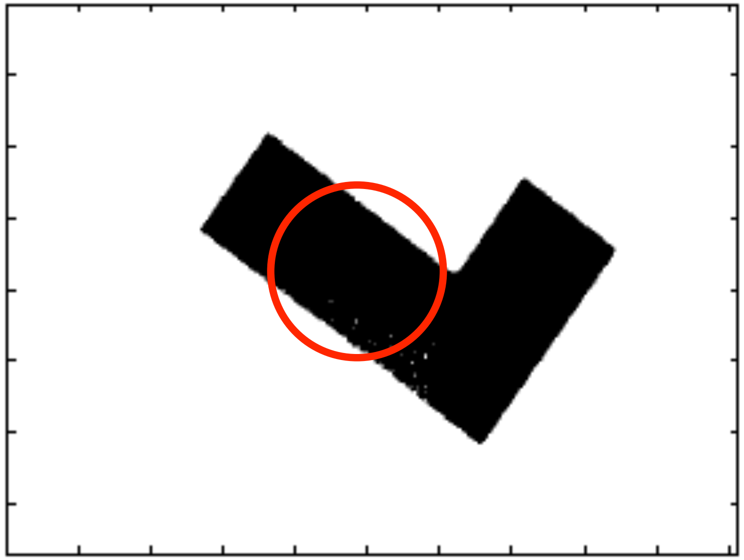
SE=



Dilation

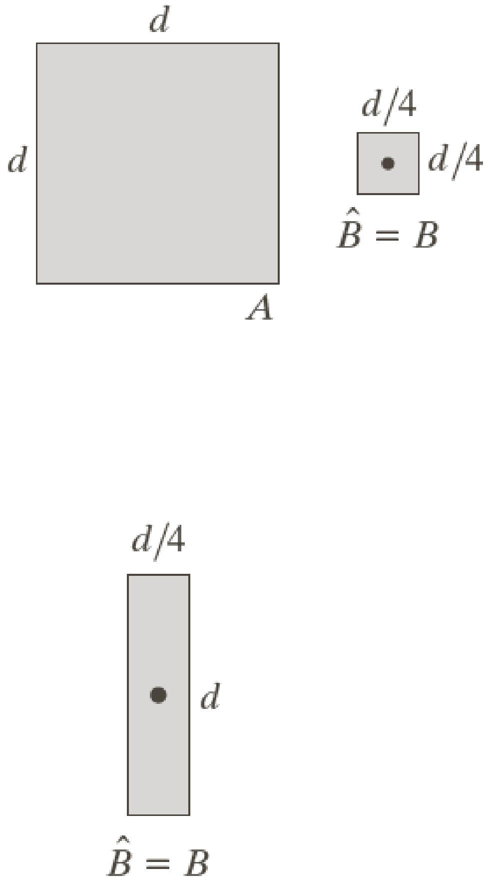
- Supponiamo ora di binarizzare l'immagine seguente utilizzando una soglia troppo bassa:





- A causa del valore troppo basso di soglia l'oggetto presenta delle lacune

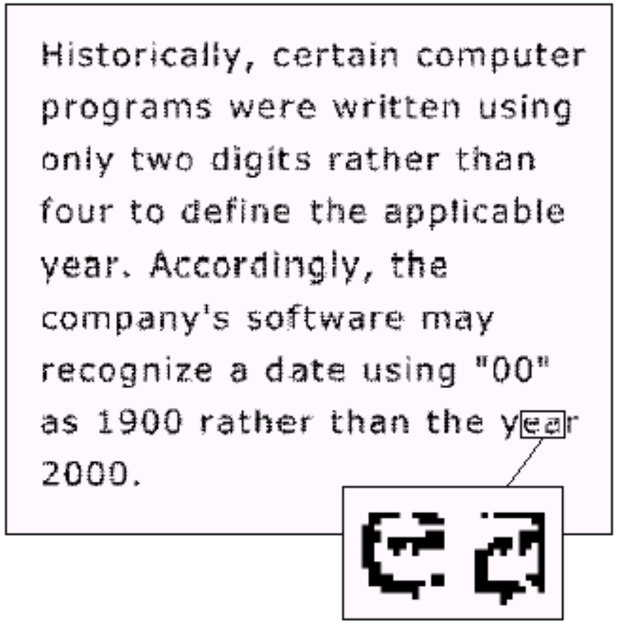
Examples of Dilation (1)

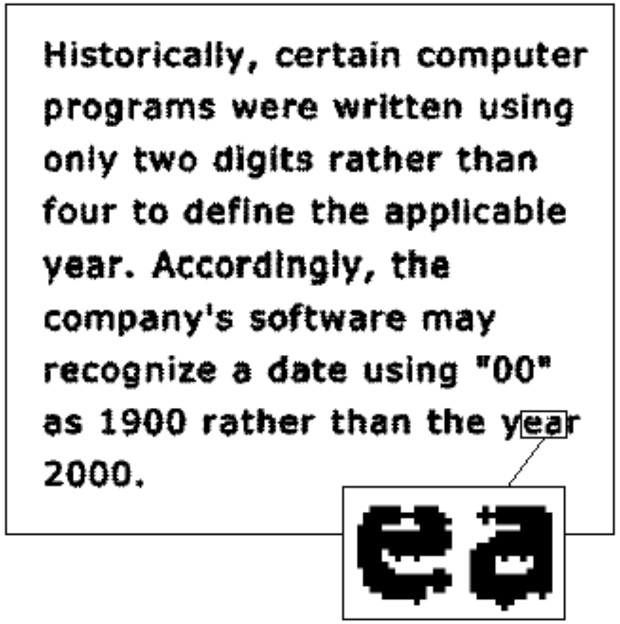


a b c
d e

FIGURE 9.6
(a) Set A.
(b) Square structuring element (the dot denotes the origin).
(c) Dilation of A by B, shown shaded.
(d) Elongated structuring element. (e) Dilation of A using this element. The dotted border in (c) and (e) is the boundary of set A, shown only for reference

Dilation : Bridging gaps





0	1	0
1	1	1
0	1	0

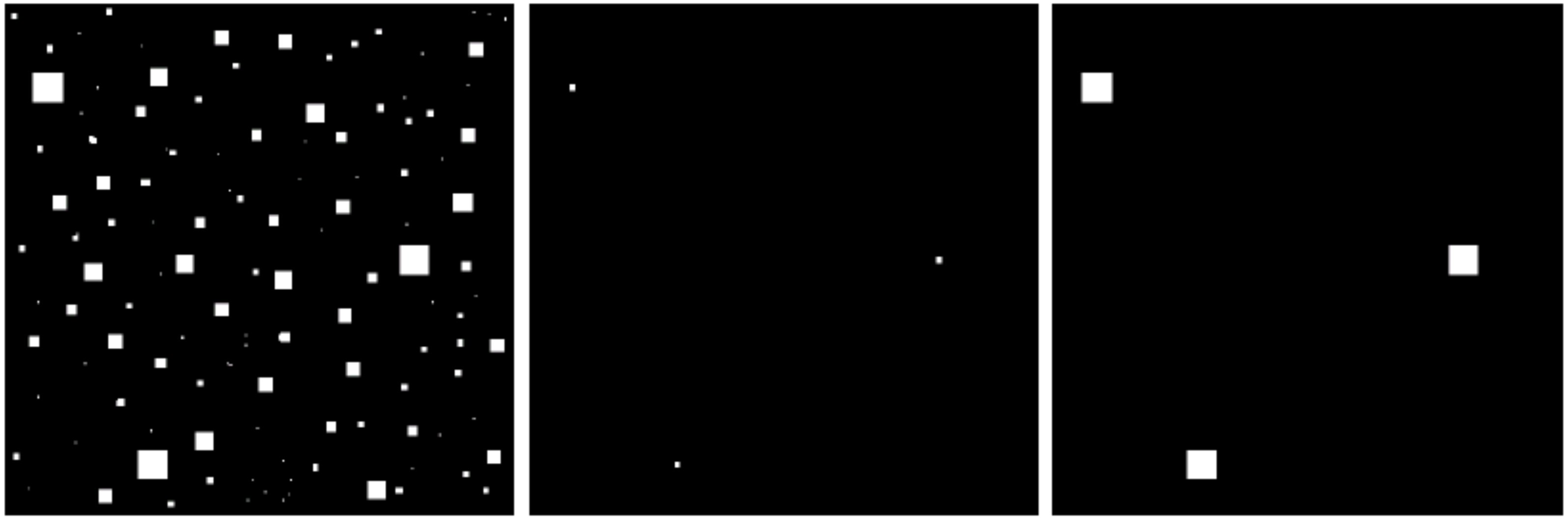
Usefulness

- Erosion
  - Removal of structures of certain shape and size, given by SE
- Dilation
  - Filling of holes of certain shape and size, given by SE

Combining erosion and dilation

- WANTED:
  - remove structures / fill holes
  - without affecting remaining parts
- SOLUTION:
  - combine erosion and dilation
  - (using same SE)

Erosion : eliminating irrelevant detail



a b c

FIGURE 9.7 (a) Image of squares of size 1, 3, 5, 7, 9, and 15 pixels on the side. (b) Erosion of (a) with a square structuring element of 1's, 13 pixels on the side. (c) Dilation of (b) with the same structuring element.

structuring element B = 13x13 pixels of gray level 1

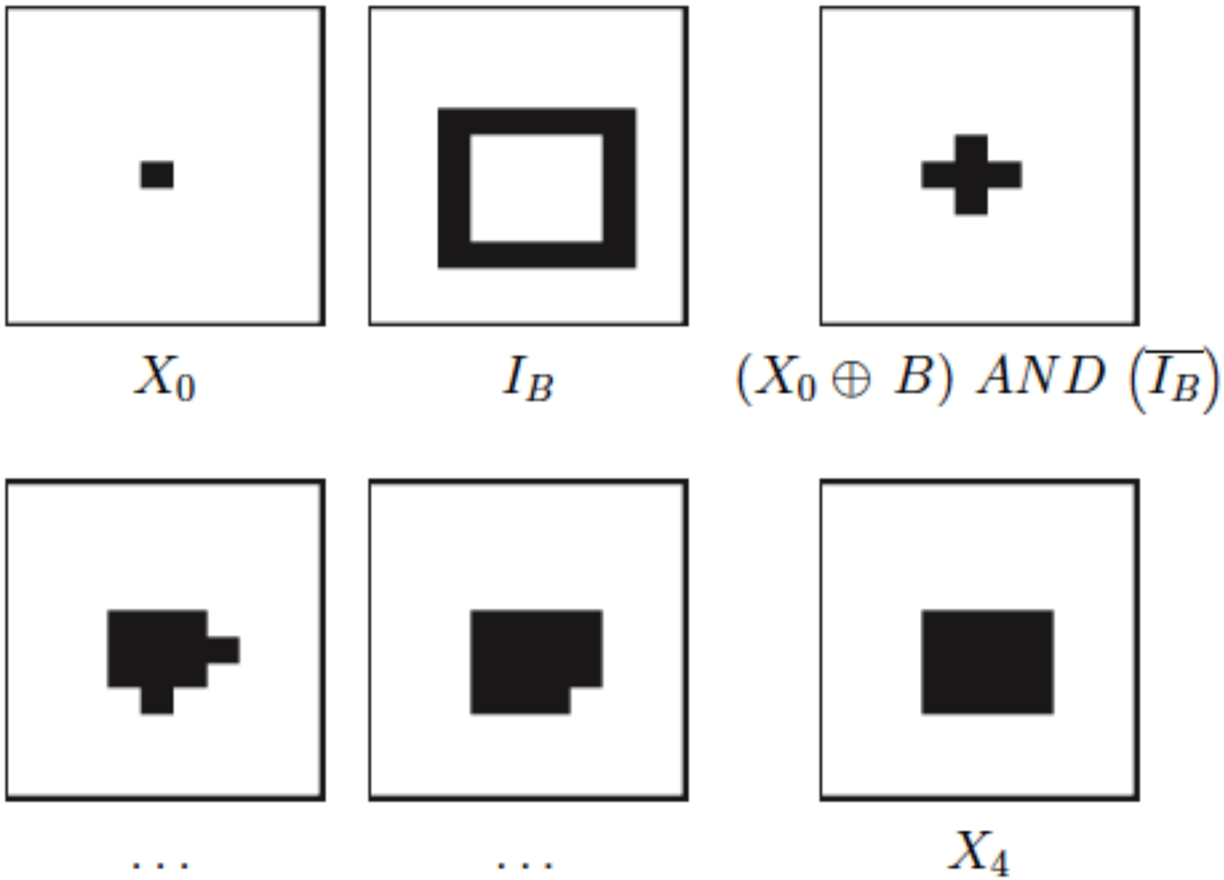
Dilation: filling

- Infine, la dilatazione viene usata insieme agli operatori logici per eseguire operazioni morfologiche più complesse.
  - Un esempio è l'operazione di filling, che ricostruisce le regioni associate agli oggetti (immagine binaria I_o) "riempiendo" i contorni estratti mediante un edge detector. Supponendo di aver estratto i contorni (immagine binaria I_B) e di conoscere almeno un pixel appartenente all'oggetto (immagine binaria X_0), è possibile ricostruire l'oggetto calcolando iterativamente la relazione:
  - dove con B si è indicato l'elemento strutturante

$$X_{n+1} = (X_n \oplus B)\ AND\ \left(\overline{I_B}\right) \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Dilation: filling (cont.)

– Quando il calcolo della relazione converge ($X_{n+1} = X_n$) *si può ottenere lo dalla relazione:* $I_o = (X_n)$ *OR* (I_B)

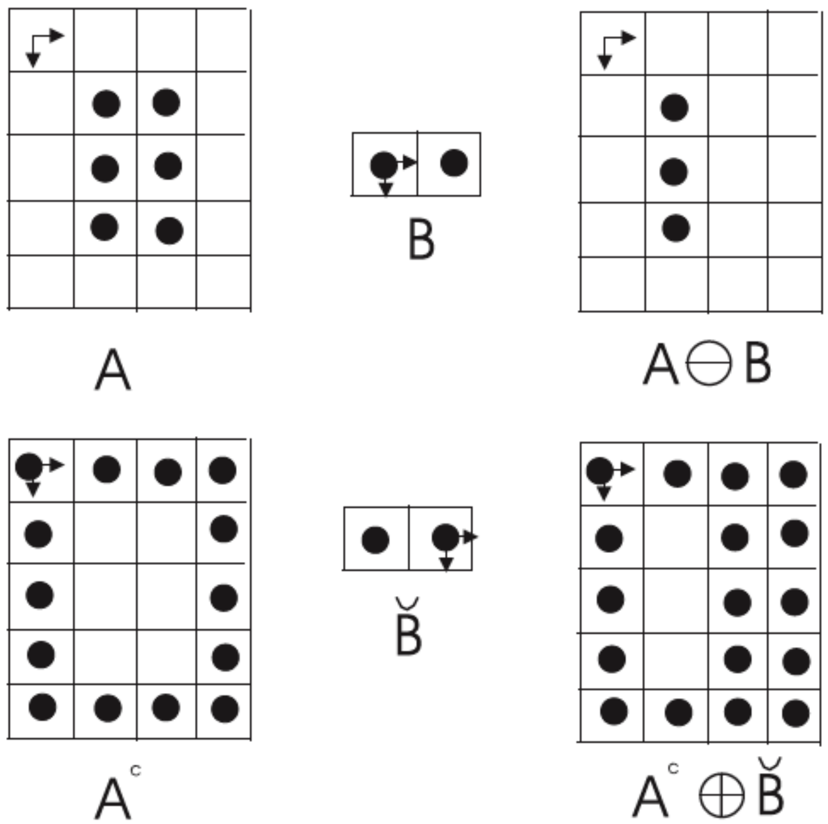


Relazione di dualità fra erosione e dilatazione

- Detto $\breve{B} = \left\{ \breve{b} : \ \breve{b} = -b, \ b \in B \right\}$

- *In generale vale che*

$$
\begin{aligned}
(A \oplus B)^c &= A^c \ominus \breve{B} \\
(A \ominus B)^c &= A^c \oplus \breve{B}
\end{aligned}
$$



Relazione di dualità fra erosione e dilatazione

- Se B è simmetrico

$$
\begin{aligned}
(A \oplus B)^c &= A^c \ominus B \\
(A \ominus B)^c &= A^c \oplus B
\end{aligned}
$$

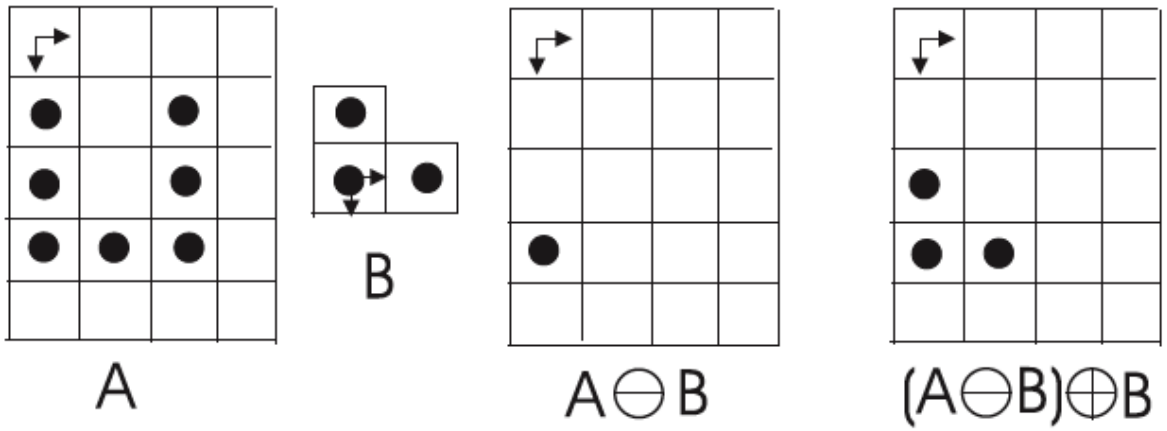
- quindi la dilatazione dell'oggetto è "equivalente" all'erosione dello sfondo e l'erosione dell'oggetto è "equivalente" alla dilatazione dello sfondo.
  - Le operazioni di erosione e dilatazione per uno stesso elemento strutturante possono essere impiegate in sequenza al fine di eliminare dall'immagine binaria le parti aventi forma "diversa" da quella dell'elemento strutturante senza distorcere le parti che invece vengono mantenute.

Opening

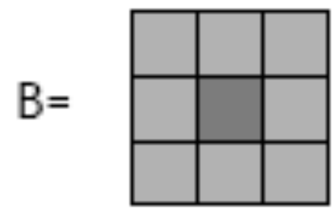
Erosion followed by dilation, denoted $\circ$

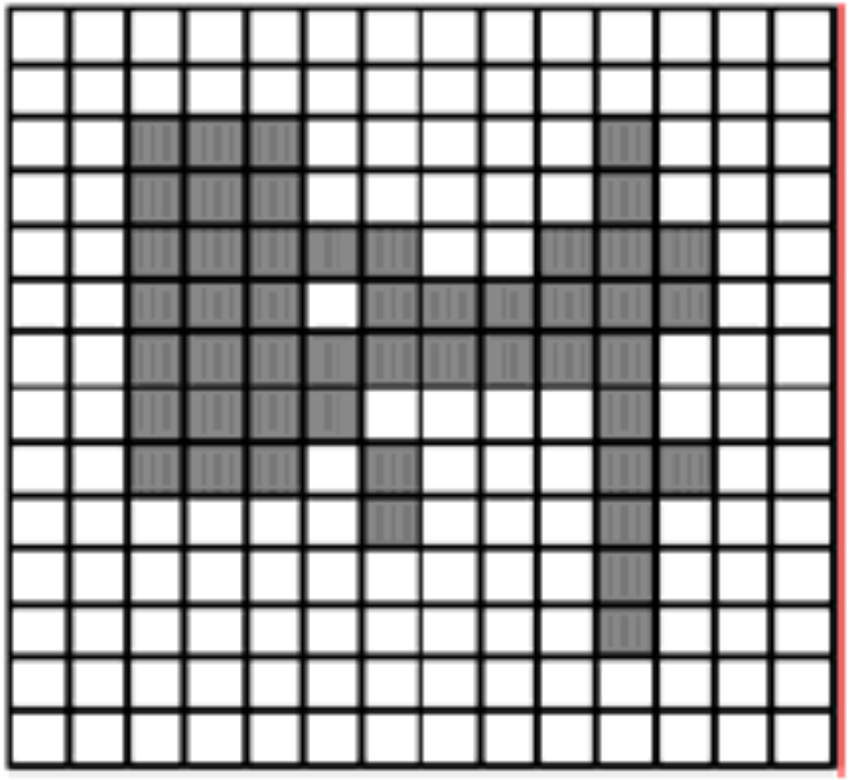
$$A \circ B = (A \ominus B) \oplus B$$

- eliminates protrusions
- breaks necks
- smoothes contour

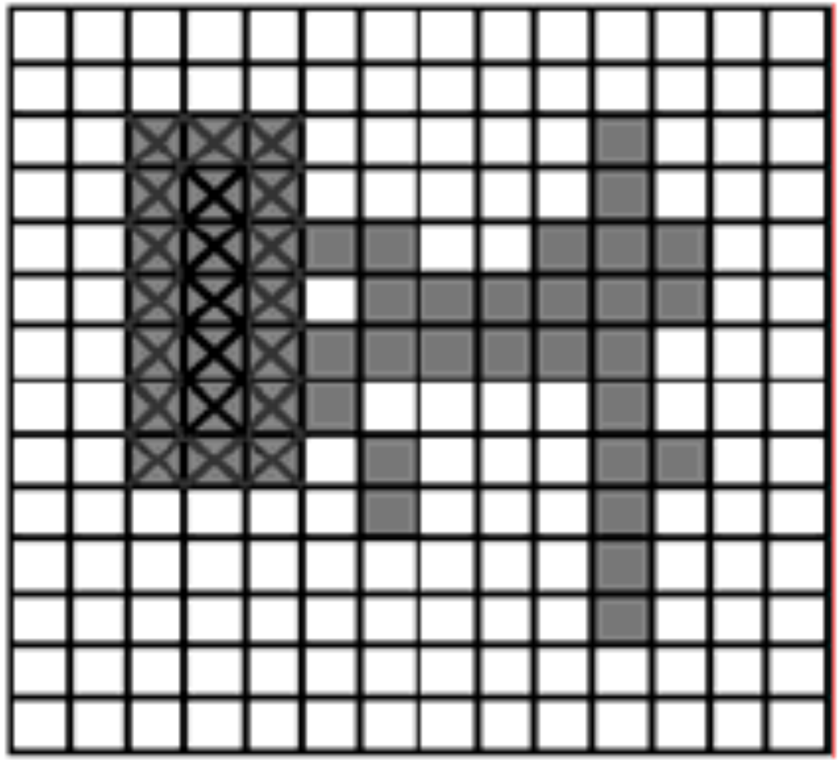


Opening



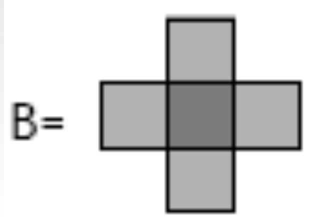


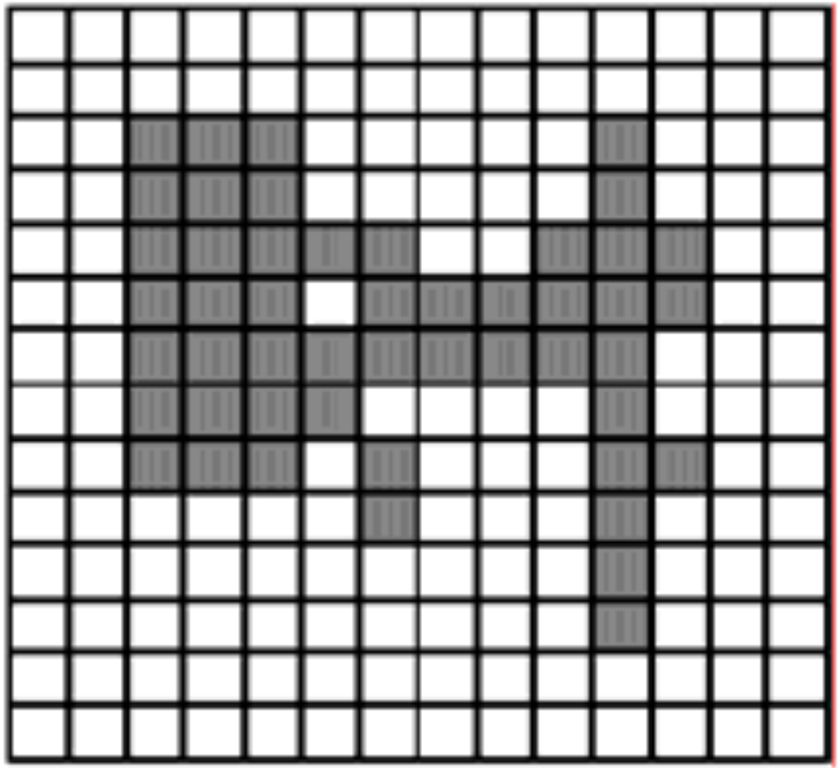
A



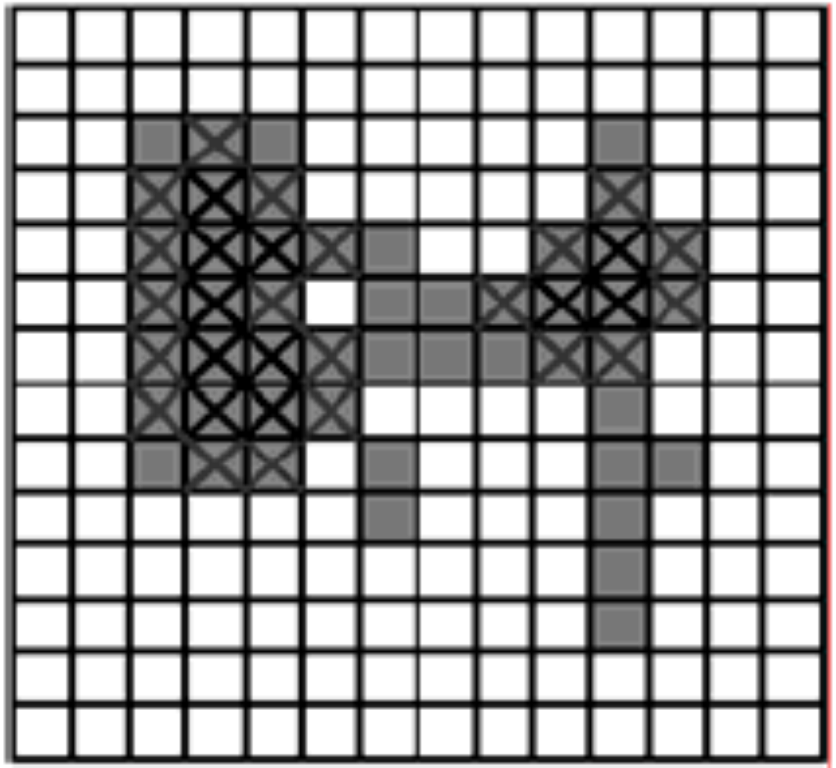
A⊖B A∘B

Opening



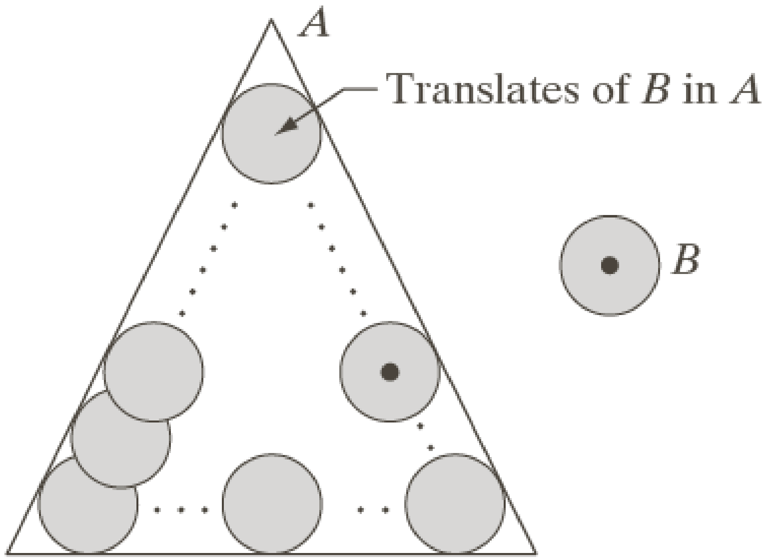


A



$A \ominus B$ $A \circ B$

Example: Opening



a b c d

FIGURE 9.8 (a) Structuring element B “rolling” along the inner boundary of A (the dot indicates the origin of B). (b) Structuring element. (c) The heavy line is the outer boundary of the opening. (d) Complete opening (shaded). We did not shade A in (a) for clarity.

Closing

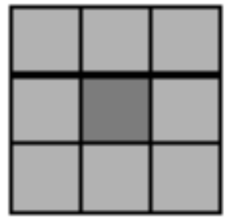
Dilation followed by erosion, denoted •

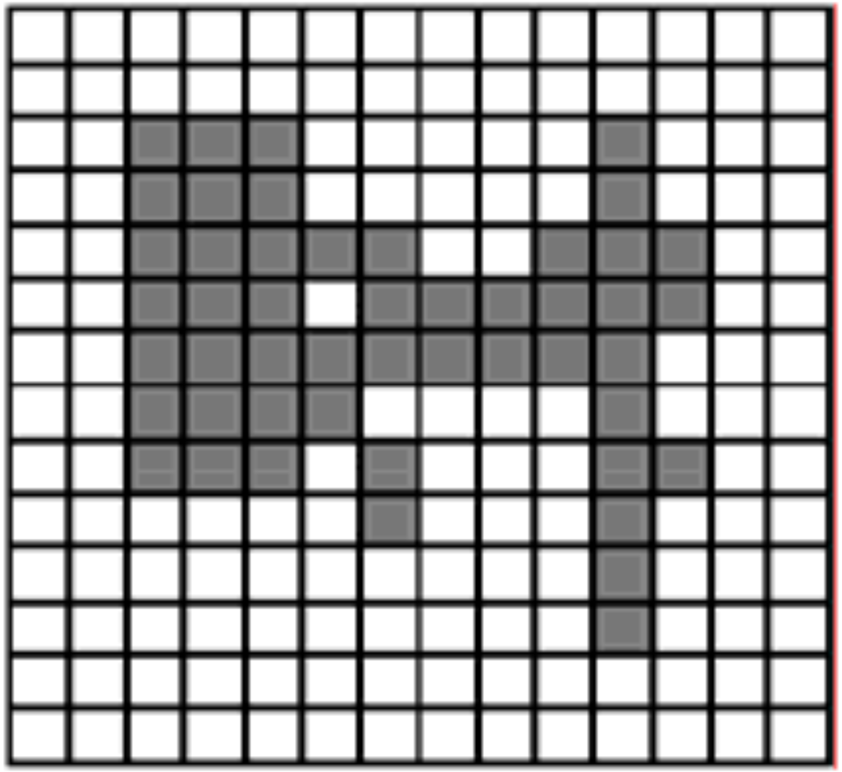
$$A \bullet B = (A \oplus B) \ominus B$$

- smooth contour
- fuse narrow breaks and long thin gulfs
- eliminate small holes
- fill gaps in the contour

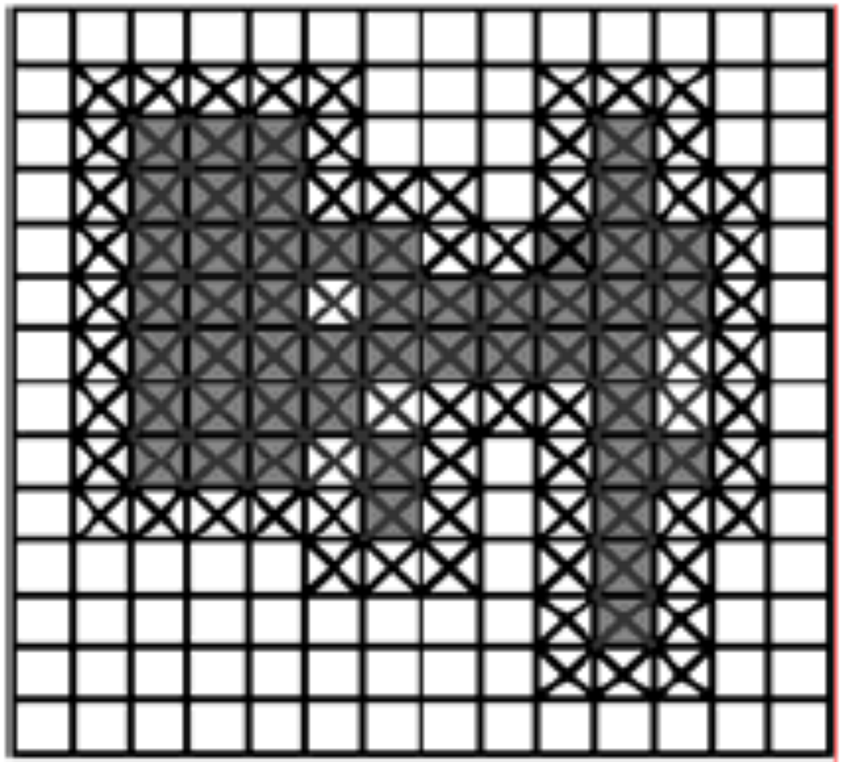
Closing

B=



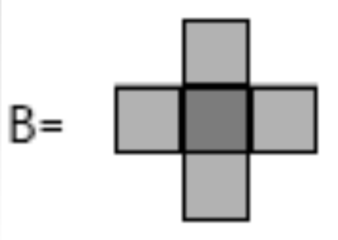


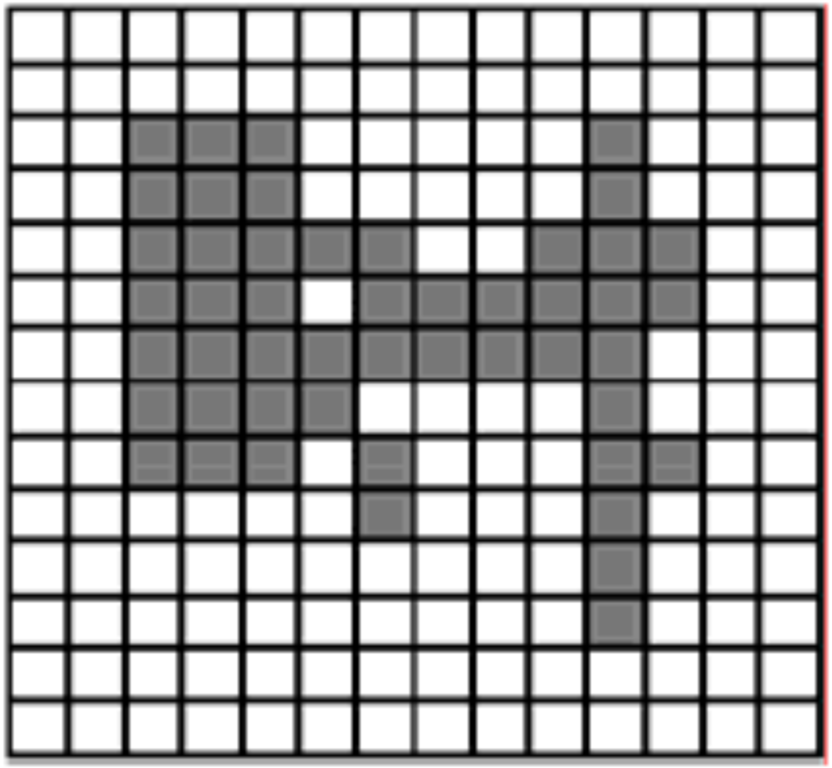
A



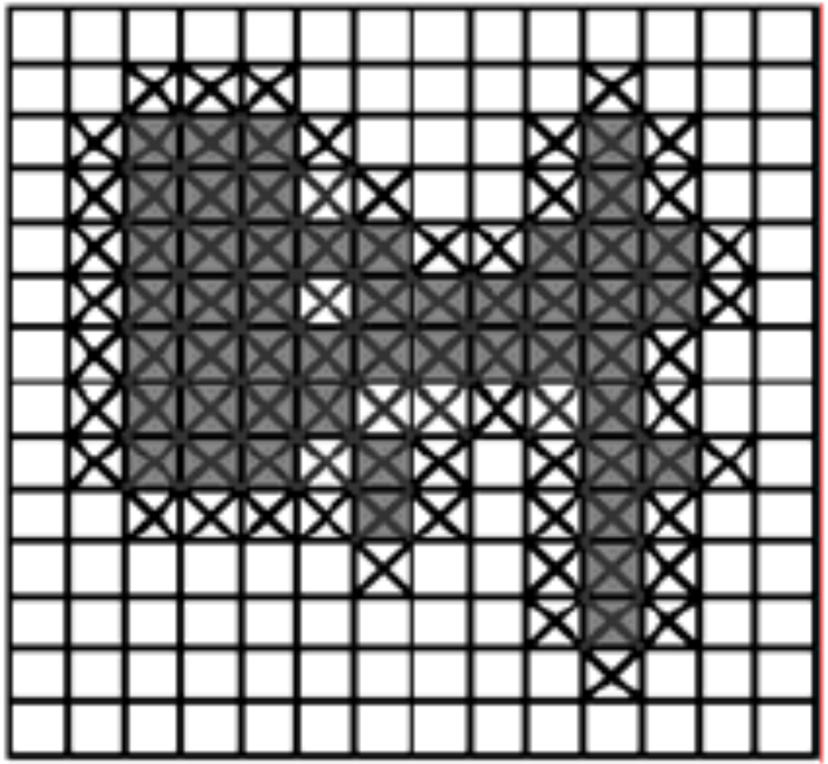
$A \oplus B$ $A \bullet B$

Closing



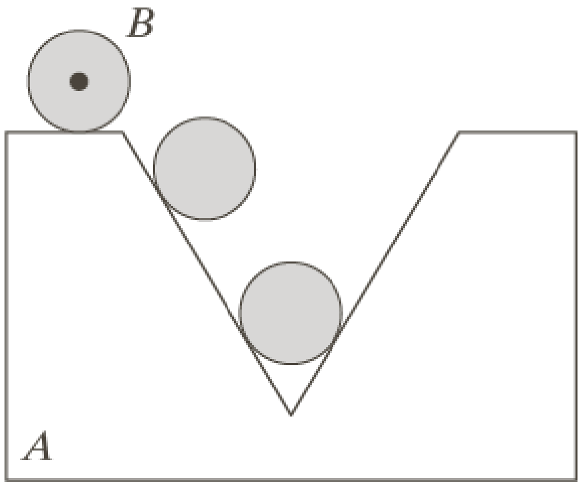


A



$A \oplus B$ $A \bullet B$

Example: Closing



a b c

FIGURE 9.9 (a) Structuring element B "rolling" on the outer boundary of set A. (b) The heavy line is the outer boundary of the closing. (c) Complete closing (shaded). We did not shade A in (a) for clarity.

Properties

Opening

(i) $A \circ B$ is a subset (subimage) of A

(ii) If C is a subset of D, then $C \circ B$ is a subset of $D \circ B$

(iii) $(A \circ B) \circ B = A \circ B$

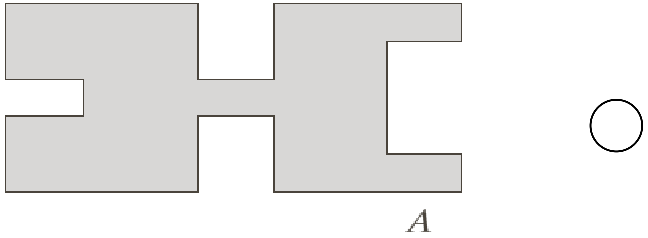
Closing

(i) A is a subset (subimage) of $A \bullet B$

(ii) If C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$

(iii) $(A \bullet B) \bullet B = A \bullet B$

Note: repeated openings/closings have no effect!



	a
b	c
d	e
f	g
h	i

FIGURE 9.10 Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The SE was not shaded here for clarity. The dark dot is the center of the structuring element.

Duality

- Opening and closing are dual with respect to complementation and reflection

$$(A \bullet B)^c = (A^c \circ \hat{B})$$

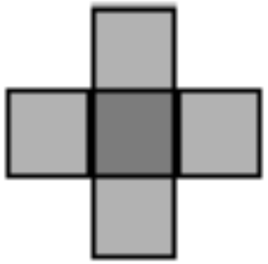
- Possiamo sfruttare la dualità per comprendere l'effetto dell'operazione di closing. Poichè il closing dell'oggetto è "equivalente" all'opening dello sfondo, l'operatore di closing esegue il "matching" fra l'elemento strutturante (o il suo riflesso) e le parti dello sfondo, preservando quelle uguali all'elemento strutturante (o al suo riflesso) ed eliminando (cioè annettendo all'oggetto) quelle diverse. Il sostanza l'oggetto viene "dilatato" annettendo le parti dello sfondo diverse da *B (o da* $\hat{B}$*).*

A

$A \ominus B$

$(A \ominus B)^C$

$$B = \hat{B}$$



A^C

$A^C \oplus B$

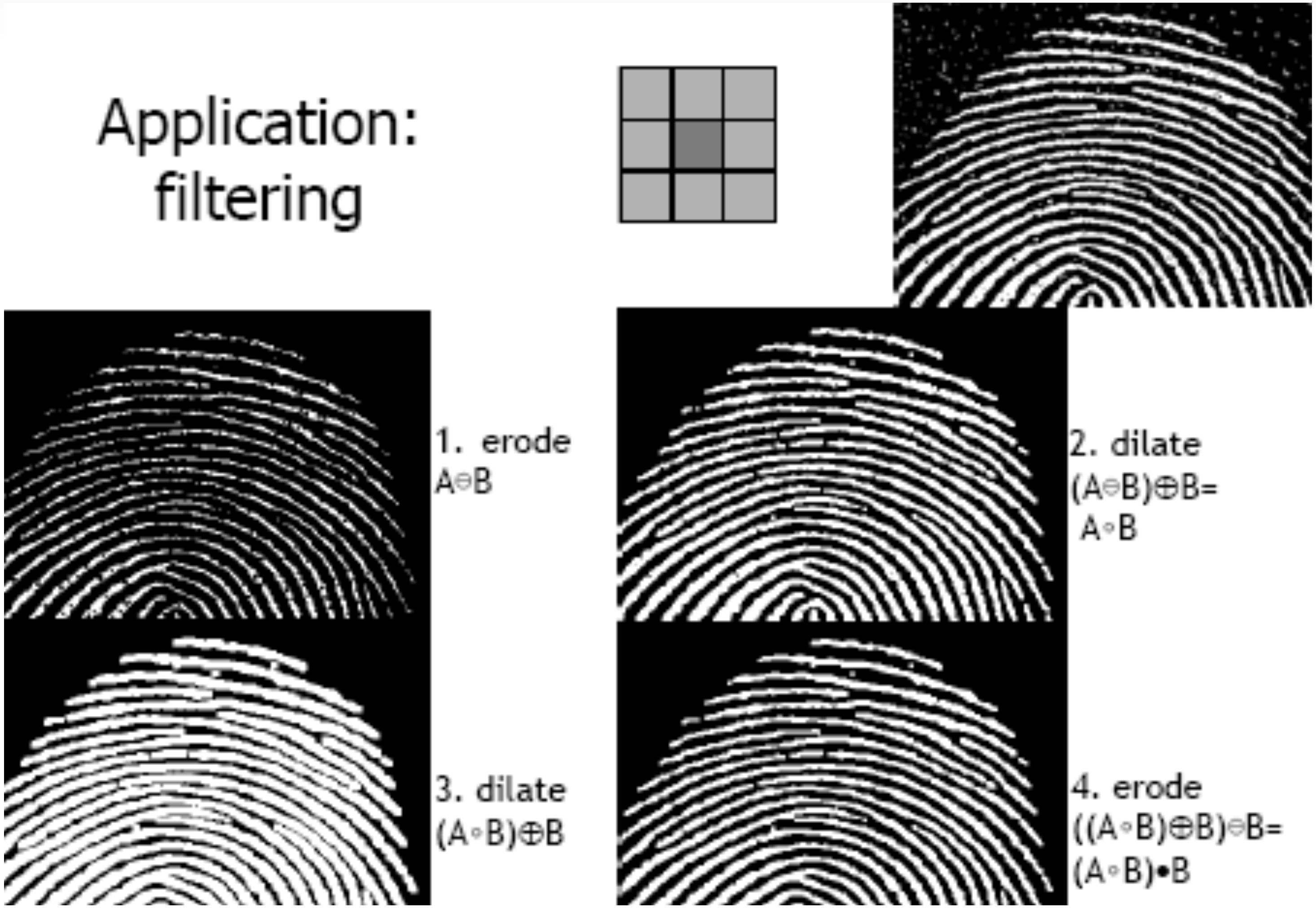
Usefulness: open & close

A

opening of A
→removal of small protrusions, thin connections, ...

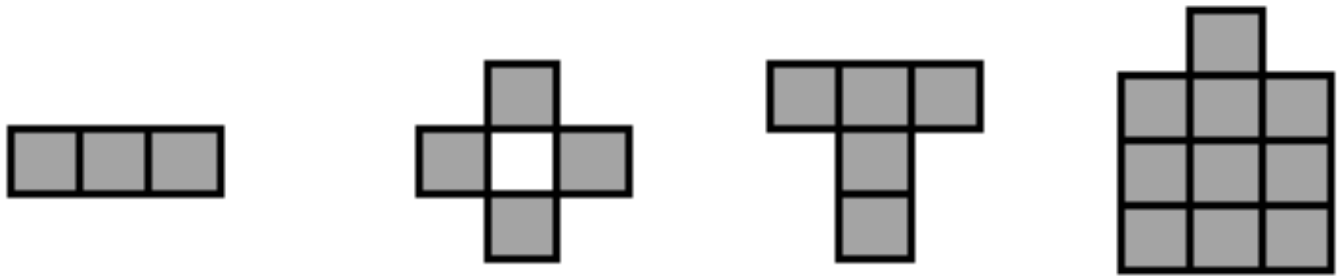
closing of A
→ removal of holes

Application: filtering



Hit-or-Miss Transformation ⊛ (HMT)

- find location of one shape among a set of shapes ”template matching”



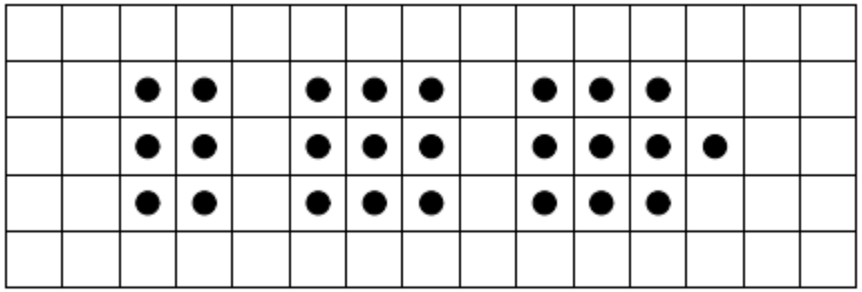
- composite SE: object part (B1) and background part (B2)
- does B1 ***fits the object while, simultaneously,*** B2 misses the object, i.e., ***fits the background?***

$$A \circledast B = (A \ominus X) \cap [A^c \ominus (W - X)]$$

Hit-or-Miss Transformation, example (1)

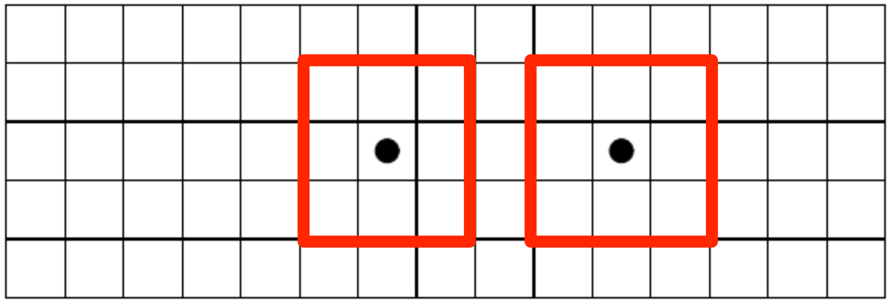
This is a powerful method for finding shapes in images. As with all other morphological algorithms, it can be defined entirely in terms of dilation and erosion; in this case, erosion only.

Suppose we wish to locate 3x3 square shapes, such as is in the centre of the following image



Hit-or-Miss Transformation, example (2)

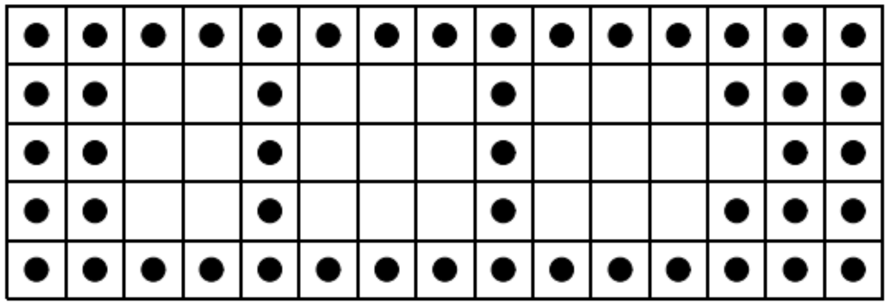
If we performed an erosion $A \ominus B$ with B being the square structuring element, we would obtain the result given in the following figure



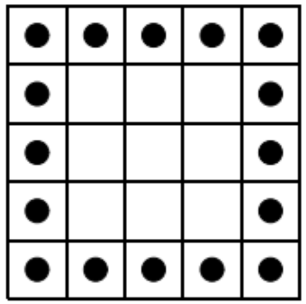
The erosion $A \ominus B$

The result contains two pixels, as there are exactly two places in A where B will fit. Now suppose we also erode the complement of A with a structuring element C which fits exactly around the 3x3 square. (we assume (0,0) is the centre of C)

$\overline{A}$:

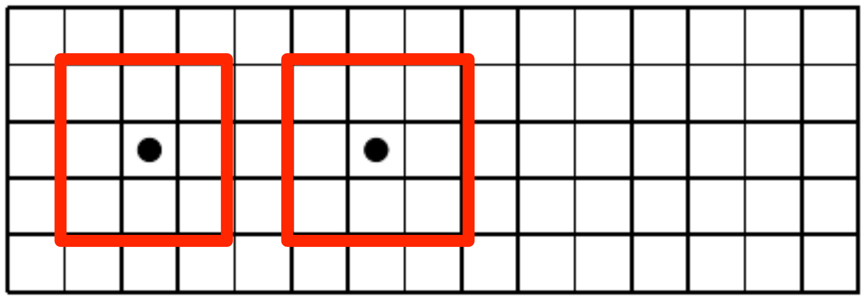


C:



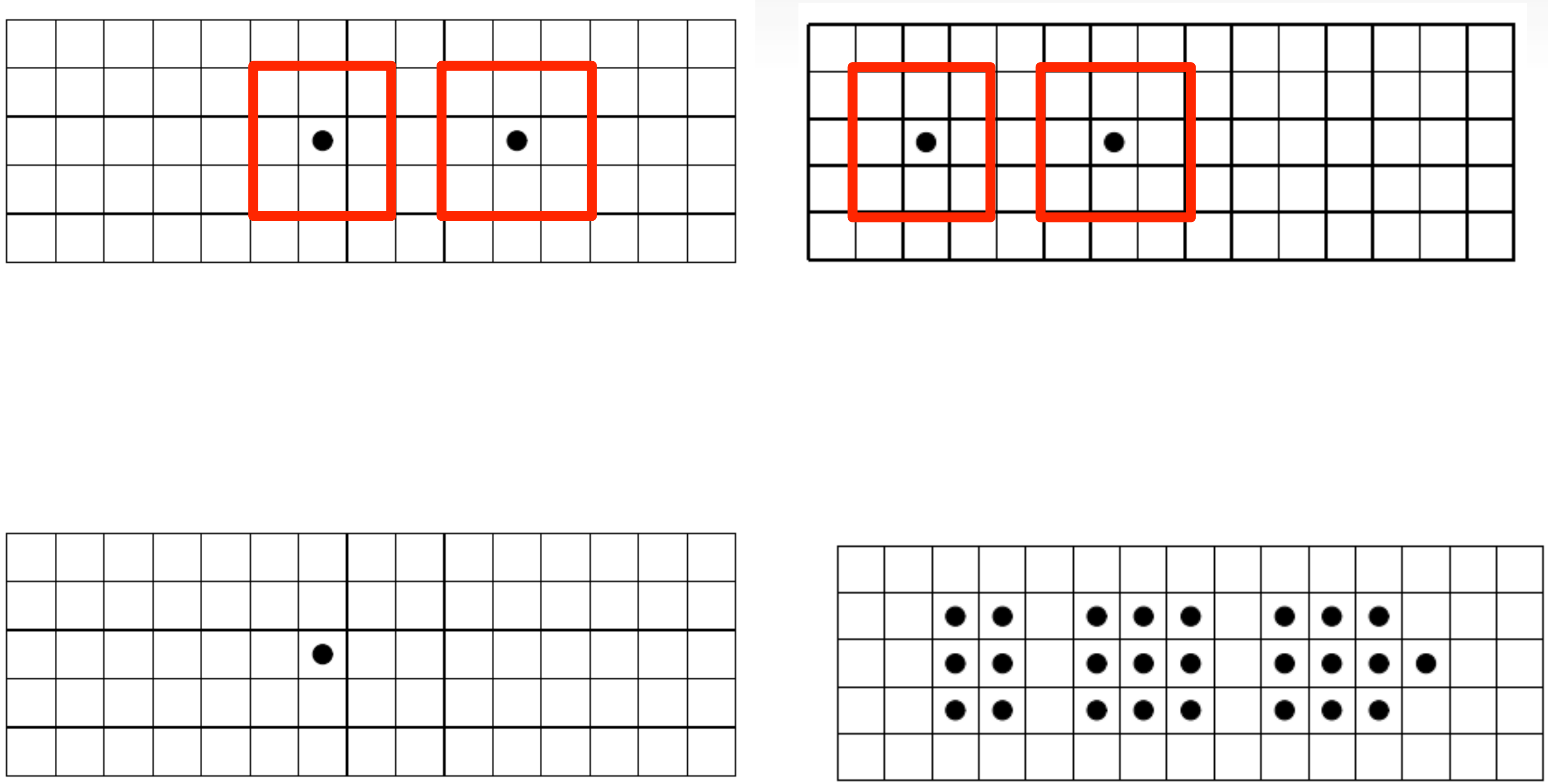
Hit-or-Miss Transformation, example (3)

If we now perform the erosion $\overline{A} \ominus C$ we would obtain the result



The **intersection** of the two erosion operations would produce just one pixel at the position of the centre of the 3x3 square in A, which is just what we want. If had contained more than one square, the final result would have been single pixels at the positions of the centres of each. This combination of erosions forms the hit-or-miss transform.

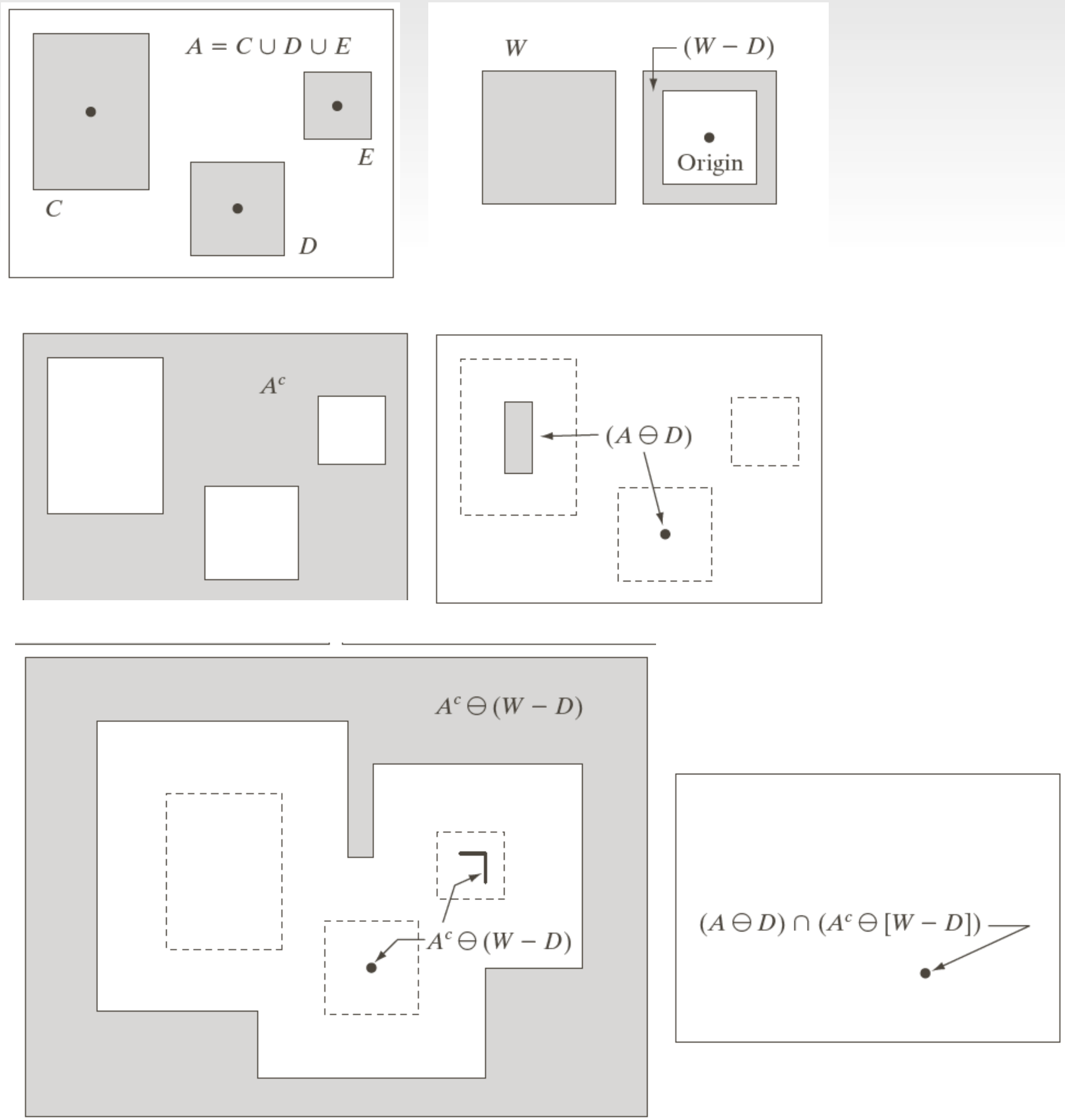
Hit-or-Miss Transformation, example (4)



Hit-or-Miss Transformation, example (5)

In general, if we are looking for a particular shape in an image, we design two structural elements: B1 which is the same shape, and B2 which fits around the shape. We then write B=(B1,B2) and the Hit-and-miss transform as

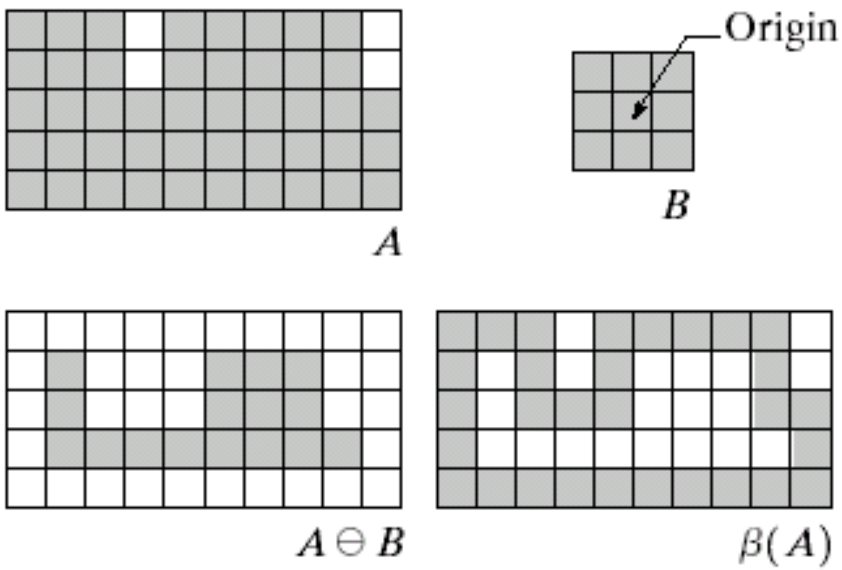
$$A \circledast B = (A \ominus B_1) \cap (\overline{A} \ominus B_2)$$



a b
c d
e
f

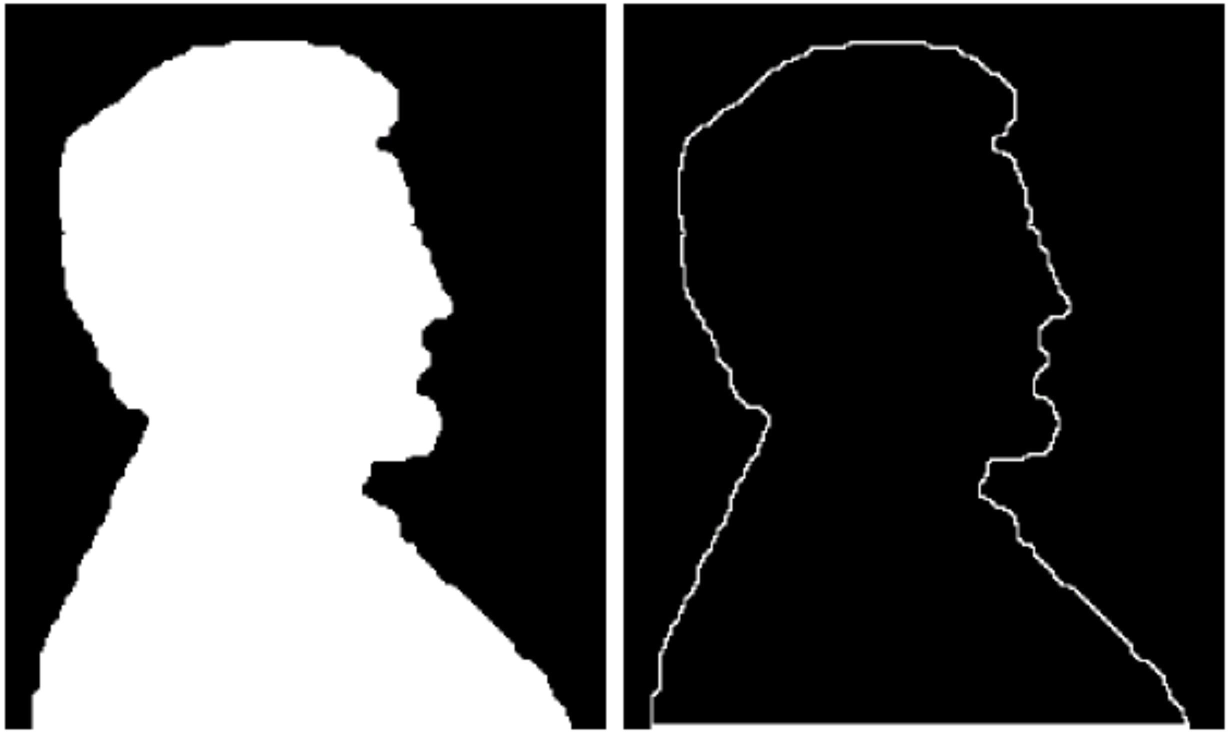
FIGURE 9.12 (a) Set A. (b) A window, W, and the local background of D with respect to W, $(W - D)$. (c) Complement of A. (d) Erosion of A by D. (e) Erosion of A^c by $(W - D)$. (f) Intersection of (d) and (e), showing the location of the origin of D, as desired. The dots indicate the origins of C, D, and E.

Boundary Extraction



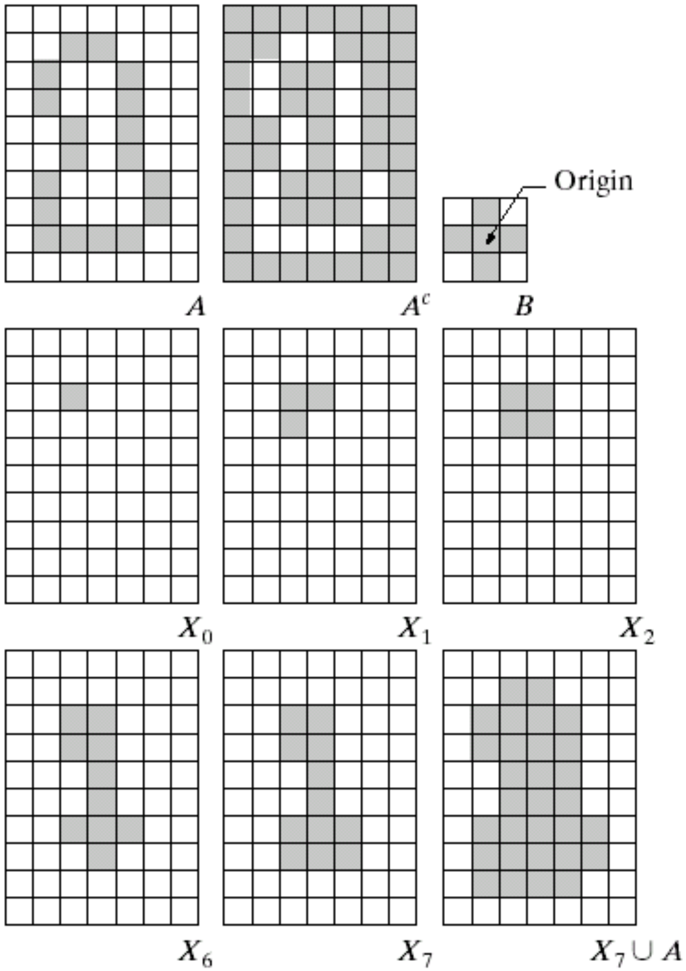
$$\beta(A) = A - (A \ominus B)$$

Example

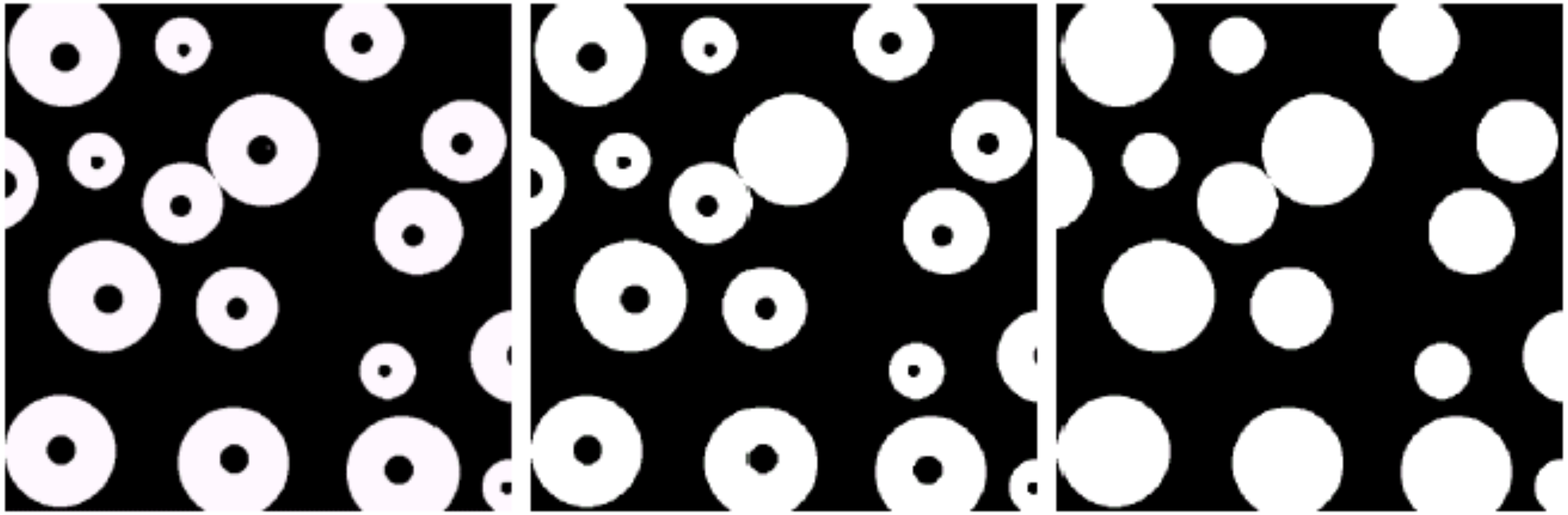


Region Filling

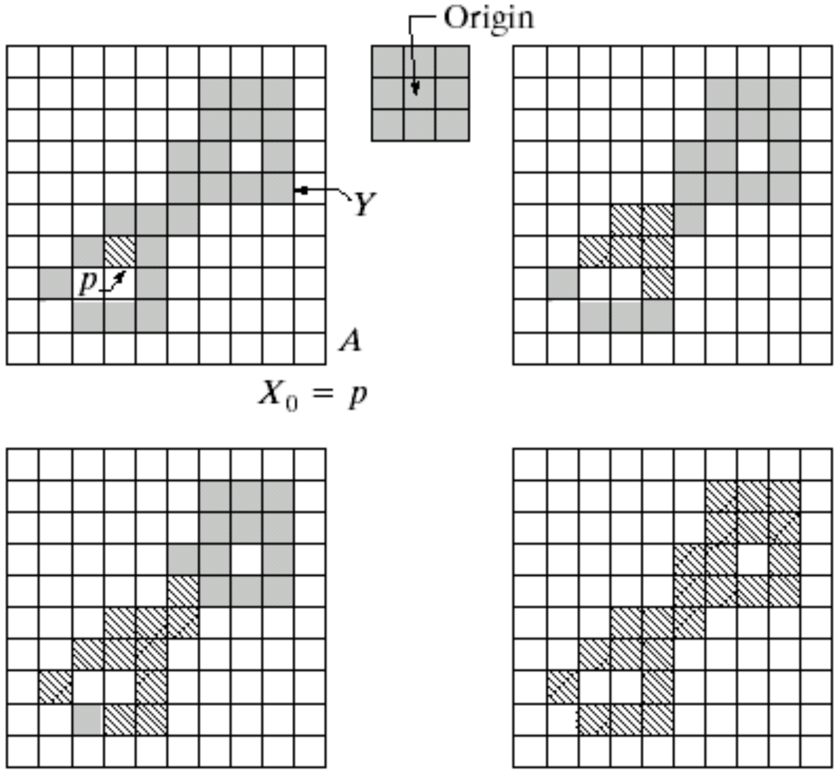
$$X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1,2,3,\ldots$$



Example



Extraction of connected components

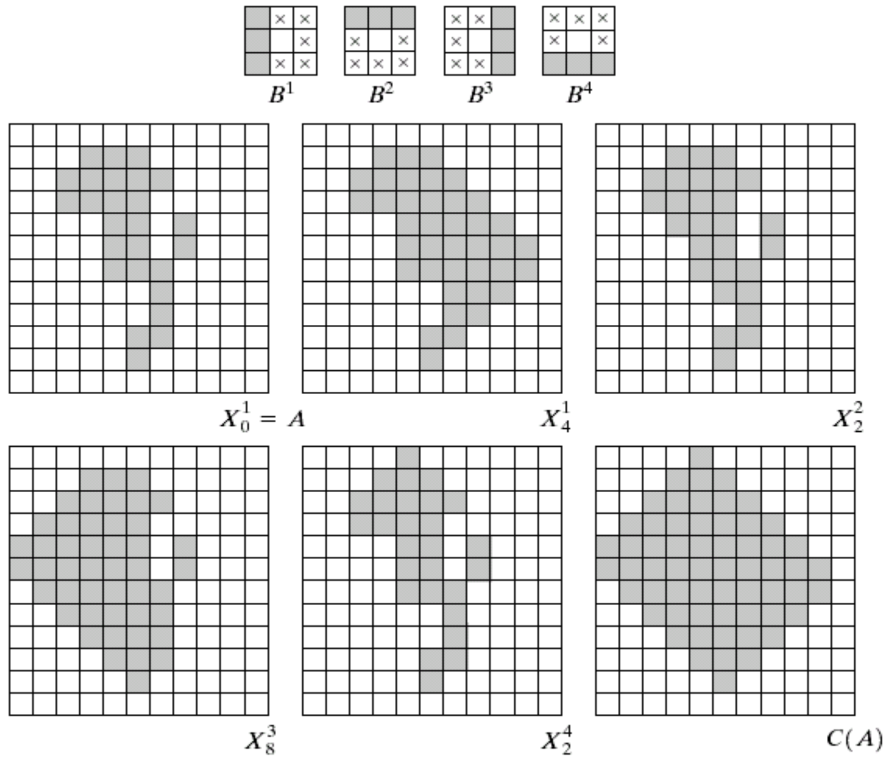


$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \ldots$$

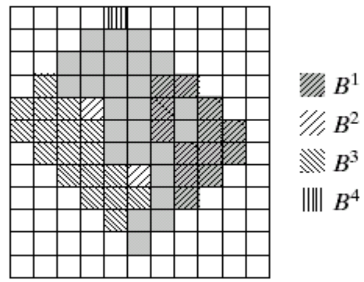
Convex hull

$$X_k^i = (X_k^i \circledast B^i) \cup A \quad i = 1,2,3,4 \;\text{ and } k = 1,2,3,\ldots$$

- A set *A* is is said to be convex if the straight line segment joining any two points in A lies entirely within A.



$$C(A) = \bigcup_{i=1}^{4} D^i$$



Thinning

- Thinning

  The thinning of a set A by a structuring element B, defined

$$A \otimes B = A - (A \circledast B)$$

$$= A \cap (A \circledast B)^c$$

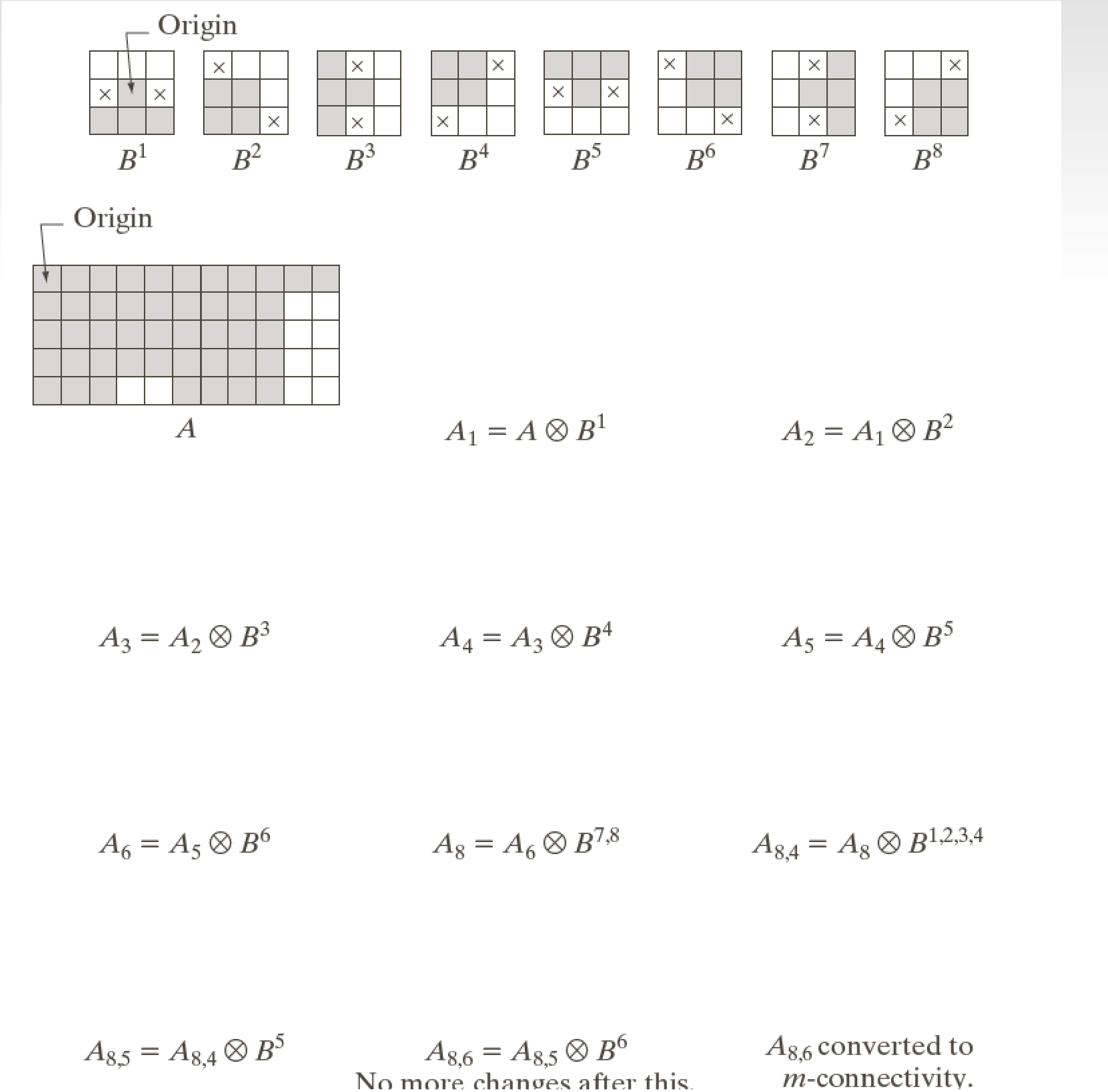
- A more useful expression for thinning A symmetrically is based on a sequence of structuring elements:

$$\{B\} = \{B^1, B^2, B^3, ..., B^n\}$$

where B^i is a rotated version of B^{i-1}

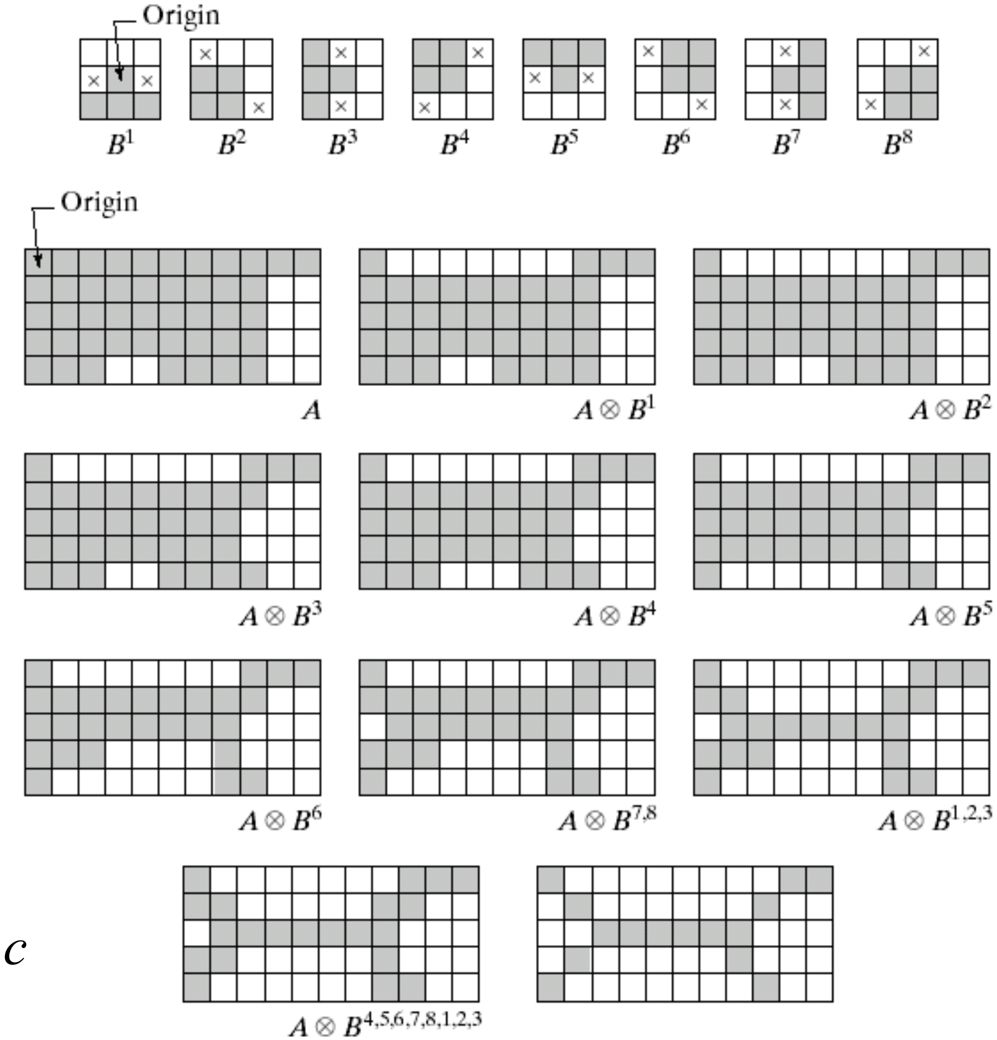
The thinning of A by a sequence of structuring element $\{B\}$

$$A \otimes \{B\} = ((...((A \otimes B^1) \otimes B^2)...) \otimes B^n)$$



a
b c d
e f g
h i j
k l m

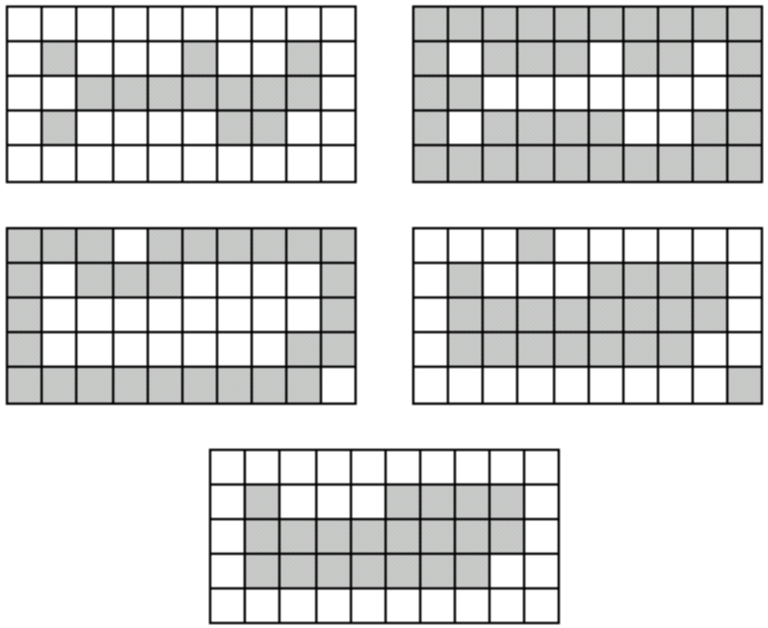
FIGURE 9.21 (a) Sequence of rotated structuring elements used for thinning. (b) Set A. (c) Result of thinning with the first element. (d)–(i) Results of thinning with the next seven elements (there was no change between the seventh and eighth elements). (j) Result of using the first four elements again. (l) Result after convergence. (m) Conversion to m-connectivity.



$$A \otimes B = A - (A \circledast B)$$
$$= A \cap (A \circledast B)^c$$

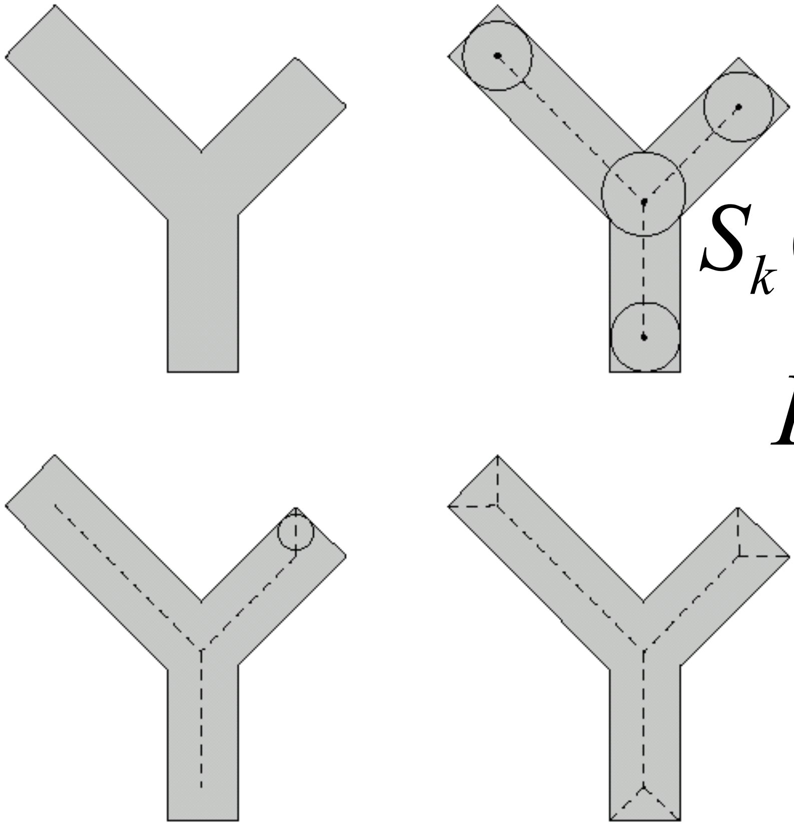
Thickening

$$A \odot B = A \cup (A \circledast B)$$



Skeletons

$$S(A) = \bigcup_{k=0}^{K} S_k(A)$$

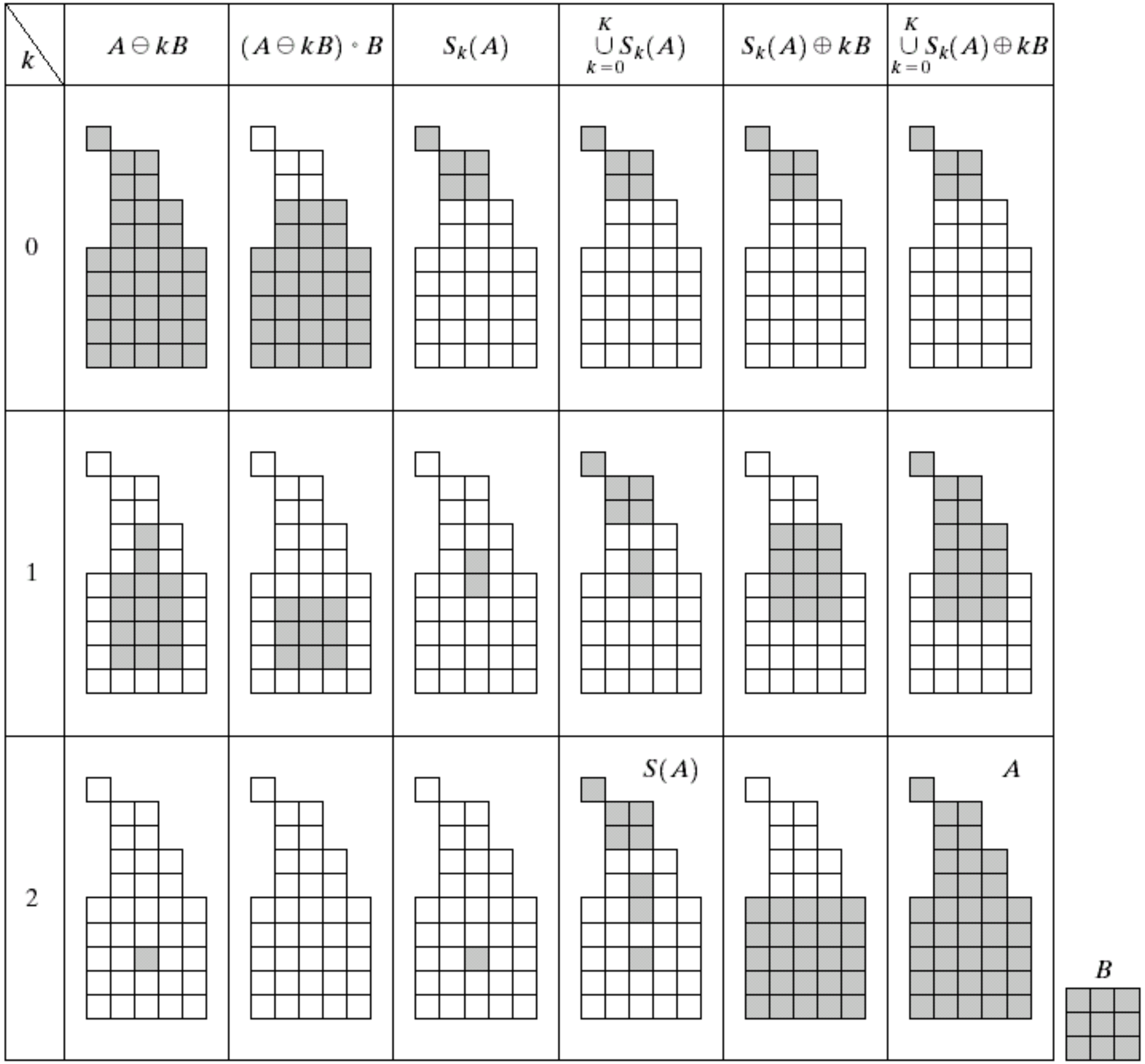


$$S_k(A) = (A \ominus kB) - (A \ominus kB) \circ B$$

$$K = \max\{k \mid (A - kB) \neq \Phi\}$$

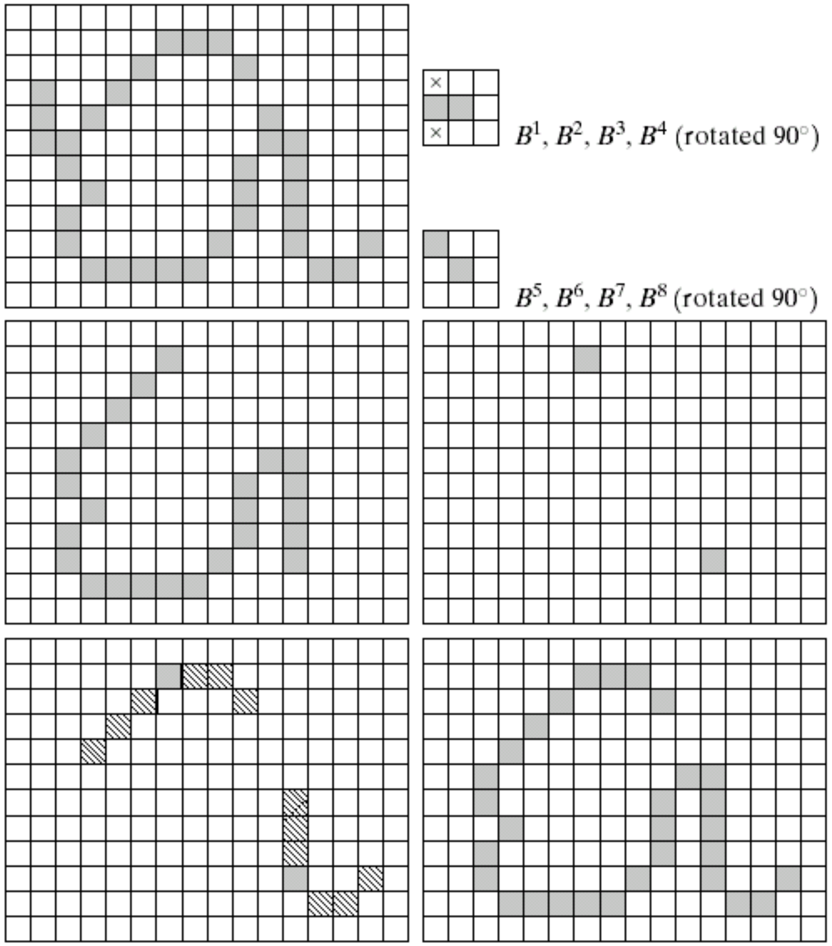
$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$$

Skeletons


k
$A \ominus kB$
$(A \ominus kB) \circ B$
$S_k(A)$
$\bigcup_{k=0}^{K} S_k(A)$
$S_k(A) \oplus kB$
$\bigcup_{k=0}^{K} S_k(A) \oplus kB$
0
1
2
$S(A)$
A
B

Pruning

H = 3x3 structuring element of 1's



$$X_1 = A \otimes \{B\}$$

$$X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$$

$$X_3 = (X_2 \oplus H) \cap A$$

$$X_4 = X_1 \cup X_3$$

Relation between sets and images

Sets and binary images.
The relation between a **binary** image and a set X: X is defined by all the pixels with grey value 1

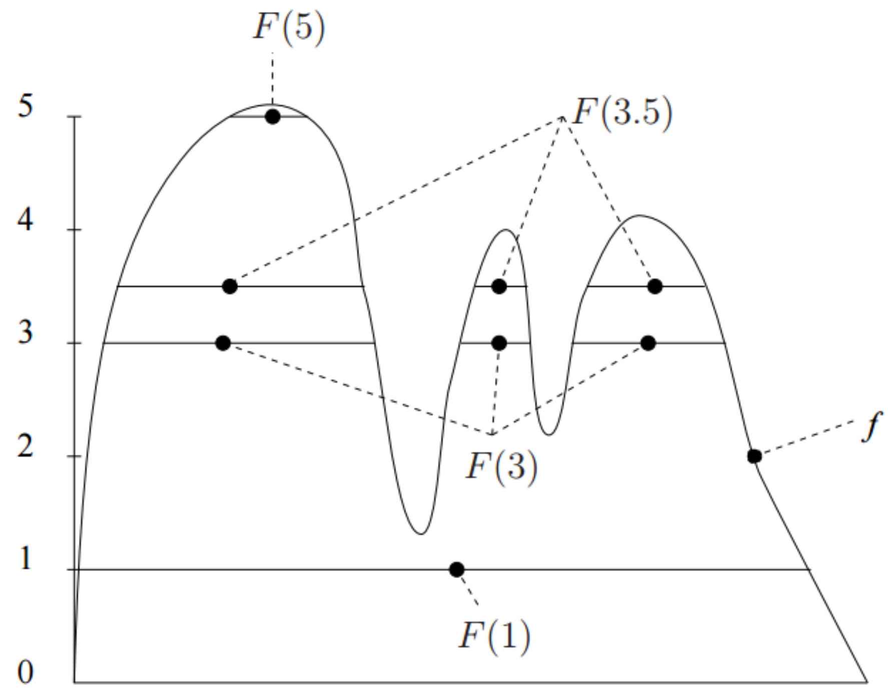
$$\{(x,y)|f(x,y) = 1\}$$

Relation between sets and images

This idea can be expanded so that we can define a relation between grey valued images and sets:

we regard each specific grey level in the image as a set; we regard the image f as a stack of sets F with the relation

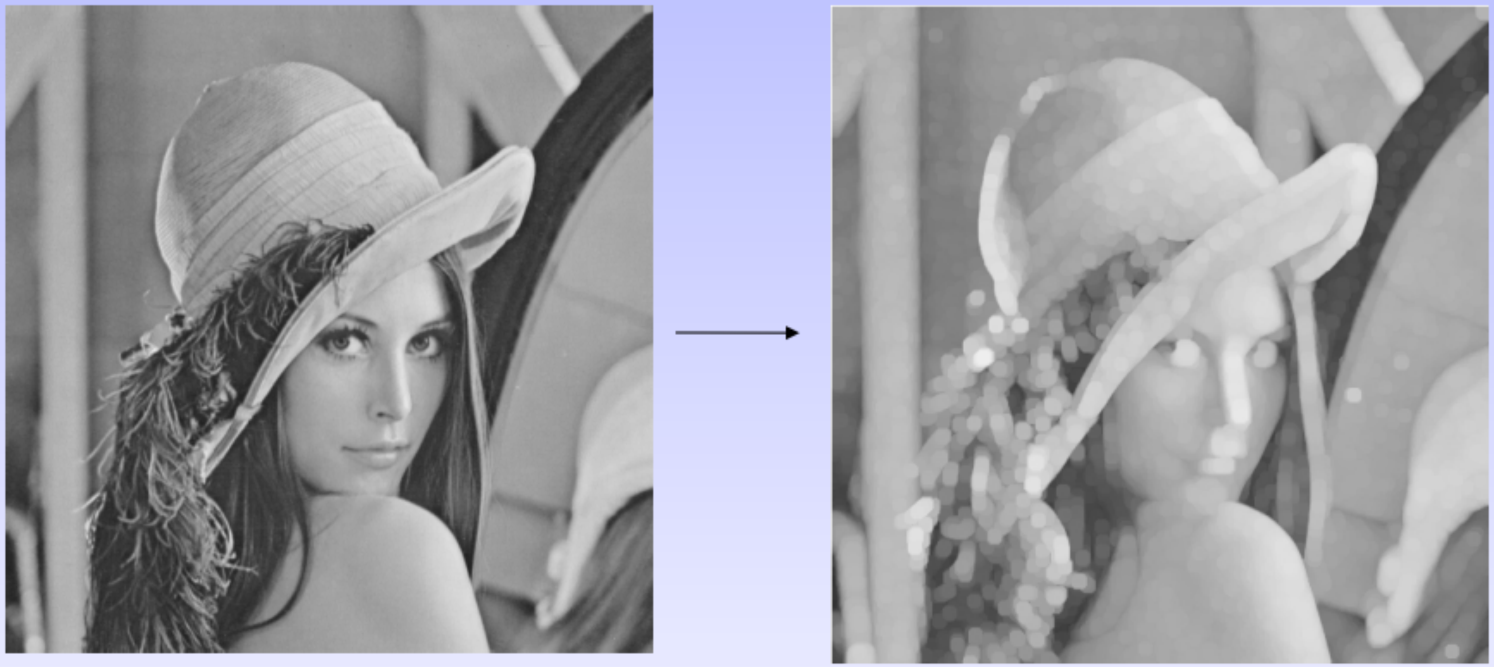
$$F(c) = \{(x, y) | f(x, y) \geqslant c\}$$



Examples with grayscale images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Dilation δ: max over the structuring element



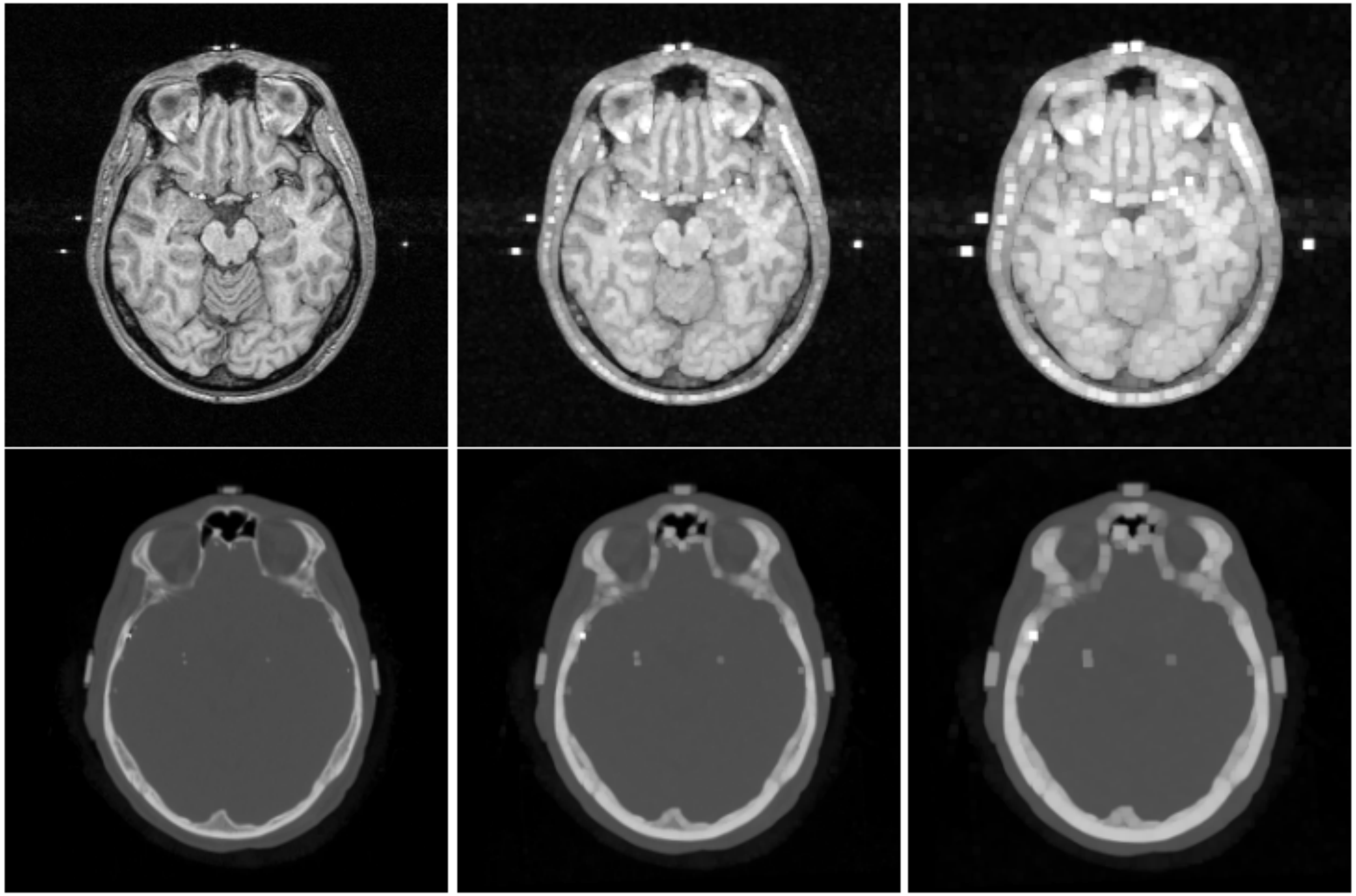


Figure 6.7 Example of dilation by a 3×3 square structuring element (middle column) and a 5×5 square structuring element (right column), applied to a 256×256 MR image (top row) and a 256×256 CT image (bottom row).

Examples with grayscale images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Erosion ε: min over the structuring element

Examples with grayscale images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Opening $\delta \circ \varepsilon$: remove light features smaller than the structuring element

Examples with grayscale images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Closing $\varepsilon \circ \delta$: remove dark features smaller than the structuring element

Examples with grayscale images

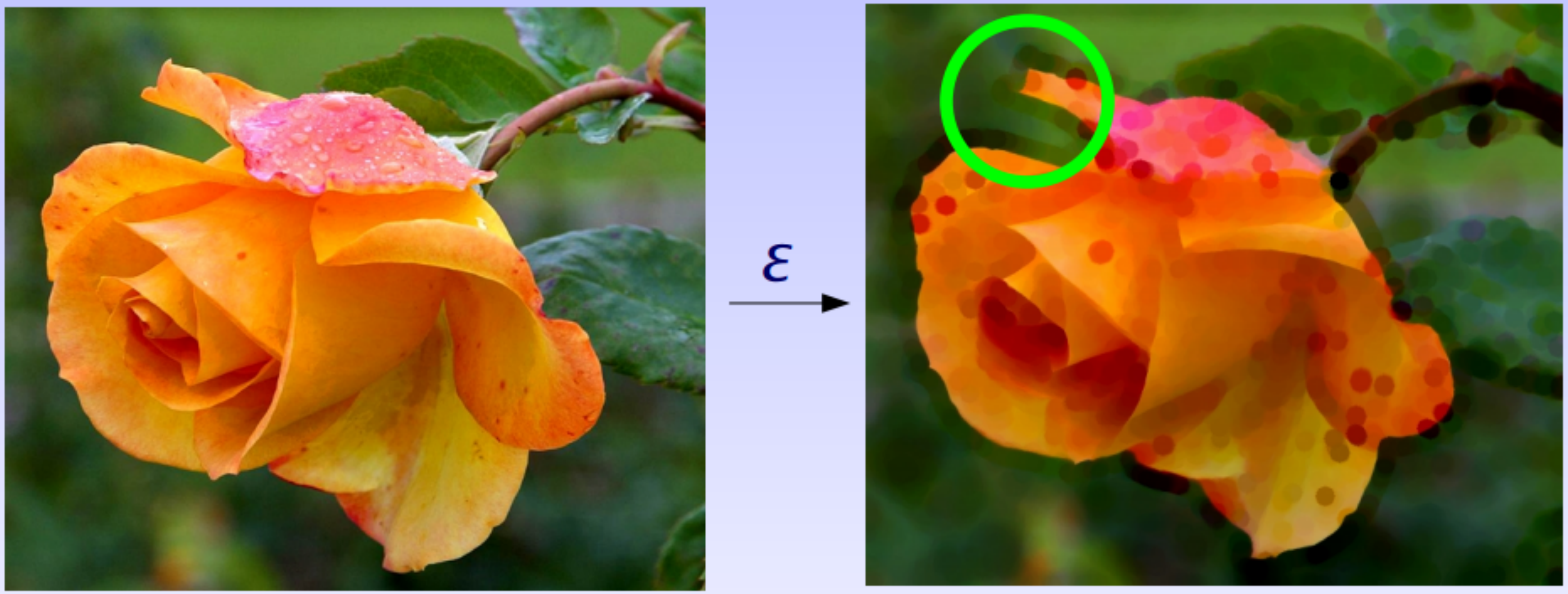
http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Sequential filter (open-close or close-open): remove both light and dark features

Examples with Color images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Process each channel separately: color ghosting with basic operators



Examples with Color images

http://maverick.inria.fr/Membres/Adrien.Bousseau/morphology/morphomath.pdf

- Process each channel separately: color ghosting unnoticeable with sequential operators

Operation	Equation	**Comments** (The Roman numerals refer to the structuring elements shown in Fig. 9.26).
Translation	$(A)_z = \{w \mid w = a + z, \quad \text{for } a \in A\}$	Translates the origin of A to point z.
Reflection	$\hat{B} = \{w \mid w = -b, \quad \text{for } b \in B\}$	Reflects all elements of B about the origin of this set.
Complement	$A^c = \{w \mid w \notin A\}$	Set of points not in A.
Difference	$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$	Set of points that belong to A but not to B.
Dilation	$A \oplus B = \left\{z \mid (\hat{B})_z \cap A \neq \varnothing\right\}$	"Expands" the boundary of A. (I)
Erosion	$A \ominus B = \left\{z \mid (B)_z \subseteq A\right\}$	"Contracts" the boundary of A. (I)
Opening	$A \circ B = (A \ominus B) \oplus B$	Smoothes contours, breaks narrow isthmuses, and eliminates small islands and sharp peaks. (I)
Closing	$A \bullet B = (A \oplus B) \ominus B$	Smoothes contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)

Hit-or-miss transform	$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$ $= (A \ominus B_1) - (A \oplus \hat{B}_2)$	The set of points (coordinates) at which, simultaneously, B_1 found a match ("hit") in A and B_2 found a match in A^c.
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set A. (I)
Region filling	$X_k = (X_{k-1} \oplus B) \cap A^c$; $X_0 = p$ and $k = 1, 2, 3, \ldots$	Fills a region in A, given a point p in the region. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap A$; $X_0 = p$ and $k = 1, 2, 3, \ldots$	Finds a connected component Y in A, given a point p in Y. (I)
Convex hull	$X_k^i = (X_{k-1}^i \circledast B^i) \cup A$; $i = 1, 2, 3, 4$; $k = 1, 2, 3, \ldots$; $X_0^i = A$; and $D^i = X_{\text{conv}}^i$.	Finds the convex hull $C(A)$ of set A, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. (III)

Operation	Equation	Comments (The Roman numerals refer to the structuring elements shown in Fig. 9.26).
Thinning	$A \otimes B = A - (A \circledast B)$ $= A \cap (A \circledast B)^c$ $A \otimes \{B\} =$ $((\ldots((A \otimes B^1) \otimes B^2)\ldots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$	Thins set A. The first two equations give the basic definition of thinning. The last two equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \circledast B)$ $A \odot \{B\} =$ $((\ldots(A \odot B^1) \odot B^2 \ldots) \odot B^n)$	Thickens set A. (See preceding comments on sequences of structuring elements.) Uses IV with 0's and 1's reversed.

Skeletons	$S(A) = \bigcup_{k=0} S_k(A)$ $S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]\}$ Reconstruction of A: $A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set A. The last equation indicates that A can be reconstructed from its skeleton subsets $S_k(A)$. In all three equations, K is the value of the iterative step after which the set A erodes to the empty set. The notation $(A \ominus kB)$ denotes the kth iteration of successive erosion of A by B. (I)
Pruning	$X_1 = A \otimes \{B\}$ $X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$ $X_3 = (X_2 \oplus H) \cap A$ $X_4 = X_1 \cup X_3$	X_4 is the result of pruning set A. The number of times that the first equation is applied to obtain X_1 must be specified. Structuring elements V are used for the first two equations. In the third equation H denotes structuring element I.