Representation theory of algebras

an introduction

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Important: These notes will be updated on a regular basis during the course.
The first part is based on previous notes by Francesca Mantese.
In the second part, many proofs are omitted or just sketched.
The complete arguments will be explained in the lectures!

Contents

1 RINGS

1.1 Reminder on rings

Recall that a *ring* $(R, +, \cdot, 0, 1)$ is given by a set R together with two binary operations, an addition $(+)$ and a multiplication $(\cdot)$, and two elements $0 \neq 1$ of R, such that $(R, +, 0)$ is an abelian group, $(R, \cdot, 1)$ is a monoid (i.e., a semigroup with unity 1), and multiplication is left and right distributive over addition. A ring whose multiplicative structure is abelian is called a *commutative ring*.
Given two rings R, S, a map $\varphi : R \to S$ is a *ring homomorphism* if for any two elements $a, b \in R$ we have $\varphi(a + b) = \varphi(a) + \varphi(b), \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$, and $\varphi(1_R) = 1_S$.

Examples:

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are commutative rings.
2. Let k be a field; the ring $k[x_1, \ldots, x_n]$ of polynomials in the variables $x_1, \ldots, x_n$ is a commutative ring.
3. Let k be a field; consider the ring $R = M_n(k)$ of $n \times n$-matrices with coefficients in k with the usual "rows times columns" product. Then R is a non-commutative ring.
4. Given an abelian group $(G, +)$, the group homomorphisms $f : G \to G$ form a ring $\operatorname{End} G$, called the *endomorphism ring* of G, with respect to the natural operations given by pointwise addition $f + g : G \to G$, $a \mapsto f(a) + g(a)$ and composition of maps $g \circ f : G \to G$, $a \mapsto g(f(a))$. The unity is given by the identity map $1_G : G \to G$, $a \mapsto a$.
5. Given a ring R, the opposite ring R^{op} has the same additive structure as R and opposite multiplication $(*)$ given by $a * b = b \cdot a$.

1.2 Finite dimensional algebras

Definition: Let k be a field. A *k-algebra* Λ is a ring with a map $k \times \Lambda \to \Lambda$, $(\alpha, a) \mapsto \alpha a$, such that Λ is a k-vector space and $\alpha(ab) = a(\alpha b) = (ab)\alpha$ for any $\alpha \in k$ and $a, b \in \Lambda$. Λ is finite dimensional if $\dim_k(\Lambda) < \infty$.
In other words, a k-algebra is a ring with a further structure of k-vector space, compatible with the ring structure.

Remark: An element $\alpha \in k$ can be identified with an element of Λ by means of the embedding $k \to \Lambda$, $\alpha \mapsto \alpha \cdot 1$. Thanks to this identification, we get that $k \leq \Lambda$.

Examples: Let k be a field.

1. The ring $M_n(k)$ is a finite dimensional k-algebra with $\dim_k(M_n(k)) = n^2$. Any element $\alpha \in k$ is identified with the diagonal matrix with α on the diagonal elements.

2. The ring $k[x]$ is a k-algebra, it is not finite dimensional.

3. Given a finite group $G = \{g_1, \dots, g_n\}$, let kG be the k-vector space with basis $\{g_1, \dots, g_n\}$ and multiplication given by $(\sum_{i=1}^n \alpha_i g_i) \cdot (\sum_{j=1}^n \beta_j g_j) = \sum_{i,j=1}^n \alpha_i \beta_j g_i g_j$. Then kG is a finite dimensional k-algebra, called the *group algebra* of G over k.

1.3 Quivers and path algebras

Definition. A *quiver* $Q = \{Q_0, Q_1\}$ is an oriented graph where Q_0 is the set of vertices and Q_1 is the set of arrows $i \xrightarrow{\alpha} j$ between the vertices. If Q_0 and Q_1 are finite sets, then Q is called a *finite* quiver.

Examples: $\mathbb{A}_n : \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \dots \bullet \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet}$, or $\bullet \circlearrowleft \alpha$, or $\bullet \rightrightarrows \bullet$

Definition. Let $Q = \{Q_0, Q_1\}$ be a finite quiver.

(1) An ordered sequence of arrows $\underset{i}{\bullet} \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \dots \bullet \xrightarrow{\alpha_n} \underset{j}{\bullet}$, denoted by $(i|\alpha_1, \dots, \alpha_n|j)$, is called a *path* in Q. A path $(i|\alpha_1, \dots, \alpha_n|i)$ starting and ending in the same vertex is called an *oriented cycle*. For each vertex i there is the *trivial (or lazy) path* $e_i = (i||i)$.

(2) For a field k, let kQ be the k-vector space having the paths of Q as k-basis. We now define an algebra structure on kQ. Hereby, the multiplication of two paths p and p' with the end point of p' coinciding with the starting point of p will correspond to the composition of arrows.

For paths $p' = (k|\beta_1, \dots, \beta_m|l)$, and $p = (i|\alpha_1, \dots, \alpha_n|j)$ of Q we set

$$p \cdot p' = \begin{cases} (k|\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_n|j) & \text{if } l = i \\ 0 & \text{else.} \end{cases}$$

In particular, the trivial paths satisfy $p \cdot e_i = e_j \cdot p = p$ and

$$e_i \cdot e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

and the unity is given by $1_{kQ} = \sum\limits_{i \in Q_0} e_i$. The algebra kQ is called the *path algebra* of Q over k. It is finite dimensional if and only if Q has no oriented cycles.

We simplify the notation and write $\alpha_n \dots \alpha_1 = (i|\alpha_1, \dots, \alpha_n|j)$.

Examples:

(1) $k\mathbb{A}_n$ is isomorphic to $\begin{pmatrix} k & & 0 \\ \vdots & \ddots & \\ k & \dots & k \end{pmatrix}$.

In fact, the only paths in $\mathbb{A}_n$ are the trivial paths and the paths $\alpha_{j-1}\dots\alpha_i = (i \mid \alpha_i\alpha_{i+1}\dots\alpha_{j-1} \mid j)$ for $1 \le i < j \le n$. So, if E_{ji} is the $n\times n$-matrix with 1 in the i-th entry of the j-th row and zero elsewhere, we obtain the desired isomorphism by mapping $e_i \mapsto E_{ii}$, and $\alpha_{j-1}\dots\alpha_i \mapsto E_{ji}$ for $1 \le i < j \le n$.

(2) The path algebra of the quiver $\bullet \circlearrowleft \alpha$ is isomorphic to $k[x]$ via the assignment $e_1 \mapsto 1$, and $\alpha \mapsto x$.

(3) The path algebra of the quiver $\bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet$ is called *Kronecker algebra*.

It is isomorphic to the triangular matrix ring $\begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$ via the assignment

$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha \mapsto \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix}$, $\beta \mapsto \begin{pmatrix} 0 & 0 \\ (0,1) & 0 \end{pmatrix}$

2 MODULES

2.1 Left and right modules

Definition: A *left R-module* is an abelian group M together with a map $R \times M \to M$, $(r,m) \mapsto rm$, such that for any $r,s \in R$ and any $x,y \in M$

(L1) $1x = x$

(L2) $(rs)x = r(sx)$

(L3) $r(x+y) = rx + ry$

(L4) $(r+s)x = rx + sx$

We write ${}_RM$ to express that M is a left R-module.

Examples:

1. Any abelian group G is a left $\mathbb{Z}$-module by defining $nx = \underbrace{x + \dots + x}_{n\ times}$ for $x \in G$ and $n > 0$, and correspondingly for $n \le 0$.
2. Given a field k, any vector space V over k is a left k-module.
3. Any ring R is a left R-module, by using the left multiplication of R on itself. It is called the *regular* module.

4. Consider the zero element of the ring R. Then the abelian group $\{0\}$ is trivially a left R-module.

Remark. Consider M an abelian group with endomorphism ring $\operatorname{End} M$. Every ring homomorphism $\lambda : R \to \operatorname{End} M, \quad r \mapsto \lambda(r)$ gives a structure of left R-module on M. Indeed, from the properties of ring homomorphisms it follows that for any $r, s \in R$ and $x, y \in M$

1. $\lambda(1)(x) = x$
2. $\lambda(rs)(x) = \lambda(r)(\lambda(s)(x))$
3. $\lambda(r)(x + y) = \lambda(r)(x) + \lambda(r)(y)$
4. $\lambda(r + s)(x) = \lambda(r)(x) + \lambda(s)(x)$

in other words, we can consider $\lambda(r)$ acting on the elements of M as a left multiplication by the element $r \in R$, and we can define $rx := \lambda(r)(x)$. Conversely, to any left R-module M, we can associate a ring homomorphism $\lambda : R \to \operatorname{End} M$ by defining $\lambda(r) : M \to M$, $x \mapsto rx$.

Similarly, we define right R-modules:

Definition: A *right R-module* is an abelian group M together with a map $M \times R \to M$, $(m, r) \mapsto mr$, such that for any $r, s \in R$ and any $x, y \in M$

(R1) $x1 = x$

(R2) $x(rs) = (xr)s$

(R3) $(x + y)r = xr + yr$

(R4) $x(r + s) = xr + xs$

We write M_R to express that M is a right R-module.

Remark (1) If R is a commutative ring, then left R-modules and right R-modules coincide. Indeed, given a left R-module M with the map $R \times M \to M$ $(r, m) \mapsto rm$, we can define a map $M \times R \to M$ $(m, r) \mapsto mr := rm$. This map satisfies the axioms (R1)–(R4) and so M is also a right R-module. The crucial point is that, in the second axiom, since R is commutative we have $x(rs) = (rs)x = (sr)x = s(rx) = (rx)s = (xr)s$.
(2) Consider M an abelian group with endomorphism ring $\operatorname{End} M$. Every ring homomorphism $\rho : R \to (\operatorname{End} M)^{op}, \quad r \mapsto \rho(r)$ gives a structure of right R-module on M, and conversely, to any right R-module M, we can associate a ring homomorphism $\rho : R \to (\operatorname{End} M)^{op}$ by defining $\rho(r) : M \to M$, $x \mapsto xr$ (check!).

We will mainly deal with left modules. So, in the following, unless otherwise is stated, with *module* we always mean *left module*.

Remark. Given ${}_RM$, for any $x \in M$ and $r \in R$, we have

1. $r0 = 0$
2. $0x = 0$
3. $r(-x) = (-r)x = -(rx)$

2.2 Submodules and quotient modules

Definition: Let ${}_RM$ be a left R-module. A subset L of M is a *submodule* of M if L is a subgroup of M and $rx \in L$ for any $r \in R$ and $x \in L$ (i.e. L is a left R-module under operations inherited from M). We write $L \leq M$.

Examples:

1. Let G be a $\mathbb{Z}$-module. The submodules of G are exactly the subgroups of G.
2. Let k a field and V a k-module. The submodules of V are exactly the k-subspaces of V.
3. Let R a ring. The submodules of the left R-module ${}_RR$ are the left ideals of R. The submodules of the right R-module R_R are the right ideals of R.

Definition: Let ${}_RM$ be a left R-module and $L \leq M$. The *quotient module* M/L is the quotient abelian group together with the map $R \times M/L \to M/L$ given by $(r, \overline{x}) \mapsto \overline{rx}$ (indeed, the map $R \times M/L \to M/L$ given by $(r, \overline{x}) \mapsto \overline{rx}$ is well-defined, since if $\overline{x} = \overline{y}$ then $x - y \in L$ and hence $rx - ry = r(x - y) \in L$, that is, $\overline{rx} = \overline{ry}$).

2.3 Homomorphisms of modules

Definition: Let ${}_RM$ and ${}_RN$ be R-modules. A map $f : M \to N$ is a *homomorphism* if $f(rx + sy) = rf(x) + sf(y)$ for any $x, y \in M$ and $r, s \in R$.

Remarks: (1) From the definition it follows that $f(0) = 0$.
(2) Clearly if f and g are homomorphisms from M to N, also $f + g$ is a homomorphism. Since the zero map is obviously a homomorphism, the set $\mathrm{Hom}_R(M, N) = \{f \mid f : M \to N \text{ is a homomorphism}\}$ is an abelian group.
(3) If $f : M \to N$ and $g : N \to L$ are homomorphisms, then $gf : M \to L$ is a homomorphism. Thus the abelian group $\mathrm{End}_R(M) = \{f \mid f : M \to M \text{ is a homomorphism}\}$ has a natural structure of ring, called the *endomorphism ring* of M. The identity homomorphism $\mathrm{id}_M : M \to M$, $m \mapsto m$, is the unity of the ring.

Definition: Given a homomorphism $f \in \mathrm{Hom}_R(M, N)$, the *kernel* of f is the set $\mathrm{Ker}\, f = \{x \in M \mid f(x) = 0\}$. The *image* of f is the set $\mathrm{Im}\, f = \{y \in N \mid y = f(x) \text{ for } x \in M\}$. It is easy to verify that $\mathrm{Ker}\, f \leq M$ and $\mathrm{Im}\, f \leq N$. Thus we can define the *cokernel* of f as the quotient module $\mathrm{Coker}\, f = N/\,\mathrm{Im}\, f$.

A homomorphism $f \in \mathrm{Hom}_R(M, N)$ is called a *monomorphism* if it is injective, that is, $\mathrm{Ker}\, f = 0$. It is called an *epimorphism* if it is surjective, that is, $\mathrm{Coker}\, f = 0$. It is is called an *isomorphism* if it is both a monomorphism and an epimorphism. If f is an isomorphism we write $M \cong N$.

Remarks: (1) For any submodule $L \leq M$ there is a canonical monomorphism $i : L \to M$, which is the usual inclusion, and a canonical epimorphism $p : M \to M/L$, $m \mapsto \overline{m}$ which is the usual quotient map.
(2) For any M the trivial map $0 \to M$, $0 \mapsto 0$, is a monomorphism, and the trivial map $M \to 0$, $m \mapsto 0$, is an epimorphism.
(3) Of course, $f \in \mathrm{Hom}_R(M, N)$ is an isomorphism if and only if there exist $g \in \mathrm{Hom}_R(N, M)$ such that $gf = \mathrm{id}_M$ and $fg = \mathrm{id}_N$. In such a case g is unique, and we usually denote it as f^{-1}.

2.4 Homomorphism theorems

Proposition 2.4.1. (Factorization of homomorphisms) *Given $f \in \mathrm{Hom}_R(M, N)$ and a submodule $L \leq M$ which is contained in $\mathrm{Ker}\, f$, there is a unique homomorphism $\overline{f} \in \mathrm{Hom}_R(M/L, N)$ such that $\overline{f}\, p = f$. We have $\mathrm{Ker}\, \overline{f} = \mathrm{Ker}\, f/L$ and $\mathrm{Im}\, \overline{f} = \mathrm{Im}\, f$.*
In particular, f induces an isomorphism $M/\,\mathrm{Ker}\, f \cong \mathrm{Im}\, f$.

Proof. The induced map $\overline{f} : M/L \to N$, $\overline{m} \mapsto f(m)$ is a homomorphism. Moreover, when $L = \mathrm{Ker}\, f$ it is clearly a monomorphism, inducing an isomorphism $M/\,\mathrm{Ker}\, f \to \mathrm{Im}\, f$. □

The usual isomorphism theorems which hold for groups hold also for homomorphisms of modules.

Proposition 2.4.2. (Isomorphism theorems) (1) *If $L \leq N \leq M$, then*

$$(M/L)/(N/L) \cong M/N.$$

(2) *If $L, N \leq M$, denote by $L + N = \{m \in M \mid m = l + n \text{ for } l \in L \text{ and } n \in N\}$. Then $L + N$ is a submodule of M and*

$$(L + N)/N \cong L/(N \cap L).$$

2.5 Bimodules

Definition: Let R and S be rings. An abelian group M is an *R-S-bimodule* if M is a left R-module and a right S-module such that the two scalar multiplications satisfy $r(xs) = (rx)s$ for any $r \in R$, $s \in S$, $x \in M$. We write ${}_RM_S$.

Examples: Let ${}_RM$ be a left R-module. Then M is a right $\mathrm{End}_R(M)^{op}$-module via the multiplication $mf = f(m)$ (check!) and we have a bimodule

$${}_RM_{\mathrm{End}_R(M)^{op}}.$$

Indeed $(rm)f = f(rm) = rf(m) = r(mf)$ for any $r \in R$, $m \in M$ and $f \in S$.

Given a bimodule ${}_RM_S$ and a left R-module N, the abelian group $\mathrm{Hom}_R(M, N)$ is naturally endowed with a structure of left S-module, by defining $(sf)(x) := f(xs)$ for any $f \in \mathrm{Hom}_R(M, N)$ and any $x \in M$. (crucial point: $(s_1(s_2f))(x) = (s_2f(xs_1)) = f(xs_1s_2) = ((s_1s_2)f)(x))$.
Similarly, if ${}_RN_T$ is a left R- right T-bimodule and ${}_RM$ is a left R-module, then $\mathrm{Hom}_R(M, N)$ is naturally endowed with a structure of right T-module, by defining $(ft)(x) := f(x)t$ (Check! crucial point: $(f(t_1t_2))(x) = f(x)(t_1t_2) = (f(x))t_1)t_2 = ((ft_1)(x))t_2 = ((ft_1)t_2)(x))$.
Moreover, if ${}_RM_S$ and ${}_RN_T$ are bimodules, we have an S-T-bimodule (check!)

$${}_S\,\mathrm{Hom}_R({}_RM_S, {}_RN_T)_T.$$

Arguing in a similar way for right R-modules, if ${}_SM_R$ and ${}_TN_R$ are bimodules, we have an T-S-bimodule

$${}_T\,\mathrm{Hom}_R({}_SM_R, {}_TN_R)_S$$

via $(tf)(x) = t(f(x))$ and $(fs)(x) = f(sx)$.

2.6 Sums and products of modules

Let I be a set and $\{M_i\}_{i\in I}$ a family of R-modules. The cartesian product

$$\prod_I M_i = \{(x_i) \mid x_i \in M_i\}$$

has a natural structure of left R-module, by defining the operations componentwise:

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}, \quad r(x_i)_{i\in I} = (rx_i)_{i\in I}.$$

This module is called the *direct product* of the modules M_i. It contains a submodule

$$\bigoplus_I M_i = \{(x_i) \mid x_i \in M_i \text{ and } x_i = 0 \text{ for almost all } i \in I\}$$

(recall that "almost all" means "except for a finite number"). The module $\oplus_I M_i$ is called the *direct sum* of the modules M_i. Clearly if I is a finite set then $\prod_I M_i = \{(x_i) \mid x_i \in M_i\} = \oplus_I M_i$. For any component $j \in I$ there are canonical homomorphisms

$$\prod_I M_i \to M_j \ , \ (x_i)_{i\in I} \mapsto x_j \quad \text{and} \quad M_j \to \prod_I M_i \ , \ x_j \mapsto (0,0,\dots,x_j,0,\dots,0)$$

called the *projection* on the j^{th}-component and the *injection* of the j^{th}-component. They are epimorphisms and monomorphisms, respectively, for any $j \in I$. The same is true for $\oplus_I M_i$.

When $M_i = M$ for any $i \in I$, we use the following notations

$$\prod_I M_i = M^I, \quad \bigoplus_I M_i = M^{(I)}, \quad \text{and if } \ I = \{1,\dots,n\}, \ \oplus_I M_i = M^n$$

Let ${}_RM$ be a module and $\{M_i\}_{i\in I}$ a family of submodules of M. We define the *sum* of the M_i as the module

$$\sum_I M_i = \{\sum_{i\in I} x_i \mid x_i \in M_i \text{ and } x_i = 0 \text{ for almost all } i \in I\}.$$

Clearly $\sum_I M_i \leq M$ and it is the smallest submodule of M containing all the M_i (notice that in the definition of $\sum_I M_i$ we need almost all the components to be zero in order to define properly the sum of elements of M).

Remark 2.6.1. Let ${}_RM$ be a module and $\{M_i\}_{i\in I}$ a family of submodules of M. Following the previous definitions we can construct both the module $\oplus_I M_i$ and module $\sum_I M_i$ (which is a submodule of M). We can define a homomorphism

$$\alpha : \oplus_I M_i \to M, \quad (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i.$$

Then $\operatorname{Im}\alpha = \sum_I M_i$. If α is a monomorphism, then $\oplus_I M_i \cong \sum_I M_i$ and we say that the module $\sum_I M_i$ is the *(innner) direct sum* of its submodules M_i. Often we omit the word "innner" and if $M = \sum_I M_i$ and α is an isomorphism, we say that M is the direct sum of the submodules M_i and we write $M = \oplus_I M_i$.

Similarly, given a family of modules $\{M_i\}_{i\in I}$ with the (outer) direct sum $M = \oplus_I M_i$, we can identify the M_i with their images under the injection in M and view M as an (inner) direct sum of these submodules.

2.7 Direct summands

Definition: (1) A submodule ${}_RL \leq {}_RM$ is a *direct summand* of M if there exists a submodule ${}_RN \leq {}_RM$ such that M is the direct sum of L and N. Then N is called a *complement* of L.
(2) A module M is said to be *indecomposable* if it only has the trivial direct summands 0 and M.

By the results in the previous section, if L is a direct summand of M and N a complement of L, any m in M can be written in a unique way as $m = l + n$ with $l \in L$ and $n \in N$. We write $M = L \oplus N$ and $L \overset{\oplus}{\leq} M$.

Remark 2.7.1. (1) Let ${}_RL, {}_RN \leq {}_RM$. Then $M = L \oplus N$ if and only if $L + N = M$ and $L \cap N = 0$.
(2) Let $f \in \mathrm{Hom}_R(L, M)$ and $g \in \mathrm{Hom}_R(M, L)$ be homomorphisms such that $gf = \mathrm{id}_L$. Then $M = \mathrm{Im}\, f \oplus \ker g$.

Examples:

1. Consider the $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$. Then $\mathbb{Z}/6\mathbb{Z} = 3\mathbb{Z}/6\mathbb{Z} \oplus 2\mathbb{Z}/6\mathbb{Z}$.
2. The regular module ${}_{\mathbb{Z}}\mathbb{Z}$ is indecomposable.
3. Let k be a field and V a k-module. Then, by a well-known result of linear algebra, any $L \leq V$ is a direct summand of V.
4. Let $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$. Then $R = P_1 \oplus P_2$, where $P_1 = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in k\}$ and $P_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in k\}$.

2.8 Representations of quivers

Definition. Let Q be a finite quiver without oriented cycles, k a field, and let $\Lambda = kQ$.

(1) A *(finite dimensional) representation* V of Q over k is given by a family of (finite dimensional) k-vector spaces $(V_i)_{i \in Q_0}$ indexed by the vertices of Q and a family of k-homomorphisms $(f_\alpha \colon V_i \to V_j)_{i \xrightarrow{\alpha} j \in Q_1}$ indexed by the arrows of Q.

(2) Given two representations V and V' of Q over k, a morphism $h : V \to V'$ is given by a family of k-homomorphism $(h_i \colon V_i \to V_i')_{i \in Q_0}$ such that the diagram

$$\begin{tikzcd} V_i \arrow[r, "f_\alpha"] \arrow[d, "h_i"'] & V_j \arrow[d, "h_j"] \\ V_i' \arrow[r, "f_\alpha'"'] & V_j' \end{tikzcd}$$

commutes for all arrows $i \xrightarrow{\alpha} j \in Q_1$.

Remark: Every representation of a quiver Q gives rise to a module over the path algebra kQ, and morphisms of representations give rise to module homomorphisms between the corresponding modules.
Indeed, if $((V_i)_{i\in Q_0}, (f_\alpha\colon V_i \to V_j)_{i\xrightarrow{\alpha} j\in Q_1})$ is a representation, we consider the vector space

$$M := \bigoplus_{i\in Q_0} V_i$$

and we define a left kQ-module structure on it. For $v = (v_i)_{i\in Q_0}$, left multiplication by the lazy path is given by $e_i \cdot v = (0, \ldots, v_i, \ldots, 0)$ and multiplication by a path $p = (i|\alpha_1, \ldots, \alpha_n|j)$ yields an element $p \cdot v$ with j-th entry $f_{\alpha_n} \ldots f_{\alpha_1}(v_i)$ and all other entries zero.
In other words, denoting by ι_j and π_i the canonical injections and projections in the j-th and on the i-th component, respectively, we have for the lazy paths

$$e_i \cdot v = \iota_i \pi_i(v)$$

and for $p = (i|\alpha_1, \ldots, \alpha_n|j)$

$$p \cdot v = \iota_j f_{\alpha_n} \ldots f_{\alpha_1} \pi_i(v).$$

Multiplication with an arbitrary linear combination of paths is defined correspondingly.

Conversely, every kQ-module gives rise to a representation, and module homomorphisms give rise to morphisms between the corresponding representations.
Indeed, if M is a left kQ-module, we set

$$V_i = e_i M$$

to get a family of vector spaces indexed over Q_0. Moreover, given an arrow $i \xrightarrow{\alpha} j$, we define a linear map

$$f_\alpha : e_i M \to e_j M,\ e_i m \mapsto e_j \alpha e_i m.$$

In this way we obtain a representation $((V_i)_{i\in Q_0}, (f_\alpha\colon V_i \to V_j)_{i\xrightarrow{\alpha} j\in Q_1})$ of Q.
The correspondence between modules and representations will be made more precise later.

Examples: (1) A representation of $\mathbb{A}_2 : 1 \xrightarrow{\alpha} 2$ has the form $V_1 \xrightarrow{f} V_2$ with k-vector spaces V_1, V_2 and a k-linear map $f : V_1 \to V_2$. The corresponding $k\mathbb{A}_2$-module is given by the vector space $M = V_1 \oplus V_2$ and the multiplication

$$e_1 \cdot (v_1, v_2) = (v_1, 0)$$

$$e_2 \cdot (v_1, v_2) = (0, v_2)$$

$$\alpha \cdot (v_1, v_2) = (0, f(v_1)).$$

Every finite dimensional representation corresponds to a matrix $A \in k^{n_2 \times n_1}$ where $n_i = \dim_k(V_i)$, and homomorphisms between two such representations, in terms of matrices A and A', are given by two matrices P, Q such that $PA = A'Q$. The representations are thus isomorphic if and only if there are matrices $P \in GL_{n_2}(K)$ and $Q \in GL_{n_1}(K)$ such that $A' = PAQ^{-1}$.

(2) A representation of the quiver $\bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet$ has the form $V_1 \underset{f_\beta}{\overset{f_\alpha}{\rightrightarrows}} V_2$ where V_1, V_2 are k-vectorspaces and $f_\alpha, f_\beta : V_1 \to V_2$ are k-linear. In other words, every finite dimensional representation of $\bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet$ corresponds to a pair of matrices (A, B) with $A, B \in k^{n_2 \times n_1}$ and $n_1, n_2 \in \mathbb{N}_0$. Moreover, isomorphism of two representations, in terms of matrix pairs (A, B) and (A', B') corresponds to the existence of two invertible matrices $P \in GL_n(K)$ and $Q \in GL_m(K)$ such that $A' = PAQ^{-1}$ and $B' = PBQ^{-1}$. So, the classification of the finite dimensional representations of $\bullet \rightrightarrows \bullet$ translates into the classification problem of "matrix pencils" considered by Kronecker in [19].

(3) A representation of Q : $\bullet \circlearrowleft \alpha$ is given as (V, f) with a vectorspace V and a linear map f. It corresponds to a module over the ring $k[x]$. Indeed, if M is a $k[x]$-module, then we obtain a representation of Q by setting $V = M$ and $f : M \to M$, $m \mapsto xm$.

2.9 Exercises - Part 1

(published on October 13, **solutions to be submitted on October 27, 2016**).

Exercise 1. (a) Let ${}_RM$ be a R-module and ${}_RR$ the regular module. Show that the abelian group $\mathrm{Hom}_R(R, M)$ is a left R-module and that the map

$$\varphi : \mathrm{Hom}_R(R, M) \to M,\ f \mapsto f(1)$$

is an isomorphism of R-modules. *(3 points)*

(b) Let $f \in \mathrm{Hom}_R(M, N)$ be a homomorphism of R-modules. Show that f is a monomorphism if and only if $fg = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(L, M)$. Show f is an epimorphism if and only if $gf = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(N, L)$. *(4 points)*

Exercise 2. (a) Let ${}_RL, {}_RN \leq {}_RM$. Show that M is the direct sum of L and N if and only if $L + N = M$ and $L \cap N = 0$. Does the same hold true for more than two summands? *(4 points)*

(b) Given $f \in \mathrm{Hom}_R(L, M)$ and $g \in \mathrm{Hom}_R(M, L)$ such that $gf = \mathrm{id}_L$, show that $M = \mathrm{Im}\, f \oplus \ker g$. *(3 points)*

Exercise 3. Given a field k, consider the ring $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in k\}$.

(a) Show that $P_1 = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in k\}$ and $P_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in k\}$ are left ideals of ${}_RR$ and that $I_1 = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in k\}$ and $I_2 = \{\begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix} \mid b, c \in k\}$ are right ideals of R_R. *(4 points)*

(b) Recall that R is isomorphic to the path algebra of the quiver $\mathbb{A}_2 \colon \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}$. Find representations of $\mathbb{A}_2$ corresponding to P_1 and P_2 under the isomorphism $k\mathbb{A}_2 \cong R$. *(4 points)*

Exercise 4. (a) Let $\varphi : S \to R$ a ring homomorphism. Show that any left R-module M is also a left S-module via the map $S \times M \to M$, $(s, m) \mapsto \varphi(s)m$. *(4 points)*

(b) Let ${}_RM$ and define $\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0 \text{ for any } m \in M\}$. M is called *faithful* if $\mathrm{Ann}_R(M) = 0$. Check that $\mathrm{Ann}_R(M)$ is a two-sided ideal of R, and set $S = R/\mathrm{Ann}_R(M)$. Verify that M has a natural structure of S-module, given by the map $S \times M \to M$, $(\overline{r}, m) \mapsto rm$. Show that M is a faithful S-module. *(4 points)*

3 PROJECTIVE MODULES, INJECTIVE MODULES

3.1 Exact sequences

Definition: A sequence of homomorphisms of R-modules

$$\cdots \to M_{i-1} \stackrel{f_{i-1}}{\to} M_i \stackrel{f_i}{\to} M_{i+1} \stackrel{f_{i+1}}{\to} \ldots$$

is called *exact* if $\operatorname{Ker} f_i = \operatorname{Im} f_{i-1}$ for any i.
An exact sequence of the form $0 \to M_1 \to M_2 \to M_3 \to 0$ is called a *short exact sequence*

Observe that if $L \leq M$, then the sequence $0 \to L \stackrel{i}{\to} M \stackrel{p}{\to} M/L \to 0$, where i and p are the canonical inclusion and quotient homomorphisms, is short exact (Check!). Conversely, if $0 \to M_1 \stackrel{f}{\to} M_2 \stackrel{g}{\to} M_3 \to 0$ is a short exact sequence, then f is a monomorphism, g is an epimorphism, and $M_3 \cong \operatorname{Coker} f$ (check!).

Example 3.1.1. (1) Consider the representations $0 \stackrel{0}{\to} K$, $K \stackrel{1}{\to} K$, and $K \stackrel{0}{\to} 0$ of $\mathbb{A}_2$ together with the morphisms

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & K \arrow[d, "1"] \\ K \arrow[r, "1"'] & K \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "1"] \arrow[d, "1"'] & K \arrow[d, "0"] \\ K \arrow[r, "0"'] & 0 \end{tikzcd}$$

They correspond to modules M_1, M_2, M_3 over $k\mathbb{A}_2$ and to homomorphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$ giving rise to a short exact sequence $0 \to M_1 \stackrel{f}{\to} M_2 \stackrel{g}{\to} M_3 \to 0$.
(2) For any $n \geq 2$ consider the short exact sequence $0 \to \mathbb{Z} \stackrel{n\cdot}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$.

The following result is very useful:

Proposition 3.1.2. *Consider the commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r] & 0 \end{tikzcd}$$

If α and γ are monomorphisms (epimorphims, or isomorphisms, respectively), so is β

Proof. (1) Suppose α and γ are monomorphisms and let m such that $\beta(m) = 0$. Then $\gamma(g(m)) = 0$ and so $m \in \operatorname{Ker} g = \operatorname{Im} f$. Hence $m = f(l)$, $l \in L$ and $\beta(m) = \beta(f(l)) = f'(\alpha(l)) = 0$. Since f' and α are monomorphism, we conclude $l = 0$ and so $m = 0$.

(2) Suppose α and γ are epimorphisms and let $m' \in M'$. Then $g'(m') = \gamma(g(m)) = g'(\beta(m))$; hence $m' - \beta(m) \in \operatorname{Ker} g' = \operatorname{Im} f'$ and so $m' - \beta(m) = f'(l')$, $l' \in L'$. Let $l \in L$ such that $l' = \alpha(l)$: then $m' - \beta(m) = f'(\alpha(l)) = \beta(f(l))$ and so we conclude $m' = \beta(m - f(l))$. □

3.2 Split exact sequences

If L and N are R-modules, there is a short exact sequence

$$0 \to L \xrightarrow{i_L} L \oplus N \xrightarrow{\pi_N} N \to 0, \text{ with } i_L(l) = (l,0) \quad \pi_N(l,n) = n, \quad \text{for any } l \in L, n \in N.$$

More generally:
Definition: A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is said to be *split exact* if there is an isomorphism $M \cong L \oplus N$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\alpha", "\cong"'] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "i_L"] & L \oplus N \arrow[r, "\pi_N"] & N \arrow[r] & 0
\end{tikzcd}$$

commutes. Then f is a *split monomorphism* and g a *split epimorphism.*

Proposition 3.2.1. *The following properties of an exact sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *are equivalent:*

1. *the sequence is split*
2. *there exists a homomorphism* $\varphi : M \to L$ *such that* $\varphi f = \mathrm{id}_L$
3. *there exists a homomorphism* $\psi : N \to M$ *such that* $g\psi = \mathrm{id}_N$

Under these conditions, L and N are isomorphic to direct summands of M.

Proof. 1 ⇒ 2. Since the sequence splits, then there exists α as in Definition 3.2. Let $\varphi = \pi_L \circ \alpha$. So for any $l \in L$ we have $\varphi f(l) = \pi_L \alpha f(l) = \pi_L(l,0) = l$.
1 ⇒ 3 Similar (Check!)
2 ⇒ 1. Define $\alpha : M \to L \oplus N$, $m \mapsto (\varphi(m), g(m))$. Since $\alpha f(l) = (\varphi(f(l)), g(f(l))) = (l,0)$ and $\pi_N \alpha(m) = g(m)$ we get that the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\alpha"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "i_L"] & L \oplus N \arrow[r, "\pi_N"] & N \arrow[r] & 0
\end{tikzcd}$$

commutes. Finally, by Proposition 3.1.2, we conclude that α is an isomorphism.
2 ⇒ 3 Similar (check!) □

Example. The short exact sequence in Example 3.1.1 is not a split exact sequence.

3.3 Free modules and finitely generated modules

Definition: A module ${}_RM$ is said to be *generated* by a family $\{x_i\}_{i\in I}$ of elements of M if every $x \in M$ can be written as $x = \sum_I r_i x_i$, with $r_i \in R$ for any $i \in I$, and $r_i = 0$ for almost every $i \in I$. Then $\{x_i\}_{i\in I}$ is called a set of *generators* of M and we write $M =< x_i, i \in I >$.
If the coefficients r_i are uniquely determined by x, the set $\{x_i\}_{i\in I}$ is called a *basis* of M. The module M is said to be *free* if it admits a basis.

Proposition 3.3.1. *A module ${}_RM$ is free if and only $M \cong R^{(I)}$ for some set I.*

Proof. The module $R^{(I)}$ is free with basis $(e_i)_{i\in I}$, where e_i is the canonical vector with all components zero except for the i-th equal to 1.
Conversely if M is free with basis $(x_i)_{i\in I}$, then we can define a homomorphism $\alpha : R^{(I)} \to M$, $(r_i)_{i\in I} \mapsto \sum_I r_i x_i$. It is easy to show that α is an isomorphism, as a consequence of the definition of a basis: indeed, it is clearly an epimorphism and if $\alpha(r_i) = \sum r_i x_i = 0$, since the r_i are uniquely determined by 0, we conclude that $r_i = 0$ for all i, i.e. α is a monomorphism. □

Given a free module M with basis $(x_i)_I$, every homomorphism $f : M \to N$ is uniquely determined by its value on the x_i, and the elements $f(x_i)$ can be chosen arbitrarily in N. Indeed, once we choose the $f(x_i)$, we define f on $x = \sum r_i x_i \in M$ as $f(x) = \sum r_i f(x_i)$ (which is well defined since $(x_i)_{i\in I}$ is a basis - notice the analogy with vector spaces!).

Proposition 3.3.2. *Any module is quotient of a free module.*

Proof. Let M be an R-module. Since we can always choose $I = M$, the module M admits a set of generators. Let $(x_i)_{i\in I}$ a set of generators for M and define a homomorphism $\alpha : R^{(I)} \to M$, $(r_i)_{i\in I} \mapsto \sum_i r_i x_i$. Clearly α is an epimorphism and so $M \cong R^{(I)}/\operatorname{Ker}\alpha$ □

Definition: A module ${}_RM$ is *finitely generated* it there exists a finite set of generators for M. A module is *cyclic* if it can be generated by a single element.

By Proposition 3.3.2, a module ${}_RM$ is finitely generated if and only if there exists an epimorphism $R^n \to M$ for some $n \in \mathbb{N}$. Similarly, ${}_RM$ is cyclic if and only if $M \cong R/J$ for a left ideal $J \leq R$.

Example 3.3.3. Let R be a ring.

1. The regular module ${}_RR$ is cyclic, generated by the unity element: ${}_RR =< 1 >$.
2. Let Λ be a finite dimensional k-algebra. Then a module ${}_\Lambda M$ is finitely generated if and only if $\dim_k(M) < \infty$.

   Indeed, assume $\dim_k(\Lambda) = n$, and let $\{a_1, \dots, a_n\}$ be a k-basis of Λ.

If $\{m_1, \ldots, m_r\}$ is a set of generators of M as Λ-module, then one verifies that $\{a_i m_j\}_{i=1,\ldots,n}^{j=1,\ldots,r}$ is a set of generators for M as k-module.

Conversely, if M is generated by $\{m_1, \ldots, m_s\}$ as k-module, since $k \leq \Lambda$, one gets that M is generated by $\{m_1, \ldots, m_s\}$ also as Λ-module.

Proposition 3.3.4. *Let* $_RL \leq {}_RM$.

1. *If M is finitely generated, then M/L is finitely generated.*
2. *If L and M/L are finitely generated, so is M*

Proof. (1) If $\{x_1, \ldots, x_n\}$ is a set of generators for M, then $\{\overline{x}_1, \ldots, \overline{x}_n\}$ is a set of generators for M/L.
(2) Let $< x_1, \ldots, x_n >= L$ and $< \overline{y}_1, \ldots, \overline{y}_m >= M/L$, where $x_1, \ldots, x_n, y_1, \ldots, y_m \in M$. Let $x \in M$ and consider $\overline{x} = \sum_{i=1,\ldots m} r_i \overline{y_i}$ in M/L. Then $x - \sum_{i=1,\ldots m} r_i y_i \in L$ and so $x - \sum_{i=1,\ldots m} r_i y_i = \sum_{j=1,\ldots,n} r_j x_j$. Hence $x = \sum_{i=1,\ldots m} r_i y_i + \sum_{j=1,\ldots,n} r_j x_j$, i.e. $\{x_1, \ldots, x_n, y_1 \ldots, y_m\}$ is a finite set of generators of M. $\square$

Notice that M finitely generated doesn't imply that L is finitely generated. For example, let R be the ring $R = k[x_i, i \in \mathbb{N}]$, and consider the regular module $_RR$ with its submodule $L =< x_i, i \in \mathbb{N} >$.

3.4 Projective modules

Definition: A module $_RP$ is *projective* if for any epimorphism $M \xrightarrow{g} N \to 0$ of left R-modules, the homomorphism of abelian groups

$$\mathrm{Hom}_R(P, g) : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N),\ \psi \mapsto g\psi$$

is surjective, that is, for any $\varphi \in \mathrm{Hom}_R(P, N)$ there exists $\psi \in \mathrm{Hom}_R(P, M)$ such that $g\psi = \phi$.

$$\begin{tikzcd} M \arrow[r, "g"] & N \arrow[r] & 0 \\ & P \arrow[ul, dashed, "\psi"] \arrow[u, "\varphi"'] & \end{tikzcd}$$

Examples: Any free module is projective. Indeed, let $R^{(I)}$ a free R-module with $(x_i)_{i \in I}$ a basis. Given homomorphisms $M \xrightarrow{g} N \to 0$ and $\varphi : R^{(I)} \to N$, let $m_i \in M$ such that $g(m_i) = \varphi(x_i)$ for any $i \in I$. Define $\psi(x_i) = m_i$ and, for $x = \sum r_i x_i$, $\psi(x) = \sum r_i m_i$. We get that $g\psi = \varphi$. It is clear from the construction that the homomorphism ψ is not unique in general.

Proposition 3.4.1. *Let P be a left R-module. The following are equivalent:*

1. *P is projective*
2. *P is a direct summand of a free module*

3. every exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ *splits.*

Proof. $1 \Rightarrow 3$ Let $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ be an exact sequence and consider the homorphism $1_P : P \to P$. Since P is projective there exists $\psi : P \to M$ such that $g\psi = 1_P$. By Proposition 3.2.1 we conclude that the sequence splits.
$3 \Rightarrow 2$ The module P is a quotient of a free module, so there exist an exact sequence $0 \to K \xrightarrow{f} R^{(I)} \xrightarrow{g} P \to 0$, which is split.
$2 \Rightarrow 1$ If $R^{(I)} = P \oplus L$, then $\mathrm{Hom}_R(R^{(I)}, N) \cong \mathrm{Hom}_R(P, N) \oplus \mathrm{Hom}_R(L, N)$ for any ${}_RN$. So let us consider the homorphisms

$$\begin{tikzcd} M \arrow[r, "g"] & N \arrow[r] & 0 \\ & P \arrow[u, "\varphi"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "g"] & N \arrow[r] & 0 \\ & R^{(I)} \arrow[u, "(\varphi,0)"'] \arrow[ul, dashed, "\alpha"] & \end{tikzcd}$$

where $(\varphi, 0)(p + l) = \varphi(p) + 0(l) = \varphi(p)$ for any $p \in P$ and $l \in L$ and α exists since $R^{(I)}$ is projective. Then $\alpha = (\psi, \beta)$, with $\psi \in \mathrm{Hom}_R(P, N)$ and $\beta \in \mathrm{Hom}_R(L, N)$, where $\alpha(p + l) = \psi(p) + \beta(l)$ for any $p \in P$ and $l \in L$. Hence $g(\psi(p)) = g(\alpha(p)) = \varphi(p)$ for any $p \in P$. So we conclude that P is projective. $\square$

Examples:

1. Let R be a principal ideal domain (for instance, $R = \mathbb{Z}$). Then any projective module is free. In particular, free abelian groups and projective abelian groups coincide.

2. Let $R = \mathbb{Z}/6\mathbb{Z}$. Then $\mathbb{Z}/6\mathbb{Z} = 3\mathbb{Z}/6\mathbb{Z} \oplus 2\mathbb{Z}/6\mathbb{Z}$. The ideals $3\mathbb{Z}/6\mathbb{Z}$ and $2\mathbb{Z}/6\mathbb{Z}$ are projective R-modules, but not free R-modules. The elements $e = \overline{3}$ and $f = \overline{4}$ are orthogonal idempotents (see Definition below) corresponding to this decomposition.

Definition. An element $e \in R$ is said to be *idempotent* if $e^2 = e$. Two idempotents $e, f \in R$ are said to be *orthogonal* if $ef = fe = 0$.

Remark 3.4.2. (1) If e is idempotent, then $(1 - e)$ is idempotent and

$$R = Re \oplus R(1 - e)$$

where Re and $R(1-e)$ denote the cyclic modules generated by e and $(1-e)$, respectively. Conversely, if $R = I \oplus J$, with I and J left ideals of R, then there exist orthogonal idempotents e and f such that $1 = e + f$, $I = Re$ and $J = Rf$.

(2) More generally, if $e_1, \ldots, e_n \in R$ are pairwise orthogonal idempotent elements such that $1 = e_1 + \ldots e_n$, then

$$R = Re_1 \oplus \ldots \oplus Re_n,$$

and every direct sum decomposition of the regular module ${}_RR$ arises in this way.

(3) If k is a field and $\Lambda = kQ$ is the path algebra of a quiver Q with $|Q_0| = n$, the lazy paths $e_1, \ldots, e_n$ are orthogonal idempotent elements of Λ as above. For each vertex

$i \in Q_0$, the paths starting in i form a k-basis of Λe_i. The representation corresponding to the module Λe_i is given by the vector spaces $V_j = e_j \Lambda e_i$ having as basis all paths starting in i and ending in j, and by the linear maps f_α corresponding to concatenation of paths with the arrow α. Moreover, $\operatorname{End}_\Lambda \Lambda e_i \cong e_i \Lambda e_i$ via $f \mapsto f(e_i)$ and if Q is acyclic, the latter is isomorphic to $ke_i \cong k$.

Example. (1) For $\Lambda = k\mathbb{A}_3$ the module Λe_1 corresponds to the representation

$$Ke_1 \xrightarrow{\alpha} K\alpha \xrightarrow{\beta} K\beta\alpha$$

which we write, up to isomorphism, as $K \to K \to K$.
(2) If $\Lambda = kQ$ is the Kronecker algebra with $Q : \bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet$, then the representations corresponding to Λe_i are

$$\Lambda e_1 : K \overset{\alpha}{\underset{\beta}{\rightrightarrows}} K^2$$

$$\Lambda e_2 : 0 \rightrightarrows K.$$

Proposition 3.4.3. (Dual Basis Lemma) *A module ${}_RP$ is projective if and only if it has a* dual basis*, that is, a pair $(\,(x_i)_{i\in I},\,(\varphi_i)_{i\in I}\,)$ consisting of elements $(x_i)_{i\in I}$ in P and homomorphisms $(\varphi_i)_{i\in I}$ in $P^* = \operatorname{Hom}_R(P,R)$ such that every element $x \in P$ can be written as*

$$x = \sum_{i\in I} \varphi_i(x)\, x_i$$

with $\varphi_i(x) = 0$ for almost all $i \in I$.

Proof. Let P be projective and let $R^{(I)} \xrightarrow{\beta} P \to 0$ be a split epimorphism. Let $(e_i)_{i\in I}$ be the canonical basis of $R^{(I)}$ and denote $x_i = \beta(e_i)$. Observe that $\beta(\sum_i r_i e_i) = \sum_i r_i \beta(e_i) = \sum_i r_i x_i$. By Proposition 3.2.1, there exists $\varphi : P \to R^{(I)}$ such that $\beta\varphi = id_P$, which induces homomorphisms $\varphi_i = \pi_i \varphi \in P^*$ where π_i is the projection on the i-th component. Then $\varphi_i(x) \in R$ is zero for almost all $i \in I$, and $\varphi(x) = \sum \varphi_i(x) e_i$. Hence for any $x \in P$ one has $x = \beta\varphi(x) = \beta(\sum_i \varphi_i(x) e_i) = \sum_i \varphi_i(x) x_i$, so $((\varphi_i)_{i\in I}, (x_i)_{i\in I})$ satisfies the stated properties.
Conversely, let $((\varphi_i)_{i\in I}, (x_i)_{i\in I})$ satisfy the statement. Define $\beta : R^{(I)} \to P$ by $e_i \mapsto x_i$. The homomorphism β is an epimorphism since the family $(x_i)_{i\in I}$ generates P, and $\beta(\sum r_i e_i) = \sum r_i x_i$. Set $\varphi : P \to R^{(I)}$, $x \mapsto \sum \varphi_i(x) e_i$. Then for any $x \in P$ one gets $\beta\varphi(x) = \beta(\sum \varphi_i(x) e_i) = \sum \varphi_i(x) x_i = x$. By Proposition 3.2.1 we conclude that β is a split epimorphism and so P is projective. $\square$

Note that, from the results in the previous sections, the projective module ${}_RR$ plays a crucial role, since for any module ${}_RM$ there exists an epimorphism $R^{(I)} \to M \to 0$, for some set I. A module with such property is called a *generator*, and so R is a *projective generator*.

In particular, for any module ${}_RM$ there exists a short exact sequence $0 \to K \to P_0 \to M \to 0$, with P_0 projective. The same holds for the module K, and so, iterating the argument, we can construct an exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all the P_i are projective. Such a sequence is called a *projective resolution* of P. It is clearly not unique.
It is natural to ask if, for a given module ${}_RM$, there exists a projective module P and a "minimal" epimorphism $P \to M \to 0$, in the sense that there is no proper direct summand P' of P with an epimorphism $f_{|P'} : P' \to M$. More precisely, we define:

Definition: (1) A homomorphism $f : M \to N$ is *right minimal* if any $g \in \mathrm{End}_R(M)$ such that $fg = f$ is an isomomorphism.
(2) A *projective cover* of M is a right minimal epimorphism $P_M \to M$ where P_M is a projective module.

Remark 3.4.4. Projective covers are "minimal" in the sense announced above. Indeed, consider another epimorphism $P \to M$ where P is a projective module. Since both P_M and P are projective, there exist φ and ψ such that the diagram

$$\begin{tikzcd} & 0 & \\ P_M \arrow[r, "f"] \arrow[dr, dashed, shift left, "\varphi"] & M \arrow[u] \arrow[r] & 0 \\ & P \arrow[u, "g"'] \arrow[ul, dashed, shift left, "\psi"] & \end{tikzcd}$$

commutes. Hence $f\psi = g$ and $g\varphi = f$, so $f\psi\varphi = f$ and, since f is right minimal, $\psi\varphi$ is an isomorphism. Then $\theta : P \to P_M$ as $\theta = (\psi\varphi)^{-1}\psi$ satisfies $\theta\varphi = id_P$, so φ is a split monomorphism and P_M is isomorphic to a direct summand of P (see Proposition 3.2.1). More precisely, $P = \mathrm{Im}\,\varphi \oplus \mathrm{Ker}\,\theta$ with $\mathrm{Im}\,\varphi \cong P_M$ and $g(\mathrm{Ker}\,\theta) = 0$.

In particular, if $g : P \to M$ is also a projective cover of M, then we can see as above that also $\varphi\psi$ is an isomorphism, so $\varphi = \psi^{-1}$ and P_M is isomorphic to P. We have shown that the projective cover is unique (up to isomorphism).

Observe that, given a module ${}_RM$, a projective cover for M need not exist. A ring over which any finitely generated module admits a projective cover is called *semiperfect*. If all modules admit a projective cover, then R is called *perfect*.

Definition. Suppose there exists a projective resolution of the module ${}_RM$

$$\ldots P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

such that P_0 is a projective cover of M and P_i is a projective cover of $\operatorname{Ker} f_{i-1}$ for any $i \in \mathbb{N}$. Such a resolution is called a *minimal projective resolution* of M.

Examples. (1) The canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not right minimal, and the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has no projective cover.
(2) The exact sequence in Example 3.1.1 is a minimal projective resolution of M_3. Indeed, by Example 3.4.2(4) we can rewrite the sequence as

$$0 \to \Lambda e_2 \xrightarrow{f} \Lambda e_1 \xrightarrow{g} M_3 \to 0$$

where the first two terms are projective modules with endomorphism ring k. It follows that g is right minimal, thus a projective cover.

3.5 Injective modules

We now turn to the dual notion of an injective module. Observe that many results will be dual to those proved for projective modules.

Definition: A module ${}_RE$ is *injective* if for any monomorphism $0 \to L \xrightarrow{f} M$ of left R-modules, the homomorphism of abelian groups $\operatorname{Hom}_R(f, E) : \operatorname{Hom}_R(M, E) \to \operatorname{Hom}_R(L, E)$ is an epimorphism, that is for any $\varphi \in \operatorname{Hom}_R(L, E)$ there exists $\psi \in \operatorname{Hom}_R(M, E)$ such that $\psi f = \varphi$.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\varphi"'] & M \arrow[dl, dashed, "\psi"] \\ & E & \end{tikzcd}$$

Any module is quotient of a projective module. Does the dual property hold? That is, is it true that every module M embeds in a injective R-module? In the sequel we will answer this crucial question.

An abelian group G is *divisible* if, for any $n \in \mathbb{Z}$ and for any $g \in G$, there exists $t \in G$ such that $g = nt$. We are going to show that an abelian group is injective if and only if it is divisible. We need the following useful criterion to check whether a module is injective.

Lemma 3.5.1. (Baer's Criterion) *A module E is injective if and only if for any left ideal I of R and for any $\varphi \in \operatorname{Hom}_R(I, E)$ there exists $\psi \in \operatorname{Hom}_R(R, E)$ such that $\psi i = \varphi$, where i is the canonical inclusion $0 \to I \xrightarrow{i} R$.*

The lemma states that it suffices to check the extending property only for the left ideals of the ring. In particular, it says that E is injective if and only if for any ${}_RI \leq {}_RR$ and for any $h \in \operatorname{Hom}_R(I, E)$ there exists $y \in E$ such that $h(a) = ay$ for any $a \in I$.

Proposition 3.5.2. *An $\mathbb{Z}$-module G is injective if and only if it is divisible.*

Proof. Let us assume G injective, consider $n \in \mathbb{Z}$ and $g \in G$ and the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}n \arrow[r, "i"] \arrow[d, "\varphi"] & \mathbb{Z} \arrow[dl, dashed, "\psi"] \\ & G & \end{tikzcd}$$

where $\varphi(sn) = sg$ for any $s \in \mathbb{Z}$ and ψ exists since G is injective. Let $t = \psi(1)$, $t \in G$. Then $\varphi(n) = \psi(i(n))$ implies $g = nt$ and we conclude that G is divisible.
Conversely, suppose G divisible and apply Baer's Criterion. The ideals of $\mathbb{Z}$ are of the form $\mathbb{Z}n$ for $n \in \mathbb{Z}$, so we have to verify that for any $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}n, G)$ there exists ψ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}n \arrow[r, "i"] \arrow[d, "\varphi"] & \mathbb{Z} \arrow[dl, dashed, "\psi"] \\ & G & \end{tikzcd}$$

commutes. Let $g \in G$ such that $\varphi(n) = g$. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, we can define ψ by setting $\psi(1) = t$ where $g = nt$, so $\psi(r) = rt$ for any $r \in \mathbb{Z}$. Hence $\varphi(sn) = sg = snt = \psi(i(sn))$. $\square$

The result stated in the previous proposition holds for any Principal Ideal Domain R.

Examples: (1) The $\mathbb{Z}$-module $\mathbb{Q}$ is injective.
(2) Let $p \in \mathbb{N}$ be a prime number and $M = \{\frac{a}{p^n} \in \mathbb{Q} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$. Then $\mathbb{Z} \leq M \leq \mathbb{Q}$, and $\mathbb{Z}_{p^\infty} = M/\mathbb{Z}$ is a divisible group, see Exercise 11.
One can show that $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$, p prime, are representatives of the indecomposable injective $\mathbb{Z}$-modules, up to isomorphism.

Remark 3.5.3. Any abelian group G embeds in an injective abelian group. Indeed, consider a short exact sequence $0 \to K \to \mathbb{Z}^{(I)} \to G \to 0$ and the canonical inclusion $0 \to \mathbb{Z} \to \mathbb{Q}$. One easily check that $\mathbb{Q}^{(I)}/K$ is divisible (check!) and so injective. Then we get the induced monomorphism $0 \to G \cong \mathbb{Z}^{(I)}/K \to \mathbb{Q}^{(I)}/K$.

Proposition 3.5.4. *Let R be a ring. If D is an injective $\mathbb{Z}$-module, then* $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ *is an injective left R-module*

Proof. First notice that, since ${}_{\mathbb{Z}}R_R$ is a bimodule, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is naturally endowed with a structure of left R-module. In order to verify that it is injective, we apply Baer's Criterion: let ${}_RI \leq {}_RR$ and $h : I \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$ be an R-homomorphism. We have to find an element $y \in \mathrm{Hom}_{\mathbb{Z}}(R, D)$ such that $h(a) = ay$ for any $a \in I$. Notice that h defines a $\mathbb{Z}$-homomorphism $\gamma : I \to D$, $a \mapsto h(a)(1)$ and, since D is an injective abelian group, there exists $\overline{\gamma} : R \to D$ which extends γ. Now we have, for any $a \in I$ and $r \in R$,

$$(a\overline{\gamma})(r) = \overline{\gamma}(ra) = \gamma(ra) = [h(ra)](1) = [rh(a)](1) = [h(a)](r)$$

so the element $\overline{\gamma} \in \mathrm{Hom}_{\mathbb{Z}}(R, D)$ satisfies $h(a) = a\overline{\gamma}$ for any $a \in I$, proving the claim. $\square$

Corollary 3.5.5. *Every module $_RM$ embeds in an injective R-module.*

Proof. As an abelian group, M embeds in an injective abelian group D by Remark 3.5.3. In other words, there is a monomorphism of $\mathbb{Z}$-modules $0 \to M \xrightarrow{g} D$, from which we obtain a monomorphism of R-modules $0 \to \mathrm{Hom}_{\mathbb{Z}}(R_R, M) \to \mathrm{Hom}_{\mathbb{Z}}(R_R, D)$ given by $f \mapsto gf$. Now $E := \mathrm{Hom}_{\mathbb{Z}}(R_R, D)$ is an injective left R-module by Proposition 3.5.4. Moreover, there is an isomorphism of R-modules $\varphi : \mathrm{Hom}_R(R, M) \to M$, $f \mapsto f(1)$ (see Exercise 1) yielding

$$_RM \cong \mathrm{Hom}_R(R_R, M) \leq \mathrm{Hom}_{\mathbb{Z}}(R_R, M) \to E = \mathrm{Hom}_{\mathbb{Z}}(R_R, D)$$

which is the desired monomorphism. □

Since any module M embeds in an injective one, it is natural to ask whether there exists a "minimal" injective module containing M.

Definition: (1) A homomorphism $f : M \to N$ is *left minimal* if any $g \in \mathrm{End}_R(N)$ such that $gf = f$ is an isomomorphism.
(2) An *injective envelope* of M is a left minimal monomorphism $M \to E_M$ where E_M is an injective module.

Remark 3.5.6. Consider a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & E_M \arrow[dl, dashed, bend left, "\varphi"] \\
 & E \arrow[ur, dashed, "\psi"] &
\end{tikzcd}$$

where $g : M \to E$ is another monomorphism where E is an injective module. Since E_M and E are both injective, there exist φ and ψ such that the diagram commutes. Hence $\psi g = f$ and $\varphi f = g$, so $\psi\varphi f = f$ and, since f is left minimal, we conclude that $\psi\varphi$ is an isomorphism. Then φ is a split monomorphism, and E_M is isomorphic to a direct summand of E.
In particular, if also g is an injective envelope of M, also $\varphi\psi$ is an isomorphism, so φ is an isomorphism and E_M is isomorphic to E. We have shown that the injective envelope is unique (up to isomorphisms).

We state a characterization of injective envelopes, for which we need the following notions.
Definition. (1) A submodule $_RN \leq {}_RM$ is *essential* if for any submodule $L \leq M$, $L \cap N = 0$ implies $L = 0$.
(2) A monomorphism $0 \to L \xrightarrow{f} M$ is *essential* if $\mathrm{Im}\, f$ is essential in M. Equivalently: every $g \in \mathrm{Hom}_R(M, N)$ with the property that gf is a monomorphism is itself a monomorphism (see Exercise 5).

Theorem 3.5.7. *Let E be an injective module. Then $0 \to M \xrightarrow{f} E$ is an injective envelope of M if and only if f is an essential monomorphism.*

Proof. Let $0 \to M \xrightarrow{f} E$ be an injective envelope and pick $L \leq E$ such that $L \cap \operatorname{Im} f = 0$. Then $\operatorname{Im} f \oplus L \leq E$, and we can consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f"'] & \operatorname{Im} f \oplus L \arrow[r, "i"] \arrow[dl, "{(\mathrm{id},0)}"'] & E \arrow[dll, dashed, "\varphi"] \\
& E & &
\end{tikzcd}$$

where i is the canonical inclusion of $\operatorname{Im} f \oplus L$ in E and φ exists since E is injective. Then $\varphi f = f$, and φ is an isomorphism, so $L = 0$.
Conversely, let $\operatorname{Im} f$ be essential in M and let $g \in \operatorname{End}_R(E)$ such that $gf = f$. Since f is an essential monomorphism, g is a monomorphism, hence a split monomorphism (see 3.5.9). Further, the direct summand $\operatorname{Im} g \overset{\oplus}{\leq} E$ of E contains the essential submodule $\operatorname{Im} f$, so it must have a trivial complement, that is, $\operatorname{Im} g = E$ and g is an isomorphism. □

Not every module has a projective cover. Thus the next result is especially remarkable

Theorem 3.5.8. *Every module has an injective envelope.*

Proof. Let ${}_RM$ be a module; by Corollary 3.5.5 there exists an injective module Q such that $0 \to M \to Q$. Consider the set $\{E' \mid M \leq E' \leq Q \text{ and } M \text{ essential in } E'\}$. One easily checks that it is an inductive set, and by Zorn's Lemma, it contains a maximal element E. Let us show that E is injective by verifying that it is a direct summand of Q (see Exercise 5). To this end, consider the set $\{F' \mid F' \leq Q \text{ and } F' \cap E = 0\}$. It is inductive so, again by Zorn's Lemma, it contains a maximal element F. We claim that $E \oplus F = Q$. Notice that there exists an obvious monomorphism $g : (E \oplus F)/F \cong E \leq Q$; further $(E \oplus F)/F \leq Q/F$ is an essential inclusion by the maximality of F (check!). We obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & (E \oplus F)/F \arrow[r, "j"] \arrow[d, "g"] & Q/F \arrow[dl, "\varphi"] \\
& Q &
\end{tikzcd}$$

where j is the canonical inclusion, φ exists since Q is injective, and moreover, φ is a monomorphism since $\varphi j = g$ is a monomorphism and j is an essential monomorphism. Then also $E = \operatorname{Im} g = \varphi(E \oplus F/F)$ is essential in $\operatorname{Im} \varphi$. Since M is essential in E, we conclude that M is essential in $\operatorname{Im} \varphi$, and by the maximality of E, it follows $E = \operatorname{Im} \varphi$. Hence $\varphi(E \oplus F/F) = \varphi(Q/F)$. Since φ is a monomorphism we conclude $E \oplus F = Q$. □

Proposition 3.5.9. *Let ${}_RE$ be a module. The following are equivalent:*

1. *E is injective*

2. every exact sequence $0 \to E \xrightarrow{f} M \xrightarrow{g} N \to 0$ *splits.*

Proof. $1 \Rightarrow 2$ Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "f"] \arrow[d, "\mathrm{id}_E"'] & M \arrow[dl, dashed, "\varphi"] \\ & E & \end{tikzcd}$$

where φ exists since E is injective. Since $\varphi f = \mathrm{id}_E$, by Proposition 3.2.1 we conclude that f is a split monomorphism.
$2 \Rightarrow 1$ By Corollary 3.5.5 there exists an exact sequence $0 \to E \to F \to N \to 0$, where F is an injective module. Since the sequence splits, we get that E is a direct summand of a injective module, and so E is injective (see Exercise 5). □

Comparing the previous proposition with the analogous one for projective modules (Proposition 3.4.1), there is an evident difference. For projective modules, we saw that a special role is played by the projective generator ${}_RR$. Does a module with the dual property exist? We will see in 4.5 that such a module always exists.

Dually to the projective case, for any module ${}_RM$ there exists a long exact sequence $0 \to M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} E_2 \to \dots$, where the E_i are injective. This is called an *injective coresolution* of M. If E_0 is an injective envelope of M and E_i in an injective envelope of $\operatorname{Ker} f_i$ for any $i \geq 1$, then the sequence is called a *minimal injective coresolution* of M.

4 ON THE LATTICE OF SUBMODULES OF M

Let R be a ring.

4.1 Simple modules

For a left R-module M, we consider the partially ordered set $\mathcal{L}_M = \{L \mid L \leq M\}$. Observe that $\mathcal{L}_M$ is a complete lattice, where for any $N, L \in \mathcal{L}$, the join is given by $\sup\{N, L\} = L + N$ and the meet by $\inf\{N, L\} = L \cap N$. The greatest element of $\mathcal{L}_M$ is M and the smallest if $\{0\}$.
Moreover, $\mathcal{L}_M$ satisfies the *Modular Law*: Given ${}_RA, {}_RB, {}_RC \leq {}_RM$ with $B \leq C$,

$$(A + B) \cap C = (A \cap C) + B.$$

It is natural to ask whether $\mathcal{L}$ has minimal or maximal elements. They are exactly the maximal submodules of M and the simple submodules of M, respectively. More precisely:

Definition: A module S is *simple* if $L \leq S$ implies $L = \{0\}$ or $L = S$.
Given a module ${}_RM$, a proper submodule ${}_RN < {}_RM$ is a *maximal submodule* of M if $N \leq L \leq M$ implies $L = N$ or $L = M$.

Examples:

1. Let k be a field. Then k is the unique simple k-module up to isomorphism.
2. Any abelian group $\mathbb{Z}/\mathbb{Z}p$ with p prime is a simple $\mathbb{Z}$-module. So there are infinitely many simple $\mathbb{Z}$-modules.
3. The regular module $\mathbb{Z}$ does not contain any simple submodule, since any ideal of $\mathbb{Z}$ is of the form $\mathbb{Z}n$ and $\mathbb{Z}m \leq \mathbb{Z}n$ whenever n divides m.
4. The $\mathbb{Z}$-module $\mathbb{Q}$ has no maximal submodules, see Exercise 7.
5. Let p be a prime number. The lattice of the subgroups of $\mathbb{Z}_{p^\infty}$ is a well-ordered chain, and $\mathbb{Z}_{p^\infty}$ has no maximal submodules, see Exercise 11.

We have just seen that in general, it is not true that any module contains a simple or a maximal submodule. Nevertheless, we have the following important result.

Proposition 4.1.1. *Let R be a ring and ${}_RI < {}_RR$ a proper left ideal. There exists a maximal left ideal $\mathfrak{m}$ of R such that $I \leq \mathfrak{m} < R$. In particular R admits maximal left ideals.*
More generally, if M is a finitely generated left R-module, then every proper submodule of M is contained in a maximal submodule.

Proof. Let $\mathcal{F} = \{L \mid I \leq L < R\}$. The set $\mathcal{F}$ is inductive since, given a sequence $L_0 \leq L_1 \leq \dots$, the left ideal $\bigcup L_i$ contains all the L_i and it is a proper ideal of R. Indeed,

if $\bigcup L_i = R$, there would exist an index $j \in \mathbb{N}$ such that $1 \in L_j$ and so $L_j = R$. So by Zorn's Lemma, $\mathcal{F}$ has a maximal element, which is clearly a maximal left ideal of R.
For the second statement, see Exercise 7. $\square$

Examples: Consider the regular module $\mathbb{Z}$. Then $\mathbb{Z}p$ is a maximal submodule of $\mathbb{Z}$ for any prime number p. Moreover the ideal $\mathbb{Z}n$ is contained in $\mathbb{Z}p$ for any p such that $p|n$.

Remark 4.1.2. Let $\mathfrak{m} \leq R$ be a maximal left ideal of R. Clearly $R/\mathfrak{m}$ is a simple R-module, and this shows that simple modules always exist over any ring R.
Conversely, if S is a simple module, any nonzero element $x \in S$ satisfies $S = Rx$, and $\mathrm{Ann}_R(x) = \{r \in R \mid rx = 0\}$ is the kernel of the epimorphism $\varphi : R \to S$, $1 \mapsto x$. Hence $\mathrm{Ann}_R(x)$ is a maximal left ideal of R and $S \cong R/\mathrm{Ann}_R(x)$.

Proposition 4.1.3. *The following statements are equivalent for a module ${}_RM$:*

1. *There is a family of simple submodules $(S_i)_{i\in I}$ of M such that $M = \sum_{i\in I} S_i$.*
2. *M is a direct sum of simple submodules.*
3. *Every submodule ${}_RL \leq {}_RM$ is a direct summand.*

Under these conditions, M is said to be *semisimple*.

Proof. Let us sketch the proof. In order to see that (1) implies (2) and (3), one uses Zorn's Lemma to show that for any ${}_RL \leq_R M$ there is a subset $J \subseteq I$ such that $M = L \oplus \bigoplus_{i\in J} S_i$.
(3)⇒(1): Using the Modular Law, we see that every submodule ${}_RN \leq {}_RM$ satisfies condition (3), that is, every submodule ${}_RL \leq {}_RN$ is a direct summand of N. Furthermore, if we consider a non-zero element $x \in M$ and choose $N = Rx$, then N contains a maximal submodule N' by Proposition 4.1.1, which then must be a direct summand of N. Since the complement of N' in N is simple, we conclude that Rx contains a simple submodule. Now consider the submodule $L = \sum_{i\in I} S_i$ defined as the sum of all simple submodules of M. We know that $M = L \oplus L'$ for some submodule L'. But by the discussion above L' cannot contain any nonzero element, hence $L' = 0$ and the claim is proven. $\square$

4.2 Socle and radical

Definition: Let M be a left R-module. The *socle* of M is the submodule

$$\mathrm{Soc}(M) = \sum \{S \mid S \text{ is a simple submodule of } M\}.$$

The *radical* of M is the submodule

$$\mathrm{Rad}(M) = \bigcap \{N \mid N \text{ is a maximal submodule of } M\}.$$

In particular, if M does not contain any simple module, $\mathrm{Soc}(M) = 0$, and if M does not contain any maximal submodule, $\mathrm{Rad}(M) = M$.

Remark 4.2.1. (1) $\operatorname{Soc}(M)$ is the largest semisimple submodule of M.
This follows immediately from Proposition 4.1.3.
(2) $\operatorname{Rad}(M) = \{x \in M \mid \varphi(x) = 0 \text{ for every } \varphi : M \to S \text{ with } S \text{ simple}\}$.
Indeed, notice that the kernel of any homomorphism $\varphi : M \to S$ with S simple is a maximal submodule of M. Conversely, if N is a maximal submodule of M, then consider $\pi : M \to M/N$, keeping in mind that M/N is simple.

In order to study $\operatorname{Rad} M$, we need the following notion, which also leads to a characterization of projective covers dual to Theorem 3.5.7.
Definition. A submodule ${}_RN \le {}_RM$ is *superfluous* if for any submodule $L \le M$, $L + N = M$ implies $L = M$.

Theorem 4.2.2. *Let P a projective module. Then $P \xrightarrow{f} M \to 0$ is a projective cover of M if and only if $\operatorname{Ker} f$ is a superfluous submodule of P.*

It follows from Proposition 4.1.1 that $\operatorname{Rad}(M)$ is a superfluous submodule of M whenever M is finitely generated. We collect some further properties of the socle and of the radical of a module in the proposition below.

Proposition 4.2.3. *Let M be a left R-module.*

1. $\operatorname{Soc}(M) = \bigcap\{L \mid L$ *is an essential submodule of* $M\}$.
2. $\operatorname{Rad}(M) = \sum\{U \mid U$ *is a superfluous submodule of* $M\}$.
3. $f(\operatorname{Soc}(M)) \le \operatorname{Soc}(N)$ *and* $f(\operatorname{Rad}(M)) \le \operatorname{Rad}(N)$ *for any* $f \in \operatorname{Hom}_R(M, N)$.
4. *If* $M = \oplus_{\lambda\in\Lambda} M_\lambda$, *then* $\operatorname{Soc}(M) = \bigoplus_{\lambda\in\Lambda} \operatorname{Soc}(M_\lambda)$ *and* $\operatorname{Rad}(M) = \bigoplus_{\lambda\in\Lambda} \operatorname{Rad}(M_\lambda)$.
5. $\operatorname{Rad}(M/\operatorname{Rad}(M)) = 0$ *and* $\operatorname{Soc}(\operatorname{Soc}(M)) = \operatorname{Soc}(M)$.

A crucial role is played by the radical of the regular module ${}_RR$.

Proposition 4.2.4. *(1)* $\operatorname{Rad}({}_RR) = \bigcap\{\operatorname{Ann}_R(S) \mid S$ *is a simple left R-module* $\}$.
(2) $\operatorname{Rad}({}_RR) = \{r \in R \mid 1 - xr$ *has a (left) inverse for any* $x \in R\}$.
(3) $\operatorname{Rad}({}_RR) = \operatorname{Rad}(R_R)$ *is a two-sided ideal.*

Proof. (1) For any simple module S, consider $\operatorname{Ann}_R(S) = \bigcap_{x\in S} \operatorname{Ann}_R(x)$ of R, which is a two-sided ideal by Exercise 4. The intersection of all annihilators $\operatorname{Ann}_R(S)$ of simple left R-modules coincides with $\operatorname{Rad}({}_RR)$ by Remarks 4.1.2 and 4.2.1.
(2) is Exercise 8. In fact, one can even show that the elements $1 - xr$ are invertible: taking $r \in \operatorname{Rad}({}_RR)$ and $x \in R$, we have $s = xr \in \operatorname{Rad}({}_RR)$, and if a is a left inverse of $1 - s$, that is, $a(1 - s) = 1$, then $a = 1 + as = 1 - (-a)s$ has again a left inverse, which must coincide with its right inverse $1 - s$, showing that a and $1 - s$ are mutually inverse.
(3) It follows from (1) that $\operatorname{Rad}({}_RR)$ is a two-sided ideal of R. So, if $r \in \operatorname{Rad}({}_RR)$, and $x \in R$, then $rx \in \operatorname{Rad}({}_RR)$, and the element $1 - rx$ has a (right) inverse by (2). From the right version of statement (2) we infer $r \in \operatorname{Rad}(R_R)$. So $\operatorname{Rad}({}_RR) \subseteq \operatorname{Rad}(R_R)$, and the other inclusion follows by symmetric arguments. □

Definition: Let R be a ring. The ideal

$$J(R) = \mathrm{Rad}({}_R R) = \mathrm{Rad}(R_R)$$

is called *Jacobson radical* of R.

Lemma 4.2.5. *(1) For every module ${}_R M$ we have* $\mathrm{J}(R)M \leq \mathrm{Rad}(M)$.
(2) **(Nakayama's Lemma)** *Let M be a finitely generated R-module. If L is a submodule of M such that $L + \mathrm{J}(R)M = M$, then $L = M$.*

Proof. (1) Since $\mathrm{J}(R)$ annihilates any simple module S, all homomorphisms $\varphi : M \to S$ vanish on $\mathrm{J}(R)M$, so $\mathrm{J}(R)M \leq \mathrm{Rad}(M)$ by Remark 4.2.1.
(2) $L + \mathrm{J}(R)M = M$ implies $L + \mathrm{Rad}(M) = M$ and since $\mathrm{Rad}(M)$ is superfluous in M by Remark 4.2.1, we get $L = M$. □

Example 4.2.6. (1) $J(\mathbb{Z}) = \bigcap_{p \text{ prime}} p\mathbb{Z} = 0$.
(2) Let $\Lambda = kQ$ be the path algebra of a finite acyclic quiver over a field k.

(i) The Jacobson radical $J(\Lambda)$ is the ideal of Λ generated by all arrows. Hence, as a k-vectorspace, $\Lambda = (\oplus_{i \in Q_0} k e_i) \oplus J(\Lambda)$. Moreover, $\Lambda / J(\Lambda) \cong k^{|Q_0|}$ as k-algebras.

(ii) Let $i \in Q_0$ be a vertex, and denote by $\alpha_1, \ldots, \alpha_t$ the arrows $i \bullet \xrightarrow{\alpha_k} \bullet\, j_k$ of Q which start in i. Then

$$\mathrm{Rad}\, \Lambda e_i = J e_i = \bigoplus_{k=1}^{t} \Lambda e_{j_k} \alpha_k \cong \bigoplus_{k=1}^{t} \Lambda e_{j_k}$$

is the unique maximal submodule of Λe_i, and it is a projective module.

(iii) Let $i \in Q_0$ be a vertex. Then $\Lambda e_i / J e_i$ is simple. In particular, the projective module Λe_i is simple if and only if i is a sink of Q, that is, there is no arrow starting in i.

Indeed, let $i \in Q_0$ be a vertex. Then the vector space generated by all paths of length at least one starting in i is the unique maximal submodule of Λe_i, so it coincides with $\mathrm{Rad}\, \Lambda e_i$. Now use that $\Lambda = \bigoplus_{i \in Q_0} \Lambda e_i$ by Remark 3.4.2, hence $J(\Lambda) = \bigoplus_{i \in Q_0} \mathrm{Rad}\, \Lambda e_i$ by Proposition 4.2.3.

4.3 Local rings

Definition:
(1) A ring R is a *skew field* (or a *division ring*) if all non-zero elements are invertible.
(2) A ring R is *local* if it satisfies the equivalent conditions in the proposition below.

Proposition 4.3.1. *The following statements are equivalent for a ring R with $J = J(R)$.*

(1) R/J is a skew field.

(2) x or $1 - x$ is invertible for any $x \in R$.

(3) R has a unique maximal left ideal.

(3') R has a unique maximal right ideal.

(4) The non-invertible elements of R form a left (or right, or two-sided) ideal of R.

Proof. (1)⇒(2): If $x \in J$, then $1 - x$ is invertible by Proposition 4.2.4. If $x \notin J$, then $\overline{x} \neq 0$ is invertible in R/J, so there is $\overline{y} \in R/J$ such that $\overline{xy} = \overline{yx} = \overline{1}$. Then $1 - xy$ and $1 - yx$ belong to J, hence xy and yx are invertible. But then x is invertible, because it has a right inverse and a left inverse.
(2)⇒(3): Any maximal left ideal $\mathfrak{m}$ contains J. Conversely, if $r \in \mathfrak{m}$ and $x \in R$, then $xr \in \mathfrak{m}$ can't be invertible, so $1 - xr$ is invertible, and $r \in J$ by Proposition 4.2.4. Hence $\mathfrak{m} = J$ is the unique maximal left ideal.
(3)⇒(1): Assume that R has a unique maximal left ideal $\mathfrak{m}$. Then $\mathfrak{m} = J$, and R/J is a simple left module. Then every non-zero element $\overline{x} \in R/J$ satisfies $Rx = R/J$, so there is $y \in R$ such that $\overline{1} = y\overline{x} = \overline{yx}$. In other words, every non-zero element in R/J has a left inverse, and therefore an inverse (because the left inverse of $\overline{y}$ must coincide with its right inverse $\overline{x}$).
(1)⇔(3') is shown symmetrically.
(3)⇒(4): J is the set of all non-invertible elements of R. Indeed, J is a maximal left ideal and therefore it consists of non-invertible elements. Conversely, if $x \in R$ has no left inverse, then Rx is a proper left ideal of R and thus it is contained in the unique maximal left ideal J. If x has no right inverse, use the equivalent condition (3').
(4)⇒(2): otherwise $1 = x + (1 - x)$ would be non-invertible. □

Remark 4.3.2. Let R be a local ring.
(1) We have seen above that J is the ideal from conditions (3), (3') and (4) above.
(2) $S = R/\operatorname{J}(R)$ is the unique simple left (or right) R-module up to isomorphism, and $E(R/\operatorname{J}(R))$ is a minimal injective cogenerator.
(3) The unique idempotent elements in R are 0 and 1. Indeed, if e is idempotent, then $e(1 - e) = 0$. So, either e is invertible, and then $e = 1$, or $1 - e$ is invertible, and then $e = 0$.
(4) ${}_RR$ is an indecomposable R-module by Remark 3.4.2.

4.4 Finite length modules

Let M be a left R-module. A sequence $0 = N_0 \leq N_1 \leq \cdots \leq N_{s-1} \leq N_s = M$ of submodules of M is called a *filtration* of M, with *factors* N_i/N_{i-1}, $i = 1, \cdots, s$. The *length* of the filtration is the number of non-zero factors.
Consider now a filtration $0 = N'_0 \leq N'_1 \leq \cdots \leq N'_{t-1} \leq N_t = M$; it is a *refinement* of the latter one if $\{N_i \mid 0 \leq i \leq s\} \subseteq \{N'_i \mid 0 \leq i \leq t\}$.
Two filtrations of M are said *equivalent* if $s = t$ and there exists a permutation $\sigma : \{0, 1, \cdots, s\} \to \{0, 1, \cdots, s\}$ such that $N_i/N_{i-1} \cong N'_{\sigma(i)}/N'_{\sigma(i-1)}$, for $i = 1, \cdots, s$.

Finally, a filtration $0 = N_0 \leq N_1 \leq \cdots \leq N_{s-1} \leq N_s = M$ of M is a *composition series* of M if the factors N_i/N_{i-1}, $i = 1, \cdots, s$, are simple modules. In such a case they are called *composition factors* of M.

Theorem 4.4.1. *Any two filtrations of M admit equivalent refinements.*

Proof. The proof follows from the following
Lemma: *Let $U_1 \leq U_2 \leq M$ and $V_1 \leq V_2 \leq M$. Then*

$$(U_1 + U_2 \cap V_2)/(U_1 + V_1 \cap U_2) \cong (U_2 \cap V_2)/(U_1 \cap V_2) + (U_2 \cap V_1) \cong (V_1 + U_2 \cap V_2)/(V_1 + U_1 \cap V_2)$$

In our setting, consider $0 = N_0 \leq N_1 \leq \cdots \leq N_{s-1} \leq N_s = M$ and $0 = L_0 \leq L_1 \leq \cdots \leq L_{s-1} \leq L_t = M$ two filtrations of M. For any $1 \leq i \leq s$ and $1 \leq j \leq t$ define $N_{i,j} = N_{i-1} + (L_j \cap N_i)$ and $L_{j,i} = L_{j-1} + (N_i \cap L_j)$. Then

$$0 = N_{1,0} \leq N_{1,1} \leq \cdots \leq N_{1,t} = N_1 = N_{2,0} \leq \cdots \leq N_{2,t} = N_2 \leq \ldots N_{s,t} = M$$

is a refinement of the first filtration with factors $F_{i,j} = N_{i,j}/N_{i,j-1}$ and

$$0 = L_{1,0} \leq L_{1,1} \leq \cdots \leq L_{1,s} = L_1 = L_{2,0} \leq \cdots \leq L_{2,s} = L_2 \leq \ldots L_{t,s} = M$$

is a refinement of the second filtration with factors $G_{j,i} = L_{j,i}/L_{j,i-1}$. Clearly the two refinements have the same length st and by the lemma above $F_{i,j} \cong G_{j,i}$. □

As a corollary of the previous Theorem, we get the following crucial result, known as Jordan-Hölder Theorem:

Theorem 4.4.2 (Jordan-Hölder). *If ${}_RM$ has a composition series of length l, then*

1. *any filtration of M has length at most l and can be refined to a composition series,*
2. *all composition series of M are equivalent and have length l.*

Proof. The proof follows by the previous proposition, since a composition series does not admit any non trivial refinement. □

This leads to the following definition:
Definition: A left R-module has *finite length* if it admits a composition series. The length l of any composition series of a module M is called the *length*, denoted by $l(M)$.

Examples:

1. Any vector space of finite dimension over a field k is a k-module of finite length. Its length coincides with its dimension.
2. The regular module ${}_{\mathbb{Z}}\mathbb{Z}$ is not of finite length.
3. Given an integer $n > 0$ with prime decomposition $n = p_1 \cdot \ldots \cdot p_r$, the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ has a compostion series

$$0 \subset \ldots \subset p_1p_2\mathbb{Z}/n\mathbb{Z} \subset p_1\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}$$

with compostion factors $\mathbb{Z}/p_i\mathbb{Z}, 1 \leq i \leq r$.

In the following proposition we collect some relevant properties of finite length modules. Most proofs are easy and left to the reader.

Proposition 4.4.3. *Let M be a left R-module of finite length. Then*

1. *M is finitely generated.*
2. *M is* noetherian, *i. e. every ascending chain of submodules $M_1 \subset M_2 \subset M_3 \subset \dots$ stabilizes: there is an integer m such that $M_m = M_{m+1} = \dots$.*
3. *M is* artinian, *i. e. every descending chain of submodules $\dots \subset M_3 \subset M_2 \subset M_1$ stabilizes: there is an integer m such that $M_m = M_{m+1} = \dots$.*
4. *If ${}_RN \leq {}_RM$, then N and M/N are of finite length.*
5. *If $0 \to N \to M \to L \to 0$ is an exact sequence, then $l(M) = l(N) + l(L)$.*
6. *M is a direct sum of indecomposable submodules.*
7. *$\mathrm{Soc}(M)$ is an essential submodule of M.*
8. *M is semisimple if and only if $\mathrm{Rad}\, M = 0$.*
9. *$M/\mathrm{Rad}(M)$ is semisimple.*

Proof. (6) If M is indecomposable the statement is trivially true. Otherwise we argue by induction on $l(M)$. If $M = V_1 \oplus V_2$, by point 5) we get that $l(V_1) < l(M)$ and $l(V_2) < l(M)$, so V_1 and V_2 are direct sums of indecomposable submodules.
(7) Any $0 \neq L \leq M$ has a composition series, so it contains a simple submodule, which is of course also a simple submodule of M.
(8) Simple modules have obviously a trivial radical, so the only-if part follows from Proposition 4.2.3(4). Conversely, assume $\mathrm{Rad}\, M = 0$ and let ${}_RL \leq {}_RM$. We have to show that L is a direct summand. Choose ${}_RN \leq {}_RM$ of minimal length such that $L + N = M$. Then $L \cap N$ is superfluous in N, because every submodule ${}_RU \leq {}_RN$ with $L \cap N + U = N$ satisfies $L + U = M$ and thus must coincide with N by length arguments. We infer that $L \cap N \subseteq \mathrm{Rad}\, N \subseteq \mathrm{Rad}\, M = 0$, so $L \oplus N = M$.
(9) Recall from Proposition 4.2.3(5) that $\mathrm{Rad}(M/\mathrm{Rad}(M)) = 0$. So the claim follows from (4) and (8). □

We want to refine statement 6. above.

Lemma 4.4.4. *A left R-module M is indecomposable if $\mathrm{End}_R(M)$ is a local ring.*

Proof. To any decomposition $M = N \oplus L$, we can associate an idempotent element $e_N = \iota_N \pi_N \in \mathrm{End}_R(M)$, given by $e_N : M \to M$, $n + l \mapsto n$. By Remark 4.3.2 it follows $e_N = 0$ or $e_N = \mathrm{id}_M$ in $\mathrm{End}_R(M)$, from which we get $N = 0$ or $N = M$, respectively. □

If M has finite length, also the converse holds true. We first need

Lemma 4.4.5 (Fitting's Lemma)**.** *Let M be a module of finite length $l(M) = n$. Then, for any $f : M \to M$, one has $M = \operatorname{Im} f^n \oplus \operatorname{Ker} f^n$.*

Proof. The descending chain $\cdots \leq \operatorname{Im} f^2 \leq \operatorname{Im} f \leq M$ stabilizes at an integer m, and of course $m \leq n$. In particular, there exists m such that $\operatorname{Im} f^m = \operatorname{Im} f^{2m}$ and we can assume $m = n$. Let now $x \in M$: hence $f^n(x) = f^{2n}(y)$ for $y \in M$ and so $x = f^n(y) - (x - f^n(y)) \in \operatorname{Im} f^n + \operatorname{Ker} f^n$.
Moreover, the ascending chain $0 \leq \operatorname{Ker} f \leq \operatorname{Ker} f^2 \leq \cdots \leq M$ stabilizes, so arguing as before we can assume $\operatorname{Ker} f^n = \operatorname{Ker} f^{2n}$. Consider now $x \in \operatorname{Im} f^n \cap \operatorname{Ker} f^n$. So $x = f^n(y)$ and $f^n(x) = f^{2n}(y) = 0$. Hence $y \in \operatorname{Ker} f^n$ and so $x = f^n(y) = 0$. □

Proposition 4.4.6. *A finite length module ${}_RM$ is indecomposable if and only if $\operatorname{End}_R(M)$ is a local ring.*

Proof. Let $f : M \to M$. Since M is indecomposable, by the previous lemma one easily conclude that f is a monomorphism if and only if it is an epimorphism if and only if it is an isomorphism if and only if $f^m \neq 0$ for any $m \in \mathbb{N}$ (see Exercise 10).
Hence, if f is not an isomorphism, $f^m = 0$ for some m, and

$$(\mathrm{id}_M - f)(\mathrm{id}_M + f + f^2 + \cdots + f^{r-1}) = \mathrm{id}_M$$

verifying condition (2) in Prpopostion 4.3.1. □

Theorem 4.4.7 (Krull-Remak-Schimdt-Azumaya)**.** *Let $M \cong A_1 \oplus A_2 \oplus \cdots \oplus A_m \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n$ where $\operatorname{End}_R(A_i)$ is a local ring for any $i = 1, \cdots, m$ and C_j is indecomposable for any $j = 1, \cdots, n$. Then $n = m$ and there exists a bijection $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $A_i \cong C_{\sigma(i)}$ for any $i = 1, \cdots, n$.*

Proof. By induction on m.
If $m = 1$, then $M \cong A_1$ is indecomposable and the calim follows.
If $m > 1$, consider the equalities

$$\mathrm{id}_{A_m} = \pi_{A_m} i_{A_m} = \pi_{A_m}\Big(\sum_{j=1}^{n} i_{C_j}\pi_{C_j}\Big) i_{A_m} = \sum_{j=1}^{n} \pi_{A_m} i_{C_j}\pi_{C_j} i_{A_m},$$

where the π's and the i's are the canonical projections and inclusions. Since $\operatorname{End}_R(A_m)$ is local, and in any local ring the sum of not invertible elements is not invertible, there exist $\bar{j}$ such that $\alpha = \pi_{A_m} i_{C_{\bar{j}}}\pi_{C_{\bar{j}}} i_{A_m}$ is invertible. We can assume $\bar{j} = n$, and consider $\gamma = \alpha^{-1}\pi_{A_m} i_{C_n} : C_n \to A_m$. Since $\gamma\pi_{C_n} i_{A_m} = \mathrm{id}_{A_m}$, we get that γ is a split epimorphism. Since C_n is indecomposable, we conclude γ is an iso, and so $C_n \cong A_m$. Then apply induction to get the claim. □

The previous theorem says that if M is a module which is a direct sum of modules with local endomorphism rings, then any two direct sum decompositions of M into indecomposable direct summands are isomorphic. We conclude

Corollary 4.4.8. *Every module of finite length admits a unique decomposition in indecomposable submodules (up to ordering and isomorphism).*

4.5 Injective cogenerators

We turn to the question posed at the end of Section 3. An injective module ${}_RE$ such that any module ${}_RM$ embeds in a product E^I of copies of E (for some set I) is called an *injective cogenerator*. It is a *minimal injective cogenerator* if it is isomorphic to a direct summand of any other injective cogenerator.

Proposition 4.5.1. *An injective module E is a cogenerator if and only if for any simple module S there exists a monomorphism $0 \to S \to E$.*

Proof. Assume E is a cogenerator, so for any simple module S there exists a monomorphism $0 \to S \overset{f_S}{\to} E^{I_S}$, for a set I_S. Then there exist $j \in I_S$ such that $\pi_j \circ f : S \to E$ is not the zero map. So, since $\mathrm{Ker}(\pi_j \circ f) \leq S$, we get that for any simple module S there exists a monomorphism $\pi_j \circ f : S \to E$. Conversely, assume the existence a monomorphism $0 \to S \to E$ for any simple module S. Let M be a module, and let $x \in M$, $x \neq 0$. So $Rx \leq M$ and $Rx \cong R/\mathrm{Ann}_R(x)$. By Proposition 4.1.1 there exists a maximal submodule $\mathfrak{m} \leq R$ such that $\mathrm{Ann}_R(x) \leq \mathfrak{m}$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & Rx \cong R/\mathrm{Ann}_R(x) \arrow[r] \arrow[d] & M \arrow[ddl, dashed, "\varphi_x"] \\
& R/\mathfrak{m} \cong S \arrow[d, "f"'] & \\
& E &
\end{tikzcd}$$

where f is a monomorphism that exists by assumption and $\varphi_x : M \to E$ exists since E is injective. In particular $\varphi_x(x) \neq 0$. Hence we can construct a monomorphism $\varphi : M \to E^M$, $x \mapsto (0, 0, \ldots, 0, \varphi_x(x), 0, \ldots, 0)$, where $\varphi_x(x)$ is the x^{th} position. $\square$

Corollary 4.5.2. *Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a set of representatives of the simple left R-modules, up to isomorphism. Then the injective envelope $E(\oplus S_\lambda)$ is a minimal injective cogenerator.*

Proof. The injective module $E(\oplus S_\lambda)$ cogenerates all the simple modules, so by the previous Proposition it is an injective cogenerator. If W is a injective cogenerator, since $S_\lambda \leq W$ for any $\lambda \in \Lambda$ (see the argument in the previous proof) one gets $\oplus S_\lambda \leq W$. Since $E(\oplus S_\lambda)$ is the injective envelope of $\oplus S_\lambda$, we conclude $E(\oplus S_\lambda) \overset{\oplus}{\leq} W$. $\square$

Remark 4.5.3. If there is only a finite number of simple left R-modules S_1, S_2, $\ldots$, S_n, up to isomorphism, then $E(\oplus S_i) = \oplus E(S_i)$ is a minimal injective cogenerator.

4.6 Exercises - Part 2

(published on October 28, **solutions to be submitted on November 10, 2016**).

Exercise 5. Let R be a ring.

(a) An idempotent element $e \in R$ is called *primitive* if it is not a sum of two non zero orthogonal idempotents. Show that Re is indecomposable if and only if e is primitive.

(b) Find the decomposition in indecomposable summands of

(i) $M_2(\mathbb{C})$= the ring of 2×2 matrices with coefficients in $\mathbb{C}$,

(ii) the path algebra of the quiver $Q : \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$ over $\mathbb{C}$.

(c) Let E_i, $i = 1, \dots, n$, be R-modules. Show: $\bigoplus_{i=1}^n E_i$ is injective if and only if E_i is injective for any $i = 1 \dots n$.

(d) Let $f \in \operatorname{Hom}_R(L, M)$ be an essential monomorphism, and $g \in \operatorname{Hom}_R(M, N)$.

Show: if gf is a monomorphism, then so is g.

Exercise 6. (a) Write the representation $K^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} K^2$ of $\mathbb{A}_2$ as a direct sum of two indecomposable representations.

(b) Find the injective envelope of the representation $0 \to K$ of $\mathbb{A}_2$.

(c) Given the path algebra $\Lambda = kQ$ of the quiver $Q : 1 \leftarrow 2 \rightarrow 3 \rightarrow 4$, compute the module Λe_2, its radical, and its socle.

Exercise 7. (a) Show: If X is a generating set of the $\mathbb{Z}$-module $\mathbb{Q}$, and $x \in X$, then $X \setminus \{x\}$ is a generating set of $\mathbb{Q}$ as well.

(b) Deduce from (a) that every finitely generated submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is superfluous.

(c) Conclude that $\operatorname{Rad}_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, and $\mathbb{Q}$ has no maximal submodules.

(d) Let M be a finitely generated left R-module iver a ring R. Show that any proper submodule $L < M$ is contained in a maximal submodule of M.

Exercise 8. Show that $\operatorname{Rad}({}_R R) = \{r \in R \mid 1 - xr \text{ has a left inverse for any } x \in R\}$. (Hint: Argue by contradiction, and use that $\operatorname{Rad}({}_R R)$ is the intersection of the annihilators of the simple left R-modules for $\supseteq$.)

5 CATEGORIES AND FUNCTORS

This is very short introduction to the basic concepts of category theory. For more details and for the set-theoretical foundation (in particular the distinction between sets and classes) we refer to S. MacLane, Category for the working mathematician, Graduate Texts in Math., Vol 5, Springer 1971.
Definition: A category $\mathcal{C}$ consists in:

1. A class $Obj(\mathcal{C})$, called the *objects* of $\mathcal{C}$;
2. for each ordered pair (C, C') of objects of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(C, C')$ whose elements are called *morphisms* from C to C';
3. for each ordered triple (C, C', C'') of objects of $\mathcal{C}$, a map
$$\mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{C}}(C', C'') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$$
called *composition* of morphisms

such that the following axioms C1, C2, C3 hold:
(before stating the axioms, we introduce the notations $\alpha : C \to C'$ for any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, and $\beta\alpha$ for the compostion of $\alpha \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\beta \in \mathrm{Hom}_{\mathcal{C}}(C', C'')$)

C1: if $(C, C') \neq (D, D')$, then $\mathrm{Hom}_{\mathcal{C}}(C, C') \cap \mathrm{Hom}_{\mathcal{C}}(D, D') = \emptyset$

C2: if $\alpha : C \to C'$, $\beta : C' \to C''$, $\gamma : C'' \to C'''$ are morphisms, then $\gamma(\beta\alpha) = (\gamma\beta)\alpha$

C3: for each object C there exists $1_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$, called *identity morphism*, such that $1_C\alpha = \alpha$ and $\beta 1_C = \beta$ for any $\alpha : C' \to C$ and $\beta : C \to C'$.

Notice that, for any $C \in Obj(\mathcal{C})$, the identity morphism 1_C is unique. Indeed, if also $1'_C$ satisfies [C3], then $1_C = 1_C 1'_C = 1'_C$.
A morphism $\alpha : C \to C'$ is an *isomorphism* if there exists $\beta : C' \to C$ such that $\beta\alpha = 1_C$ and $\alpha\beta = 1_{C'}$. If α is an isomorphism, C and C' are called *isomorphic* and we write $C \cong C'$.
Examples:

1. The category **Sets**: the class of objects is the class of all sets; the morphisms are the maps between sets with the usual compositions.
2. The category **Ab**: the objects are the abelian groups; the morphisms are the group homomorphisms with the usual compositions.
3. The category $R\,\mathrm{Mod}$ for a ring R: the objects are the left R-modules and the morphisms are the module homomorphisms with the usual compositions.
4. The category **Mod-**R for a ring R: the objects are the right R-modules and the morphisms are the module homomorphisms with the usual compositions.

Notice that, given a category $\mathcal{C}$, we can construct the *dual* category $\mathcal{C}^{op}$, with $Obj(\mathcal{C}^{op}) = Obj(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{op}}(C, C') = \mathrm{Hom}_{\mathcal{C}}(C', C)$, and $\alpha * \beta = \beta \cdot \alpha$, where $*$ denotes the composition in $\mathcal{C}^{op}$ and $\cdot$ the composition in $\mathcal{C}$ ($\mathcal{C}^{op}$ is obtained from $\mathcal{C}$ by "reversing the arrows"). Any statement regarding a category $\mathcal{C}$ dualizes to a corresponding statement for $\mathcal{C}^{op}$.

Definition: Let $\mathcal{B}$ and $\mathcal{C}$ be two categories. A *functor* $F : \mathcal{B} \to \mathcal{C}$ assigns to each object $B \in \mathcal{B}$ an object $F(B) \in \mathcal{C}$, and assigns to any morphism $\beta : B \to B'$ in $\mathcal{B}$ a morphism $F(\beta) : F(B) \to F(B')$ in $\mathcal{C}$, in such a way:

F1: $F(\beta\alpha) = F(\beta)F(\alpha)$ for any $\alpha : B \to B'$, $\beta : B' \to B''$ in $\mathcal{B}$

F2: $F(1_B) = 1_{F(B)}$ for any B in $\mathcal{B}$.

By construction, a functor $F : \mathcal{B} \to \mathcal{C}$ defines a map for any B, B' in $\mathcal{B}$

$$\eta_{B,B'} : \mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{C}}(F(B), F(B')), \quad \beta \mapsto F(\beta)$$

The functor F is called *faithful* if all these maps are injective and is called *full* it they are surjective. If F is full and faithful, then all the maps $\eta_{B,B'}$ are bijective and so the morphisms in the two categories are the same.

A functor $F : \mathcal{B}^{op} \to \mathcal{C}$ is called a *contravariant* functor from $\mathcal{B}$ to $\mathcal{C}$. In particular a contravariant functor F assigns to any morphism $\beta : B \to B'$ in $\mathcal{B}$ a morphism $F(\beta) : F(B') \to F(B)$ in $\mathcal{C}$.

Examples:

1. Let $\mathcal{B}$ and $\mathcal{C}$ two categories. $\mathcal{B}$ is a *subcategory* of $\mathcal{C}$ if $Obj(\mathcal{B}) \subseteq Obj(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{B}}(B, B') \subseteq \mathrm{Hom}_{\mathcal{C}}(B, B')$ for any B, B' objects of $\mathcal{B}$, and the compositions in $\mathcal{B}$ and $\mathcal{C}$ are the same. In this case there is a canonical functor $\mathcal{B} \to \mathcal{C}$ which is clearly faithful. If this functor is also full, $\mathcal{B}$ is said a *full subcategory* of $\mathcal{C}$.

2. Let $M \in R\,\mathrm{Mod}$. As we have already observed $\mathrm{Hom}_R(M, N)$ is an abelian group for any $N \in R\,\mathrm{Mod}$. So we can define a functor (Check the axioms!)

$$\mathrm{Hom}_R(M, -) : R\,\mathrm{Mod} \to \mathbf{Ab}, \quad N \mapsto \mathrm{Hom}_R(M, N)$$

such that for any $\alpha : N \to N'$,

$$\mathrm{Hom}_R(M, \alpha) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'), \ \varphi \mapsto \alpha\varphi$$

3. Let $M \in R\,\mathrm{Mod}$ and consider the abelian group $\mathrm{Hom}_R(N, M)$ for any $N \in R\,\mathrm{Mod}$. So we can define a contravariant functor (Check the axioms!)

$$\mathrm{Hom}_R(-, M) : (R\,\mathrm{Mod})^{op} \to \mathbf{Ab}, \quad N \mapsto \mathrm{Hom}_R(N, M)$$

such that for any $\alpha : N \to N'$,

$$\mathrm{Hom}_R(\alpha, M) : \mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N, M), \ \psi \mapsto \psi\alpha$$

In these lectures we will deal mainly with categories having some kind of additive structure. For instance in the category $R\,\mathrm{Mod}$, any set of morphisms $\mathrm{Hom}_R(M,N)$ is an abelian group and the composition preserves the sums.

Definition: A category $\mathcal{C}$ is called *preadditive* if each set $\mathrm{Hom}_{\mathcal{C}}(C,C')$ is an abelian group and the compositions maps $\mathrm{Hom}_{\mathcal{C}}(C,C')\times\mathrm{Hom}_{\mathcal{C}}(C',C'')\to\mathrm{Hom}_{\mathcal{C}}(C,C'')$ are bilinear.
If $\mathcal{B}$ and $\mathcal{C}$ are preadditive categories, a functor $F:\mathcal{B}\to\mathcal{C}$ is *additive* if $F(\alpha+\alpha')=F(\alpha)+F(\alpha')$ for $\alpha,\alpha':C\to C'$.

Examples: The category $R\,\mathrm{Mod}$ is a preadditive category. If $M\in R\,\mathrm{Mod}$, then $\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,M)$ are additive functors.

Definition: Let R and S two rings and let $F:R\,\mathrm{Mod}\to S\,\mathrm{Mod}$ be an additive functor. F is called *left exact* if, for any exact sequence $0\to L\to M\to N\to 0$ in $R\,\mathrm{Mod}$, the sequence $0\to F(L)\to F(M)\to F(N)$ in $S\,\mathrm{Mod}$ is exact. F is called *right exact* if, for any exact sequence $0\to L\to M\to N\to 0$ in $R\,\mathrm{Mod}$, the sequence $F(L)\to F(M)\to F(N)\to 0$ in $S\,\mathrm{Mod}$ is exact. The functor F is *exact* if it is both left and right exact.

In particular, if F is exact then for any exact sequence in $R\,\mathrm{Mod}$ $0\to L\to M\to N\to 0$, the corresponding sequence $0\to F(L)\to F(M)\to F(N)\to 0$ in $S\,\mathrm{Mod}$ is exact.

Proposition 5.0.1. *Let $X\in R\,\mathrm{Mod}$. The functor $\mathrm{Hom}_R(X,-)$ is left exact*

Proof. Let $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ be an exact sequence in $R\,\mathrm{Mod}$. Denoted by $f^*=\mathrm{Hom}_R(X,f)$ and $g^*=\mathrm{Hom}_R(X,g)$, we have to show that the sequence of abelian groups $0\to\mathrm{Hom}_R(X,L)\xrightarrow{f^*}\mathrm{Hom}_R(X,M)\xrightarrow{g^*}\mathrm{Hom}_R(X,N)$ is exact. In particular, we have to show that f^* is a monomorphism and that $\mathrm{Im}\,f^*=\mathrm{Ker}\,g^*$.
Let us start considering $\alpha:X\to L$ such that $f^*(\alpha)=0$. So for any $x\in X$ $f^*(\alpha)(x)=f\alpha(x)=0$. Since f is a monomorphism we conclude $\alpha(x)=0$ for any $x\in X$, that is $\alpha=0$.
Consider now $\beta\in\mathrm{Im}\,f^*$; then there exists $\alpha\in\mathrm{Hom}_R(X,L)$ such that $\beta=f^*(\alpha)=f\alpha$. Hence $g^*(\beta)=g\beta=gf\alpha=0$, since $gf=0$. So we get $\mathrm{Im}\,f^*\leq\mathrm{Ker}\,g^*$.
Finally, let $\beta\in\mathrm{Ker}\,g^*$, so that $g\beta=0$ This means $\mathrm{Im}\,\beta\leq\mathrm{Ker}\,g=\mathrm{Im}\,f$. For any $x\in X$ define α as $\alpha(x)=f^{\leftarrow}(\beta(x))$: α is well-defined since f is a monomorphism and clearly $\beta=f\alpha=f^*(\alpha)$. So we get $\mathrm{Ker}\,g^*\leq\mathrm{Im}\,f^*$ □

In a similar way one prove that the functor $\mathrm{Hom}_R(-,X)$ is left exact. Notice that, since $\mathrm{Hom}_R(-,X)$ is a contravariant functor, left exact means that for any exact sequence in $R\,\mathrm{Mod}$ $0\to L\to M\to N\to 0$, the corresponding sequence of abelian groups $0\to\mathrm{Hom}_R(N,X)\to\mathrm{Hom}_R(M,X)\to\mathrm{Hom}_R(L,X)$ is exact.

Remark 5.0.2. Notice that if F is an additive functor and $0\to L\xrightarrow{f} M\xrightarrow{g} N\to 0$ is a split exact sequence in $R\,\mathrm{Mod}$, then $0\to F(L)\xrightarrow{F(f)} F(M)\xrightarrow{F(g)} F(N)\to 0$ is split exact. Indeed, since there exists φ such that $\varphi f=id_L$ (see Proposition 3.2.1), $F(\varphi)F(f)=id_{F(L)}$, so $F(f)$ is a split mono. Similarly one show that $F(g)$ is a split epi.

In particular, for a given module $X \in R\,\mathrm{Mod}$ the functors $\mathrm{Hom}_R(X,-)$ and $\mathrm{Hom}_R(-,X)$ could be not exact. Nevertheless, if $0 \to L \to M \to N \to 0$ is a split exact sequence in $R\,\mathrm{Mod}$, then the sequence $0 \to \mathrm{Hom}_R(X,L) \to \mathrm{Hom}_R(X,M) \to \mathrm{Hom}_R(X,N) \to 0$ and the sequence $0 \to \mathrm{Hom}_R(N,X) \to \mathrm{Hom}_R(M,X) \to \mathrm{Hom}_R(L,X) \to 0$ are split exact. In particular $\mathrm{Hom}_R(X, L \oplus N) \cong \mathrm{Hom}_R(X,L) \oplus \mathrm{Hom}_(X,N)$ and $\mathrm{Hom}_R(L \oplus N, X) \cong \mathrm{Hom}_R(L,X) \oplus \mathrm{Hom}_R(N,X)$

One often wishes to compare two functors with each other. So we introduce the notion of *natural transformation*:
Definition: Let F and G two functors $\mathcal{B} \to \mathcal{C}$. A natural transformation $\eta : F \to G$ is a family of morphisms $\eta_B : F(B) \to G(B)$, for any $B \in \mathcal{B}$, such that for any morphism $\alpha : B \to B'$ in $\mathcal{B}$ the following diagram in $\mathcal{C}$ is commutative

$$\begin{tikzcd} F(B) \arrow[r, "\eta_B"] \arrow[d, "F(\alpha)"'] & G(B) \arrow[d, "G(\alpha)"] \\ F(B') \arrow[r, "\eta_{B'}"] & G(B') \end{tikzcd}$$

If η_B is an isomorphism in $\mathcal{C}$ for any $B \in \mathcal{B}$, then η is called a *natural equivalence.*

Two categories $\mathcal{B}$ and $\mathcal{C}$ are *isomorphic* if there exist functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ such that $GF = 1_{\mathcal{B}}$ and $FG = 1_{\mathcal{C}}$. This is a very strong notion, in fact there are several and relevant examples of categories $\mathcal{B}$ and $\mathcal{C}$ which have essentially the same structure, but where there is a bijective correspondence between the isomorphism classes of objects rather than between the individual objects. Therefore we define the following concept:
Definition: A functor $F : \mathcal{B} \to \mathcal{C}$ is an *equivalence* if there exists a functor $G : \mathcal{C} \to \mathcal{B}$ and natural equivalences $GF \to 1_{\mathcal{B}}$ and $FG \to 1_{\mathcal{C}}$

If the functor F is contravariant and gives an equivalence between $\mathcal{B}^{op}$ and $\mathcal{C}$, we say that F is a *duality.*

Proposition 5.0.3. *A functor $F : \mathcal{B} \to \mathcal{C}$ is an equivalence if and only if it is full and faithful, and every object of $\mathcal{C}$ is isomorphic to an object of the form $F(B)$, with $B \in \mathcal{B}$.*

Thanks to the previous proposition and its analogous for any duality, one can prove the following properties (we state everything in case of a duality, since we will deeply deal with this setting in the final section):

Proposition 5.0.4. *Let R and S be two rings and $F : R\,\mathrm{Mod} \to S\,\mathrm{Mod}$ be a duality. Then:*

1. $0 \to M \stackrel{f}{\to} N$ *is a monomorphism in* $R\,\mathrm{Mod}$ *if and only if* $F(N) \stackrel{F(f)}{\to} F(M) \to 0$ *is an epimorphism in* $S\,\mathrm{Mod}$.
2. $M \stackrel{f}{\to} N \to 0$ *is an epimorphism in* $R\,\mathrm{Mod}$ *if and only if* $0 \to F(N) \stackrel{F(f)}{\to} F(M)$ *is an epimorphism in* $S\,\mathrm{Mod}$.

3. $M \xrightarrow{f} N$ *is an iso in* $R\,\mathrm{Mod}$ *if and only if* $F(N) \xrightarrow{F(f)} F(M)$ *is an iso in* $S\,\mathrm{Mod}$.

4. *The sequence* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is exact in* $R\,\mathrm{Mod}$ *if and only if the sequence* $0 \to F(N) \xrightarrow{F(g)} F(M) \xrightarrow{F(f)} F(L) \to 0$ *is exact in* $S\,\mathrm{Mod}$

5. *an object* $B \in R\,\mathrm{Mod}$ *is projective if and only if* $F(B) \in S\,\mathrm{Mod}$ *is injective.*

6. *An object* $B \in R\,\mathrm{Mod}$ *is injective if and only if* $F(B) \in S\,\mathrm{Mod}$ *is projective.*

7. *An object* $B \in R\,\mathrm{Mod}$ *is indecomposable if and only if* $F(B) \in S\,\mathrm{Mod}$ *is indecomposable.*

8. *An object* $B \in R\,\mathrm{Mod}$ *is simple if and only if* $F(B) \in S\,\mathrm{Mod}$ *is simple.*

6 MODULES OVER FINITE DIMENSIONAL ALGEBRAS

Throughout this chapter, we fix a field k and a finite dimensional algebra Λ over k. We denote by $\Lambda\,\mathrm{mod}$ the full subcategory of $\Lambda\,\mathrm{Mod}$ consisting on the finitely generated Λ-modules.

6.1 Basic and indecomposable algebras

(1) Λ is Morita equivalent to a *basic* finite dimensional algebra, that is, the category $\Lambda\,\mathrm{Mod}$ is equivalent to $S\,\mathrm{Mod}$ where S is a finite dimensional algebra with the property that ${}_S S$ is a direct sum of *pairwise nonisomorphic* projectives, or equivalently, $S/J(S)$ is a product of division rings, see [1, p. 309] or [13, II.2].

(2) The ring Λ has a block decomposition $\Lambda = \bigoplus_{i=1}^{m} b_i \Lambda b_i$ with central orthogonal idempotents $b_1, \ldots, b_m \in \Lambda$ such that $\sum_{i=1}^{m} b_i = 1$. The blocks $b_i \Lambda b_i$ are then *indecomposable* finite dimensional algebras, that is, they do not admit a non-trivial decomposition in a direct product of subalgebras, see [1, 7.9] or [13, II.5].
Note that this induces a decomposition on the Λ-modules. In fact, for $M, N \in \Lambda\,\mathrm{Mod}$ we have $M = \bigoplus_{i=1}^{m} b_i M$ and $\mathrm{Hom}_\Lambda(M,N) \cong \prod_{i=1}^{m} \mathrm{Hom}_{b_i\Lambda b_i}(b_i M, b_i N)$. In particular, every indecomposable Λ-module belongs to the module category $\mathrm{Mod} b_i \Lambda b_i$ of a single block $b_i \Lambda b_i$.

For the purpose of studying $\Lambda\,\mathrm{Mod}$, we can thus assume w.l.o.g. that Λ is indecomposable and basic.

6.2 The Gabriel-quiver of an algebra

Over an algebraically closed field, every finite dimensional algebra is a quotient of a path algebra.

Theorem 6.2.1 (Gabriel 1972)**.** *Let Λ be an indecomposable basic finite dimensional algebra over an algebraically closed field k. Then there are a connected finite quiver Q and a finitely generated ideal $\mathcal{I} \subset kQ$ such that*

> *(i) $\Lambda \cong kQ/\mathcal{I}$;*
> *(ii) if $\mathcal{A}$ is the ideal of kQ generated by the arrows of Q, then there is $t \in \mathbb{N}$ such that $\mathcal{A}^t \subset \mathcal{I} \subset \mathcal{A}^2$.*

The quiver Q is uniquely determined by Λ and is called the Gabriel-quiver *of Λ.*

Proof: We only sketch the arguments and refer to [13, III.1] for a complete proof. Let $e_1, \ldots, e_n \in \Lambda$ be primitive orthogonal idempotents with $\sum_{i=1}^{n} e_i = 1$. For distinct indices $1 \leq i, j \leq n$ set $d_{ji} = \dim_k e_j(J/J^2)e_i$ and choose elements $b_1, \ldots, b_{d_{ji}} \in e_j J e_i$ that induce a k-basis $\overline{b_1}, \ldots, \overline{b_{d_{ji}}}$ of $e_j J e_i / e_j J^2 e_i \cong e_j(J/J^2)e_i$.
The quiver Q is constructed as follows: It has n vertices $1, \ldots, n$ corresponding to $e_1, \ldots, e_n$. Given two vertices $i, j \in Q_0$, there are d_{ji} arrows $\alpha_1, \ldots, \alpha_{d_{ji}} \colon i \to j$ corresponding to $b_1, \ldots, b_{d_{ji}} \in e_j J e_i$.
The fact that k is algebraically closed and Λ is basic implies $\Lambda/J \cong k^n$. Then it can be shown that the map

$$kQ \to \Lambda\,, \qquad (i\|i) \mapsto e_i\,, \quad \text{and} \quad (i \xrightarrow{\alpha_l} j) \mapsto b_l$$

is a surjective homomorphism of k-algebras.
Let $\mathcal{I}$ be its kernel. Then there is $s \in \mathbb{N}$ such that $J(kQ)^s \subset \mathcal{I} \subset J(kQ)^2$. So $\mathcal{I}$ is finitely generated, and of course, $\Lambda \cong kQ/\mathcal{I}$. □

Remarks: (1) In general, the finitely generated ideal $\mathcal{I}$ in the Theorem above defines finitely many *relations* on Q. In fact, for each element y in a finite set of generators of $\mathcal{I}$, we can write $y = \sum_{1 \leq i,j \leq n} e_j\, y\, e_i$ and replace y by the elements $e_j\, y\, e_i$ which are linear combinations of paths from i to j of length at least two. So, every finite dimensional algebra over an algebraically closed field is given by a finite quiver with finitely many relations.
(2) The numbers d_{ji} have various interpretations. With the notation of **??** we have $d_{ji} = \dim_k \operatorname{Ext}^1_\Lambda(S_i, S_j)$. Moreover, if $P \to P_i \to S_i \to 0$ is a minimal projective presentation of S_i, then d_{ji} equals the multiplicity of P_j as a direct summand of P, see [13, III, 1.14].

6.3 Modules and representations

Proposition 6.3.1. *Let Q be a finite quiver without oriented cycles, k a field, and let $\Lambda = kQ/\mathcal{I}$. The category $\Lambda\,\mathrm{mod}$ of finitely generated Λ-modules is equivalent to the category of finite dimensional representations of Q over k which are bound by $\mathcal{I}$.*

Proof: For a module $M \in \Lambda\,\mathrm{mod}$, the corresponding representation is given by the family of k-vectorspaces $(e_i M)_{i \in Q_0}$ and the family of k-homomorphisms $(f_\alpha \colon e_i M \to e_j M, \quad e_i x \mapsto \alpha e_i x = e_j \alpha e_i x)_{i \xrightarrow{\alpha} j \in Q_1}$. For a detailed treatment, we refer to [13, III.1]. □

6.4 Finite dimensional modules

Corollary 6.4.1. *Any finitely generated module $M \in \Lambda\,\mathrm{mod}$ is a finite length module, and $l(M) \leq \dim_k(M)$.*

Proof. Since any $M \in \Lambda \operatorname{mod}$ is a finite dimensional vector space, M admits a composition series in $k \operatorname{mod}$ of length n, where $\dim_k(M) = n$. So any filtration of M in $\Lambda \operatorname{Mod}$ is at most of length n and any refinement is a refinement also in $k \operatorname{mod}$. This gives the claim. $\square$

Proposition 6.4.2. *If M, N are finitely generated Λ-modules, then $\operatorname{Hom}_\Lambda(M,N)$ is a finitely generated k-module via the multiplication*

$$\alpha \cdot f \colon m \mapsto \alpha f(m) \quad \textit{for} \quad \alpha \in k,\ f \in \operatorname{Hom}_\Lambda(M,N)$$

In particular, $\operatorname{End}_\Lambda N$ and $(\operatorname{End}_\Lambda N)^{\mathrm{op}}$ are again finite dimensional k-algebras, and N is a Λ-$(\operatorname{End}_\Lambda N)^{\mathrm{op}}$-bimodule via the multiplication

$$n \cdot s := s(n) \quad \textit{for} \quad n \in N,\ s \in \operatorname{End}_\Lambda N$$

Moreover,

$${}_{\operatorname{End} N} \operatorname{Hom}_\Lambda(M,N)_{\operatorname{End} M}$$

is an $\operatorname{End} N$-$\operatorname{End} M$-bimodule which has finite length on both sides.

Proof. The k-module $\operatorname{Hom}_\Lambda(M,N)$ is a k-submodule of $\operatorname{Hom}_k(M,N)$, and the latter is finitely generated by a well-known result of linear algebra. Thus $\operatorname{Hom}_\Lambda(M,N)$ is finitely generated as k-module. In particular, $\Gamma = \operatorname{Hom}_\Lambda(M,M)$ is a finite dimensional k-algebra. Since M has a natural structure of right Γ-module and it is a finitely generated k-module, it is also a finitely generated Γ-module. $\square$

In the sequel, let Λ be a *finite dimensional, indecomposable, basic k-algebra* over a field k. We want to determine the simple, the indecomposable projective and the indecomposable injective left modules over Λ.

Proposition 6.4.3. *There are primitive orthogonal idempotents*

$$e_1, \ldots, e_n \in \Lambda \textit{ such that } 1 = \sum_{i=1}^{n} e_i,$$

and $e_i \Lambda e_i$ is a local ring for every $1 \le i \le n$, yielding indecomposable decompositions

$${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i \quad \textit{and} \quad {}_\Lambda\Lambda/J \cong \bigoplus_{i=1}^{n} \Lambda e_i/Je_i,$$

and similarly for Λ_Λ and Λ/J_Λ.

Proof. Since ${}_\Lambda\Lambda$ is of finite length, it admits a unique decomposition in indecomposable submodules. The indecomposable summands of a ring are in correspondence with the idempotent elements, so there exists a set $\{e_1, e_2, \ldots, e_n\}$ of pairwise orthogonal idempotents of Λ such that ${}_\Lambda\Lambda = \Lambda e_1 \oplus \ldots \Lambda e_n$, and $1 = e_1 + \cdots + e_n$, see Remark 3.4.2. Finally since Λe_i are indecomposable, each idempototent e_i is *primitive*, i.e. it cannot be a sum of two non-zero orthogonal idempotents, see Exercise 5. Moreover, $e_i \Lambda e_i \cong \operatorname{End}_\Lambda \Lambda e_i$ is local. Notice finally that $\Lambda_\Lambda = e_1\Lambda \oplus \cdots \oplus e_n\Lambda$ is a decomposition in indecomposable summands of the regular right module Λ_Λ. $\square$

From this discussion it follows that, for $i = 1, \dots, n$, the $P_i = \Lambda e_i$ are indecomposable projective left Λ-modules and the $e_i\Lambda$ are indecomposable projective right Λ-modules. Moreover, if $P \in \Lambda\,\mathrm{mod}$ is an indecomposable projective, then P is a direct summand of Λ^m for a suitable $m > 0$. Since $\Lambda^m = P_1^m \oplus \dots P_n^m$, we conclude from Theorem 4.4.7 that P is isomorphic to P_j for a suitable $j \in \{1, \dots, n\}$

Proposition 6.4.4. *There is a duality*

$$D : \Lambda\,mod \longrightarrow \quad \mathrm{mod}\ \Lambda,\ M \mapsto \mathrm{Hom}_k(M, k),$$

and ${}_\Lambda D(\Lambda_\Lambda)$ *is an injective cogenerator of* $\Lambda\,Mod$.

Proof. The functor $D : \Lambda\,\mathrm{mod} \to \mathrm{mod}\text{-}\Lambda$, $M \mapsto D(M) = \mathrm{Hom}_k({}_\Lambda M, k)$ is well-defined, since $\mathrm{Hom}_k({}_\Lambda M, k)$ is a finitely generated right Λ-module with $\dim_k(\mathrm{Hom}_k({}_\Lambda M, k)) < \infty$. For simplicity, we denote by D the analogous functor $D : \mathrm{mod}\text{-}\Lambda \to \Lambda\,\mathrm{mod}$, $N \mapsto D(N) = \mathrm{Hom}_k(N_\Lambda, k)$. For any $M \in \Lambda\,\mathrm{mod}$ define the *evaluation morphism* $\delta_M : M \to D^2(M)$, $x \mapsto \delta_M(x)$, where $\delta_M(x) : D(M) \to k$, $\varphi \mapsto \varphi(x)$. One easily verifies by dimension arguments that δ_M is an isomorphism for any $M \in \Lambda\,\mathrm{mod}$. Similarly one defines δ_N for any $N \in \mathrm{mod}\text{-}\Lambda$, which is an iso for any N.
It turns out that $\delta : 1 \to D^2$ is a natural transformation (see Definition 5) which defines a duality between $\Lambda\,\mathrm{mod}$ and mod-Λ. Thanks to the properties of dualities described at the end of Section 5 , we get in particular that P is indecomposable projective in $\Lambda\,\mathrm{mod}$ if and only if $D(P)$ is indecomposable injective in mod-Λ; dually, E is indecomposable injective in $\Lambda\,\mathrm{mod}$ if and only if $D(E)$ is indecomposable projective in mod-Λ. Moreover S is simple in $\Lambda\,\mathrm{mod}$ if and only if $D(S)$ is simple in mod-Λ.
Notice the the concepts of cover and generator are dual to the concepts of envelope and cogenerator, respectively. So, thanks to the duality (D, D), we conclude that $D(\Lambda_\Lambda)$ is the minimal injective cogenerator of $\Lambda\,\mathrm{mod}$, and the $I_i = D(e_i\Lambda)$ are the unique indecomposable injective left Λ-modules up to isomorphism. $\square$

Observe that if S and T are non isomorphic simple modules in $\Lambda\,\mathrm{mod}$, then their injective envelopes $E(S)$ and $E(T)$ are non isomorphic indecomposable injective modules; moreover any indecomposable injective module E is the injective envelope of its simple socle (see Exercise 9). We conclude that in $\Lambda\,\mathrm{mod}$ there are exactly n non-isomorphic simple modules, which are the socles of the indecomposable injectives $I_1, \dots, I_n$.

One can easily verify that $P(M)$ is a projective cover of a module $M \in \Lambda\,\mathrm{mod}$ if and only if $D(P(M))$ is an injective envelope of $D(M)$. Since in mod-Λ there exist injective envelopes, thanks to the duality, we get that any module in $\Lambda\,\mathrm{mod}$ has a projective cover (i.e., Λ is a semiperfect ring, see Section 3.4).
Let us see how to compute injective envelopes and projective covers. In the sequel denote by $\mathrm{J} = \mathrm{J}(\Lambda) = \mathrm{Rad}({}_\Lambda\Lambda)$ the Jacobson radical of Λ.

Proposition 6.4.5. *The Jacobson radical* $J = J(\Lambda)$ *is nilpotent, i.e.* $J^r = 0$ *for some* $r \in \mathbb{N}$*, and* Λ/J *is semisimple. Further,* $\mathrm{Rad}\, M = JM$ *for every* $M \in \Lambda\,mod$.

Proof. The first statement is due to the fact that the descending chain $\ldots \subset J^3 \subset J^2 \subset J$ stabilizes in some J^r which must be zero by Nakayama's Lemma, and the second statement is Proposition 4.4.3(7). Next observe that, by Lemma 4.2.5, the two-sided ideal J satisfies $J\Lambda e_i = Je_i \leq \mathrm{Rad}(\Lambda e_i)$ for any $i = 1, \ldots, n$. Moreover recall that $J = \mathrm{Rad}({}_\Lambda\Lambda) = \mathrm{Rad}(\Lambda e_1) \oplus \cdots \oplus \mathrm{Rad}(\Lambda e_n)$ (see Proposition 4.2.3). Since the sum of the $\mathrm{Rad}(\Lambda e_i)$ is direct and $Je_i \leq \mathrm{Rad}(\Lambda e_i)$, we also get $J = J1 = J(e_1 + \ldots e_n) = Je_1 \oplus \ldots Je_n$. Thus, $\dim_k(J) = \dim_k(Je_1) + \cdots \dim_k(Je_n) \leq \dim_k(\mathrm{Rad}(\Lambda e_1)) + \cdots + \dim_k(\mathrm{Rad}(\Lambda e_n)) = \dim_k(\mathrm{Rad}(\Lambda))$, from which we infer $\dim_k(Je_i) = \dim_k(\mathrm{Rad}(\Lambda e_i))$ for any $i = 1, \ldots, n$. We conclude that $Je_i = \mathrm{Rad}(\Lambda e_i)$ for any $i = 1, \ldots, n$. It can be proved that the same holds true for any $M \in \Lambda\,\mathrm{mod}$. □

In particular, $\mathrm{J}\, e_1$ is superfluous in Λe_i, so Λe_i is the projective cover of $\Lambda e_i / \mathrm{J}\, e_i$ (see Theorem 4.2.2). Moreover, $\Lambda e_i / \mathrm{J}\, e_i$ is semisimple by Proposition 4.4.3, and it is even simple, because Λe_i is indecomposable (see Exercise 9). Notice that, since $\Lambda e_i \not\cong \Lambda e_j$ for $i \neq j$, we get $\Lambda e_i / \mathrm{J}\, e_i \not\cong \Lambda e_j / \mathrm{J}\, e_j$ for $i \neq j$. Then the $S_i = \Lambda e_i / \mathrm{J}\, e_i$, $i = 1, \ldots n$, are non-isomorphic simple modules in $\Lambda\,\mathrm{mod}$. Since we already know that there are exactly n non-isomorphic simple modules, we conclude that $S_1, \cdots, S_n$ is a complete list, up to isomorphism, of the simple left Λ-modules.
Similarly, $e_i\Lambda / e_i\, \mathrm{J}$, $1 \leq i \leq n$, is a complete list of the simple right Λ-modules. Arguing on the annihilators of the simple modules, it is not difficult to show that the action of the functor D on the simple modules respects the idempotents, that is

$$D(\Lambda e_i / Je_i) \cong e_i\Lambda / e_i J.$$

Summarizing:

$$\Lambda e_1, \ldots, \Lambda e_n$$

are representatives of the isomorphism classes of the indecomposable projectives in $\Lambda\,\mathrm{Mod}$,

$$\Lambda e_1 / Je_1, \ldots, \Lambda e_n / Je_n$$

are representatives of the isomorphism classes of the simples in $\Lambda\,\mathrm{Mod}$, and

$$D(e_1\Lambda), \ldots, D(e_n\Lambda)$$

are representatives of the isomorphism classes of the indecomposable injectives in $\Lambda\,\mathrm{Mod}$, where for all $1 \leq i \leq n$

$P_i = \Lambda e_i$ is a projective cover of $S_i = \Lambda e_i / Je_i$,

$I_i = {}_\Lambda D(e_i\Lambda)$ is an injective envelope of S_i,

and the analogous statements hold true for right Λ-modules.

How to compute injective envelopes and projective covers for an arbitrary $M \in \Lambda\,\mathrm{mod}$? Since M is of finite length, $M/\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ are semisimple. Let $M/\mathrm{Rad}(M) = {S_1}^{r_1} \oplus \cdots \oplus {S_n}^{r_n}$ with multiplicities $r_1, \ldots, r_n \geq 0$. Then $P(M) = {P_1}^{r_1} \oplus \cdots \oplus {P_n}^{r_n}$. Similarly, if $\mathrm{Soc}(M) = {S_1}^{s_1} \oplus \cdots \oplus {S_n}^{s_n}$, then $E(M) = {I_1}^{s_1} \oplus \cdots \oplus {I_n}^{s_n}$. (see Exercises 9 and **??**).

6.5 Exercises - Part 3

(published on November 10, **solutions to be submitted on November 24, 2016**).

Exercise 9. (a) Let $M = N_1 \oplus N_2$ be a module and let P_1 and P_2 be projective covers of N_1 and N_2, respectively. Show that $P_1 \oplus P_2$ is a projective cover of M.

(b) Let M be a module of finite length with $M/\operatorname{Rad}(M) = S_1 \oplus \dots S_r$. Show that there exists a superfluous epimorphism $P(S_1) \oplus \cdots \oplus P(S_r) \to M$ and conclude that $P(M) = P(M/\operatorname{Rad}(M)) = P(S_1) \oplus \cdots \oplus P(S_r)$.

(Hint: $\operatorname{Rad}(M)$ is superfluous in M, so...)

(c) Prove that the injective envelope $E(S)$ of any simple module S is indecomposable.

(d) Show that any indecomposable injective module E is the injective envelope of its socle. Deduce that $\operatorname{Soc} E$ is a simple module.

Exercise 10. (a) Let M be an indecomposable left R-module of finite length, and let $f \in \operatorname{End}_R(M)$. Show that the following statements are equivalent.
(i) f is a monomorphism,
(ii) f is an epimorphism,
(iii) f is an isomorphism,
(iv) f is not nilpotent.

In particular, if f is not invertible, then gf is not invertible for any $g \in \operatorname{End}_R(M)$.

(b) Prove Schur's Lemma: If S is a simple module, then $\operatorname{End}_R S$ is a skew field.
Is the converse true?

Exercise 11. Let $p \in \mathbb{N}$ a prime and $M = \{\frac{a}{p^n} \in \mathbb{Q} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$.

(a) Verify that $\mathbb{Z} \leq M \leq \mathbb{Q}$ in $\mathbb{Z}\,\mathrm{Mod}$.

(b) Let $\mathbb{Z}_{p^\infty} = M/\mathbb{Z}$. Show that $\mathbb{Z}_{p^\infty}$ is a divisible group.

(c) show that any $L \leq \mathbb{Z}_{p^\infty}$ is cyclic, generated by an element $\frac{1}{p^l}$, $l \in \mathbb{N}$.

Conclude the the lattice of the subgroups of $\mathbb{Z}_{p^\infty}$ is a well-ordered chain, and $\mathbb{Z}_{p^\infty}$ does not have any maximal subgroup.

Exercise 12. (a) Let $F : \mathcal{B} \longrightarrow \mathcal{C}$ be a functor and let B and B' be two objects in $\mathcal{B}$. Show that:
- if B and B' are isomorphic in $\mathcal{B}$, then the objects $F(B)$ and $F(B')$ are isomorphic in $\mathcal{C}$;
- if F is full and faithful, then the converse is also true.

(b) Let R and S be two rings and let $G : Mod(R) \longrightarrow Mod(S)$ be an equivalence of categories. Show that G is an exact functor.

7 CONSTRUCTING NEW MODULES

Throughout this chapter, we fix a field k and a finite dimensional algebra Λ over k. We assume that Λ is basic and indecomposable.
We have seen above how to determine the indecomposable projective, or injective or simple modules over Λ. Starting from these known modules, we want to construct new indecomposable Λ-modules. We first need some preliminaries.

7.1 Reminder on projectives and minimal projective resolutions.

Recall that every Λ-module M has a *projective cover* $p : P \to M$, that is, p is an epimorphism with P being projective and $\operatorname{Ker} p$ being superfluous. Then $\operatorname{Ker} p \subset J\,P$, and no non-zero summand of P is contained in $\operatorname{Ker} p$.
We infer that every Λ-module M has a *minimal projective presentation*

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

and a *minimal projective resolution*

$$\cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

that is, a long exact sequence where p_0 is a projective cover of M, p_1 is a projective cover of $\operatorname{Ker} p_0$, and so on. In other words, for all $i \geq 0$

$$\operatorname{Im} p_{i+1} = \operatorname{Ker} p_i \subset \operatorname{Rad} P_i = J\,P_i.$$

We will often just consider the *complex* of projectives

$$P^{\cdot} : \qquad \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to 0 \to \cdots$$

and will also call it a projective resolution of M (see Section 8.3).

Proposition 7.1.1. *Let M, N be two modules with projective resolutions $P^{\cdot}$ and $Q^{\cdot}$, respectively, and let $f : M \to N$ be a homomorphism.*

1. *There are homomorphisms $f_0, f_1, \ldots$ making the following diagram commutative*

$$\begin{array}{ccccccc} \ldots & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow 0 \\ & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & \\ \ldots & Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & N & \longrightarrow 0 \end{array}$$

   Then $f^{\cdot} = (f_n)_{n\geq 0} : P^{\cdot} \to Q^{\cdot}$ is called a chain map.

2. *If $g^{\cdot} = (g_n)_{n\geq 0} : P^{\cdot} \to Q^{\cdot}$ is another chain map as above, then there are homomorphisms $s_n : P_n \to Q_{n+1}, n \geq 0$ such that, setting $h_n = f_n - g_n$, we have*

$$h_0 = q_1 s_0,$$

$$h_n = s_{n-1} p_n + q_{n+1} s_n \quad \textit{for } n \geq 1.$$

   Then $s = (s_n)_{n\geq 0}$ is called a homotopy between $P^{\cdot}$ and $Q^{\cdot}$, and we say that the chain maps $f^{\cdot}$ and $g^{\cdot}$ are homotopic *(or that $h^{\cdot} = (h_n)_{n\geq 0}$ is* homotopic to zero*).*

7.2 The Auslander-Bridger transpose

As a consequence of the Dual Basis Lemma 3.4.3, we obtain the following properties of the contravariant functor $^* = \mathrm{Hom}(-,\Lambda) : \Lambda\mathrm{Mod} \longrightarrow \mathrm{Mod}\,\Lambda$.

Proposition 7.2.1. *Let P be a finitely generated projective left Λ-module. Then P^* is a finitely generated projective right Λ-module, and $P^{**} \cong P$. Moreover, if I is an ideal of Λ, then $\mathrm{Hom}_\Lambda(P, I) = P^* \cdot I$.*

Proof. We only sketch the arguments. First of all, note that the evaluation map $c : P \to P^{**}$ defined by $c(x)(\varphi) = \varphi(x)$ on $x \in P$ and $\varphi \in P^*$ is a monomorphism. Further, if $(\,(x_i)_{1\leq i\leq n},\,(\varphi_i)_{1\leq i\leq n}\,)$ is a dual basis of P, then it is easy to see that $(\,(\varphi_i)_{1\leq i\leq n},\,(c(x_i))_{1\leq i\leq n}\,)$ is a dual basis of P^*. This shows that P^* is finitely generated projective. The isomorphism $P^{**} \cong P$ is proved by showing that the assignment $P^{**} \ni f \mapsto \sum_{i=1}^n f(\varphi_i)\,x_i \in P$ is inverse to c.
For the second statement, the inclusion $\subset$ follows immediately from the fact that $\varphi \in \mathrm{Hom}_\Lambda(P, I)$ satisfies $\varphi(x_i) \in I$ for all $1 \leq i \leq n$, and $\supset$ follows from the fact that for $\varphi \in P^*$ and $a \in I$ we have $(\varphi \cdot a)(x) = \varphi(x) \cdot a \in I$. $\square$

So, the functor $^* = \mathrm{Hom}(-,\Lambda) : \Lambda\mathrm{Mod} \longrightarrow \mathrm{Mod}\,\Lambda$ induces a duality between the full subcategories of finitely generated projective modules in $\Lambda\,\mathrm{Mod}$ and $\mathrm{Mod}\Lambda$. The following construction from [10] can be viewed as a way to extend this duality to all finitely presented modules.
We denote by $\Lambda\mathrm{mod}_{\mathcal{P}}$ the full subcategory of $\Lambda\,\mathrm{mod}$ consisting of the modules without non-zero projective summands.
Let $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. Applying the functor $^* = \mathrm{Hom}_\Lambda(-,\Lambda)$ on it, we obtain a minimal projective presentation

$$P_0^* \xrightarrow{p_1^*} P_1^* \to \mathrm{Coker}\,p_1^* \to 0\,.$$

Set $\mathrm{Tr}\,M = \mathrm{Coker}\,p_1^*$. Then $\mathrm{Tr}\,M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$. Moreover, the following hold true.

(1) The isomorphism class of $\mathrm{Tr}\,M$ does not depend on the choice of $P_1 \to P_0 \to M \to 0$.

(2) There is a natural isomorphism $\mathrm{Tr}^2(M) \cong M$.

Let us now consider a homomorphism $f \in \mathrm{Hom}_\Lambda(M, N)$ with $M, N \in \Lambda\,\mathrm{mod}$. It induces a commutative diagram

$$\begin{array}{ccccccc}
P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & N & \longrightarrow & 0
\end{array}$$

Applying $^* = \mathrm{Hom}(-,\Lambda)$, we can construct $\widetilde{f} \in \mathrm{Hom}(\mathrm{Tr}\,N, \mathrm{Tr}\,M)$ as follows:

$$\begin{tikzcd}
P_0^* \arrow[r, "p_1^*"] & P_1^* \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \\
Q_0^* \arrow[u, "f_0^*"'] \arrow[r, "q_1^*"] & Q_1^* \arrow[u, "f_1^*"'] \arrow[r] & \operatorname{Tr} N \arrow[u, "\widetilde{f}"'] \arrow[r] & 0
\end{tikzcd}$$

Note that this construction is not unique since $\widetilde{f}$ depends on the choice of f_0, f_1. However, if we choose another factorization of f, say by maps g_0 and g_1, and conctruct $\widetilde{g}$ correspondingly, then the difference $f_0 - g_0 \in \operatorname{Ker} q_0 = \operatorname{Im} q_1$ factors through Q_1, and so $\widetilde{f} - \widetilde{g}$ factors through P_1^*, as illustrated below:

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] \arrow[d, dotted, bend right, "g_1"'] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] \arrow[d, dotted, bend right, "g_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & N \arrow[r] & 0
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
P_0^* \arrow[r, "p_1^*"] & P_1^* \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \\
Q_0^* \arrow[u, "f_0^*"] \arrow[u, dotted, bend right, "g_0^*"'] \arrow[r, "q_1^*"'] & Q_1^* \arrow[u, "f_1^*"] \arrow[u, dotted, bend right, "g_1^*"'] \arrow[r] & \operatorname{Tr} N \arrow[u, "\widetilde{f}"] \arrow[u, dotted, bend right, "\widetilde{g}"'] \arrow[r] \arrow[ul] & 0
\end{tikzcd}$$

In other words, if we consider the subgroups
$P(M, N) = \{f \in \operatorname{Hom}(M, N) \mid f \text{ factors through a projective module}\} \leq \operatorname{Hom}_\Lambda(M, N)$,
then $\widetilde{f}$ is uniquely determined modulo $P(\operatorname{Tr} N, \operatorname{Tr} M)$.

We set $\underline{\operatorname{Hom}}_\Lambda(M, N) = \operatorname{Hom}_\Lambda(M, N)/P(M, N)$, and let $\Lambda\, \underline{\operatorname{mod}}$ be the category with the same objects as $\Lambda \operatorname{mod}$ and morphisms $\underline{\operatorname{Hom}}_\Lambda(M, N)$. It is called the *stable category* of $\Lambda \operatorname{mod}$ modulo projectives. We obtain the following.

Proposition 7.2.2.

(1) There is a group isomorphism $\underline{\operatorname{Hom}}(M, N) \to \underline{\operatorname{Hom}}(\operatorname{Tr} N, \operatorname{Tr} M)$, $\underline{f} \mapsto \underline{\widetilde{f}}$.

(2) $\operatorname{End}_\Lambda M$ *is local if and only if* $\operatorname{End} \operatorname{Tr} M_\Lambda$ *is local.*

(3) Tr *induces a duality* $\Lambda\, \underline{\operatorname{mod}} \to \underline{\operatorname{mod}}\, \Lambda$.

7.3 The Nakayama functor

We now combine the transpose with the duality D. Denote by

$$\nu : \Lambda \operatorname{Mod} \to \Lambda \operatorname{Mod},\ X \mapsto D(X^*)$$

the *Nakayama functor.*

Lemma 7.3.1. *The functor ν has the following properties.*

1. *ν is covariant and right exact.*
2. *$\nu(\Lambda e_i) = D(e_i \Lambda)$ is the injective envelope of $\Lambda e_i / J e_i$ for $1 \leq i \leq n$.*
3. *$\nu({}_\Lambda \Lambda) = D(\Lambda_\Lambda)$ is an injective cogenerator of Λ Mod.*

4. For $M \in \Lambda\,\mathrm{mod}$ with minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ there is an exact sequence

$$0 \to D\,\mathrm{Tr}\,M \to \nu(P_1) \xrightarrow{\nu(p_1)} \nu(P_0) \to \nu(M) \to 0$$

7.4 The Auslander-Reiten translation

We denote

$$\tau(M) = D\,\mathrm{Tr}\,M = \mathrm{Ker}\,\nu(p_1).$$

The functor τ is called *Auslander-Reiten translation*

Denote by $\Lambda\mathrm{mod}_{\mathcal{I}}$ the full subcategory of Λmod consisting of the modules without non-zero injective summands. For $M, N \in \Lambda\,\mathrm{mod}$ consider further the subgroup
$I(M,N) = \{f \in \mathrm{Hom}_\Lambda(M,N) \mid f \text{ factors through an injective module}\} \leq \mathrm{Hom}_\Lambda(M,N)$, set $\overline{\mathrm{Hom}}_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)/I(M,N)$, and let $\Lambda\,\overline{\mathrm{mod}}$ be the category with the same objects as $\Lambda\,\mathrm{mod}$ and morphisms $\overline{\mathrm{Hom}}_\Lambda(M,N)$.

Proposition 7.4.1. *(1) The duality D induces a duality $\Lambda\,\underline{\mathrm{mod}} \to \overline{\mathrm{mod}}\,\Lambda$.*

(2) The composition $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ is an equivalence with inverse $\tau^- = \mathrm{Tr}\,D\colon \Lambda\,\overline{\mathrm{mod}} \to \Lambda\,\underline{\mathrm{mod}}$.

Example 7.4.2. *Let $\Lambda = kA_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$.*
The indecomposable projectives are $P_1,\ P_2 = JP_1,\ P_3 = S_3 = JP_2$, and the indecomposable injectives are $I_1 = S_1 = I_2/S_2,\ I_2 = I_3/S_3,\ I_3 = P_1$.
We compute τS_2. Taking the minimal projective resolution $0 \to P_3 \to P_2 \to S_2 \to 0$, and using that $S_2^* = 0$ and thus $\nu(S_2) = 0$, we obtain an exact sequence

$$0 \to \tau S_2 \to I_3 \to I_2 \to 0$$

showing that $\tau S_2 = S_3$.

7.5 Exercises - Part 4

(published on November 29, **solutions to be submitted on December 13, 2016**).

Exercise 13. Let K be a field and Q the quiver
$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & 3 \end{tikzcd}$$

(a) Determine all indecomposable projective representations and their radicals.

(b) Determine all indecomposable injective representations and their socles.

(c) Determine the minimal projective resolutions of the simple modules.

(d) Compute the representation $\nu(S_1)$.

(e) Compute the representation $\tau(S_1)$.

Exercise 14. Let Λ be a finite dimensional algebra over a field k, let M, N be finitely generated Λ-modules without projective summands, and let $f \in \mathrm{Hom}_\Lambda(M, N)$. Show

(a) $f \in P(M, N)$ if and only if $\mathrm{Tr}\, f \in P(\mathrm{Tr}\, N, \mathrm{Tr}\, M)$.

(b) f is an isomorphism if and only if so is $\mathrm{Tr}\, f$.

Show further that $\underline{\mathrm{Hom}}_\Lambda(M, N) \to \underline{\mathrm{Hom}}_\Lambda(\mathrm{Tr}\, N, \mathrm{Tr}\, M)$, $\underline{f} \mapsto \underline{\mathrm{Tr}\, f}$ is an isomorphism of k-vector spaces.

Exercise 15. Given a pair of homomorphisms in RMod
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$
consider the cokernel L of the map $A \to B \oplus C$, $a \mapsto (f(a), -g(a))$. Prove that
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "\tau"] \\ C \arrow[r, "\sigma"] & L \end{tikzcd}$$
is a push-out, where $\sigma : C \to L$, $c \mapsto \overline{(0, c)}$, and $\tau : B \to L$, $b \mapsto \overline{(b, 0)}$.

Exercise 16. Given a pair of homomorphisms in RMod
$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$
construct the pull-back of f and g.

8 SOME HOMOLOGICAL ALGEBRA

Throughout this chapter, let R be a ring, and denote by $R\,\mathrm{Mod}$ the category of all left R-modules.

8.1 Push-out and Pull-back

Proposition 8.1.1. [27, pp. 41] *Consider a pair of homomorphisms in RMod*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

There is a module $_RL$ together with homomorphisms $\sigma : C \to L$ and $\tau : B \to L$ such that (i) the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau \\ C & \xrightarrow{\sigma} & L \end{array}$$

commutes; and
(ii) given any other module $_RL'$ together with homomorphisms $\sigma' : C \to L'$ and $\tau' : B \to L'$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau' \\ C & \xrightarrow{\sigma'} & L' \end{array}$$

commute, there exists a unique homomorphism $\gamma : L \to L'$ such that $\gamma\sigma = \sigma'$ and $\gamma\tau = \tau'$.

The module L together with σ, τ is unique up to isomorphism and is called push-out *of f and g.*

Proof. We just sketch the construction. The module L is defined as the quotient $L = B \oplus C\,/\,\{\,\overline{(f(a), -g(a))} \mid a \in A\}$, and the homomorphisms are given as $\sigma : C \to L$, $c \mapsto \overline{(0, c)}$, and $\tau : B \to L$, $b \mapsto \overline{(b, 0)}$. $\square$

Remark 8.1.2. If f is a monomorphism, also σ is a monomorphism, and $\mathrm{Coker}\,\sigma \cong \mathrm{Coker}\,f$.

Dually, one defines the *pull-back* of a pair of homomorphisms

$$\begin{array}{ccc} & & B \\ & & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$$

8.2 A short survey on Ext^1

Aim of this section is to give a brief introduction to the functor Ext^1, as needed in the sequel. For a comprehensive treatment we refer to textbooks in homological algebra, e.g. [27].

Definition. Let A, B be two R-modules. We define a relation on short exact sequences of the form $\mathfrak{E}\colon 0 \to B \to E \to A \to 0$ by setting

$$\mathfrak{E}_1 : 0 \to B \to E_1 \to A \to 0 \ \sim\ \mathfrak{E}_2 : 0 \to B \to E_2 \to A \to 0$$

if there is $f \in \mathrm{Hom}_R(E_1, E_2)$ making the following diagram commute.

$$\begin{array}{lccccccccc}
\mathfrak{E}_1\colon & 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \| & & \downarrow f & & \| & & \\
\mathfrak{E}_2\colon & 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Exercise. Show that $\sim$ is an equivalence relation (Hint: symmetry is the only nontrivial condition). We denote the set of all equivalence classes of short exact sequences starting at B and ending at A by $\mathrm{Ext}^1_R(A, B)$.

Definition. Let $A, B, B' \in R\ \mathrm{Mod}$ and $\beta \in \mathrm{Hom}_R(B, B')$. We define a map

$$\begin{aligned}
\mathrm{Ext}^1_R(A, \beta)\colon\ \mathrm{Ext}^1_R(A, B) &\to \mathrm{Ext}^1_R(A, B') \\
[\mathfrak{E}] &\mapsto [\beta\mathfrak{E}]
\end{aligned}$$

as follows. For a short exact sequence $\mathfrak{E}\colon 0 \to B \to E \to A \to 0$ define $\beta\mathfrak{E}$ via the pushout

$$\begin{array}{lccccccccc}
\mathfrak{E}\colon & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & & {\scriptstyle\beta}\downarrow & & \downarrow & & \| & & \\
\beta\mathfrak{E}\colon & 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Definition. Let $A, A', B \in R\ \mathrm{Mod}$ and $\alpha \in \mathrm{Hom}_R(A', A)$. We define a map

$$\begin{aligned}
\mathrm{Ext}^1_R(\alpha, B)\colon\ \mathrm{Ext}^1_R(A, B) &\to \mathrm{Ext}^1_R(A', B) \\
[\mathfrak{E}] &\mapsto [\mathfrak{E}\alpha]
\end{aligned}$$

as follows. For a short exact sequence $\mathfrak{E}\colon 0 \to B \to E \to A \to 0$ define $\mathfrak{E}\alpha$ via the pullback

$$\begin{array}{lccccccccc}
\mathfrak{E}\alpha\colon & 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
\mathfrak{E}\colon & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

One can check that the maps in the definitions above are well defined, that is, if $\mathfrak{E} \sim \mathfrak{E}'$ then $\beta\mathfrak{E} \sim \beta\mathfrak{E}'$ and $\mathfrak{E}\alpha \sim \mathfrak{E}'\alpha$. Moreover, one can verify that for R-homomorphisms $\beta_1\colon B \to B'$ and $\beta_2\colon B' \to B''$ that $\mathrm{Ext}^1_R(A, \beta_2)\,\mathrm{Ext}^1_R(A, \beta_1) = \mathrm{Ext}^1_R(A, \beta_2\beta_1)$. Analogously given two R-homomorphisms $\alpha_1\colon A'' \to A'$ and $\alpha_2\colon A' \to A$, we the equality $\mathrm{Ext}^1_R(\alpha_1, B)\,\mathrm{Ext}^1_R(\alpha_2, B) = \mathrm{Ext}^1_R(\alpha_2\alpha_1, B)$.

Proposition 8.2.1. *Let* $A, A', B, B' \in R\,\mathrm{Mod}$*,* $\alpha\colon A' \to A$ *and* $\beta\colon B \to B'$ *be* R*-homomorphisms. Then*

$$\mathrm{Ext}^1_R(\alpha, B')\,\mathrm{Ext}^1_R(A, \beta)[\mathfrak{E}] = \mathrm{Ext}^1_R(A', \beta)\,\mathrm{Ext}^1_R(\alpha, B)[\mathfrak{E}].$$

In particular, we get a well-defined map $\mathrm{Ext}^1_R(\alpha, \beta) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A', B')$.

Proof. Given a short exact sequence $\mathfrak{E}\colon 0 \longrightarrow B \xrightarrow{f} M \xrightarrow{g} A \longrightarrow 0 \in \mathrm{Ext}^1_R(A, B)$ we must show that $[\beta(\mathfrak{E}\alpha)] = [(\beta\mathfrak{E})\alpha]$, i.e. that the extensions $\beta(\mathfrak{E}\alpha)$ and $(\beta\mathfrak{E})\alpha$ are equivalent. Consider the following commutative diagram involving the respective pushout and pullback squares.

$$\begin{tikzcd}
\beta(\mathfrak{E}\alpha)\colon & 0 \arrow[r] & B' \arrow[r, "\tilde{\tilde{f}}"] & \tilde{E} \arrow[r, "\tilde{\tilde{g}}"] & A' \arrow[r] & 0 \\
\mathfrak{E}\alpha\colon & 0 \arrow[r] & B \arrow[r, "\tilde{f}"] \arrow[u, "\beta"] \arrow[d, equal] & E \arrow[r, "\tilde{g}"] \arrow[u, "\tilde{\beta}"'] \arrow[d, "\tilde{\alpha}"'] & A' \arrow[r] \arrow[u, equal] \arrow[d, "\alpha"'] & 0 \\
\mathfrak{E}\colon & 0 \arrow[r] & B \arrow[r, "f"] \arrow[d, "\beta"] & M \arrow[r, "g"] \arrow[d, "\beta'"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\beta\mathfrak{E}\colon & 0 \arrow[r] & B' \arrow[r, "f'"] & F \arrow[r, "g'"] & A \arrow[r] & 0 \\
(\beta\mathfrak{E})\alpha\colon & 0 \arrow[r] & B' \arrow[r, "f''"'] \arrow[u, equal] & F' \arrow[r, "g''"'] \arrow[u, "\alpha'"'] & A' \arrow[r] \arrow[u, "\alpha"'] & 0
\end{tikzcd}$$

Step 1. *There exists* $\rho\colon E \to F'$ *such that* $\alpha'\rho = \beta\tilde{\alpha}$ *and* $g''\rho = \tilde{g}$. This is immediate by considering the pullback of the maps $g'\colon F \to A$ and $\alpha\colon A' \to A$. The pullback diagram is the following.

$$\begin{tikzcd}
E \arrow[dr, dashed, "\rho", "\exists!"'] \arrow[drr, bend left, "\tilde{g}"] \arrow[ddr, bend right, "\beta'\tilde{\alpha}"'] & & \\
& F' \arrow[r, "g''"] \arrow[d, "\alpha"'] & A' \arrow[d, "\alpha"] \\
& F \arrow[r, "g'"'] & A
\end{tikzcd}$$

Step 2. *There exists* $\gamma\colon \tilde{E} \to F$ *such that* $\gamma\tilde{\tilde{f}} = f'$ *and* $\gamma\tilde{\beta} = \alpha'\rho$. This is immediate by considering the pushout of the maps $\tilde{f}\colon B \to E$ and $\beta\colon B \to B'$.
Step 3. *We have* $g'\gamma = \alpha\tilde{\tilde{g}}$. By Steps 1 and 2 above we have

(a) $g'\gamma\tilde{\tilde{f}} = g'f' = 0$ and $\alpha\tilde{\tilde{g}}\tilde{\tilde{f}} = 0$ (consecutive maps in a short exact sequence); and

(b) $g'\gamma\tilde{\beta} = g'\alpha'\rho = g'\beta'\tilde{\alpha} = \alpha\tilde{g} = \alpha\tilde{\tilde{g}}\tilde{\beta}$.

Therefore we have a commutative diagram,

$$\begin{tikzcd}
B \arrow[r, "\tilde{f}"] \arrow[d, "\beta"'] & E \arrow[d, "\tilde{\beta}"] \arrow[ddr, bend left, "\alpha\tilde{g}"] & \\
B' \arrow[r, "\tilde{\tilde{f}}"'] \arrow[drr, bend right, "0"'] & \tilde{E} \arrow[dr, dashed, "\delta", "\exists!"'] & \\
 & & A
\end{tikzcd}$$

whence the universal property of the pushout asserts the existence of a unique map δ such that $\delta\tilde{\beta} = \alpha\tilde{g}$ and $\delta\tilde{\tilde{f}} = 0$. However, by (a) and (b) above, the maps $g'\gamma$ and $\alpha\tilde{\tilde{g}}$ are two further maps satisfying the commutativity relations that δ satisfies. Therefore, the uniqueness of δ means that $g'\gamma = \delta = \alpha\tilde{\tilde{g}}$, as claimed.

Step 4. *There exists $\tau\colon \tilde{E} \to F$ such that $\alpha'\tau = \gamma$ and $g''\tau = \tilde{\tilde{g}}$.* This is immediate by considering the pullback of the maps $g'\colon F \to A$ and $\alpha\colon A' \to A$. This gives the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r, "\tilde{\tilde{f}}"] \arrow[d, equal] \arrow[dr, phantom, "(*)"] & \tilde{E} \arrow[r, "\tilde{\tilde{g}}"] \arrow[d, "\tau"] \arrow[dr, phantom, "\checkmark"] & A' \arrow[r] \arrow[d, equal] & 0\,, \\
0 \arrow[r] & B' \arrow[r, "f''"'] & F' \arrow[r, "g''"'] & A' \arrow[r] & 0
\end{tikzcd}$$

where Step 4 shows that the right hand square marked ✓ commutes. We need to show that the square marked $(*)$ commutes.

Step 5. *The square marked $(*)$ commutes.* Similar to Step 3 above, we observe that $g''f'' = 0$ and $g''\tau\tilde{\tilde{f}} = \tilde{\tilde{g}}\tilde{\tilde{f}} = 0$, and $\alpha'f'' = f'$ and $\alpha'\tau f = \gamma\tilde{\tilde{f}} = f'$. Then using the universal property of the pullback (of the maps $g'\colon F \to A$ and $\alpha\colon A' \to A$) we deduce, as in Step 3 above, that $f'' = \tau\tilde{\tilde{f}}$, as required. □

Definition (Baer sum)**.** Let $\mathfrak{E}_1\colon 0 \to B \to E_1 \to A \to 0$ and $\mathfrak{E}_2\colon 0 \to B \to E_2 \to A \to 0$ be two elements of $\operatorname{Ext}^1_R(A, B)$. Consider the direct sum

$$\mathfrak{E}_1 \oplus \mathfrak{E}_2\colon 0 \to B \oplus B \to E_1 \oplus E_2 \to A \oplus A \to 0$$

together with the diagonal map $\Delta_A\colon A \to A \oplus A$, $a \mapsto (a, a)$, and the summation map $\nabla_B\colon B \oplus B \to B$, $(b_1, b_2) \mapsto b_1 + b_2$. Set

$$[\mathfrak{E}_1] + [\mathfrak{E}_2] = \operatorname{Ext}^1_R(\Delta_A, \nabla_B)([\mathfrak{E}_1 \oplus \mathfrak{E}_2]) \ \in \operatorname{Ext}^1_R(A, B)$$

Theorem 8.2.2. *For $A, B \in R$ Mod, $\operatorname{Ext}^1_R(A, B)$ has the structure of an abelian group with*

- *addition given by the Baer sum;*
- *neutral element given by the equivalence class of split exact sequences;*

- *for* $\mathfrak{E}\colon 0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0$ *the inverse of the equivalence class* $[\mathfrak{E}]$ *is the equivalence class of* $0 \longrightarrow B \xrightarrow{-f} E \xrightarrow{g} A \longrightarrow 0$, *i.e.* $-[\mathfrak{E}] = [-1_B\mathfrak{E}]$ *where* $1_B \in \mathrm{Hom}_R(B, B)$ *is the identity map.*

Moreover, the maps $\mathrm{Ext}^1_R(A, \beta)$ *and* $\mathrm{Ext}^1_R(\alpha, B)$ *defined above become group homomorphisms.*

Corollary 8.2.3. *For* $A, B \in R\ \mathrm{Mod}$ *we have defined*

(a) a covariant functor $\mathrm{Ext}^1_R(A, -)\colon\ R\ \mathrm{Mod} \to \mathrm{Ab}$*; and*

(b) a contravariant functor $\mathrm{Ext}^1_R(-, B)\colon\ R\ \mathrm{Mod} \to \mathrm{Ab}$,

where Ab *denotes the category of abelian groups (=* $\mathbb{Z}$*-modules).*

8.3 The category of complexes

Let R be a ring.

Definition. (1) A *(co)chain complex* of R-modules $A^\bullet = (A^n, d^n)$ is given by a sequence

$$A^\bullet\colon \qquad \cdots \to A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

of R-modules A^n with R-homomorphisms $d^n : A^n \to A^{n+1}$, called *differentials*, satisfying

$$d^{n+1} \circ d^n = 0$$

for all $n \in \mathbb{Z}$. Given two complexes $A^\bullet, B^\bullet$, a *(co)chain map* $f^\bullet : A^\bullet \to B^\bullet$ is given by a family of R-homomorphisms $f^n : A^n \to B^n$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"'] & \cdots
\end{tikzcd}$$

Complexes and (co)chain maps form the category of complexes $\mathcal{C}(R\ \mathrm{Mod})$.

(2) Given a *complex* of R-modules $A^\bullet = (A^n, d^n)$, the abelian group

$$H^n(A^\bullet) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$$

is called n-th *(co)homology group*. Note that $H^n(A^\bullet)$ is an R-module, and every cochain map $f^\bullet\colon A^\bullet \to B^\bullet$ induces R-homomorphisms $H^n(f^\bullet)\colon H^n(A^\bullet) \to H^n(B^\bullet)$. So, for every $n \in \mathbb{Z}$ there is a functor

$$H^n\colon \mathcal{C}(R\ \mathrm{Mod}) \to R\ \mathrm{Mod}\,.$$

(3) A cochain map $h^\bullet\colon A^\bullet \to B^\bullet$ is *null-homotopic* (or *homotopic to zero*) if there is a *homotopy* $s = (s^n)$ with homomorphisms $s^n\colon A^n \to B^{n-1}$, $n \in \mathbb{Z}$, such that

$$h^n = s^{n+1}d^n + d'^{n-1}s^n \ \text{ for } n \in \mathbb{Z}.$$

Two cochain maps $f^\bullet, g^\bullet \colon A^\bullet \to B^\bullet$ are *homotopic* if the cochain map $h^\bullet = f^\bullet - g^\bullet$ given by $h^n = f^n - g^n$ is null-homotopic.

Lemma 8.3.1. *Let $f^\bullet, g^\bullet \colon A^\bullet \to B^\bullet$ be two cochain maps.*

(1) If $f^\bullet$ and $g^\bullet$ are homotopic, then $H^n(f^\bullet) = H^n(g^\bullet)$ for all $n \in \mathbb{Z}$.

(2) If $g^\bullet f^\bullet$ is homotopic to $\mathrm{id}_{A^\bullet}$ and $f^\bullet g^\bullet$ is homotopic to $\mathrm{id}_{B^\bullet}$, then $H^n(f^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

Theorem 8.3.2. *Let $0 \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \longrightarrow 0$ be a short exact sequence in $\mathcal{C}(R\,\mathrm{Mod})$, that is, $f^\bullet, g^\bullet$ are cochain maps inducing short exact sequences in each degree. Then there is a long exact sequence of cohomology groups*

$$\cdots \to H^{n-1}(C^{\cdot}) \xrightarrow{\delta^{n-1}} H^n(A^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(B^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(C^\bullet) \xrightarrow{\delta^n} H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} \cdots$$

given by natural connecting homomorphisms

$$\delta^n \colon H^n(C^{\cdot}) \to H^{n+1}(A^{\cdot}).$$

Proof. The diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "d_A^{n-1}"'] & \vdots \arrow[d, "d_B^{n-1}"'] & \vdots \arrow[d, "d_C^{n-1}"'] & \\
0 \arrow[r] & A^n \arrow[r, "f^n"] \arrow[d, "d_A^n"'] & B^n \arrow[r, "g^n"] \arrow[d, "d_B^n"] & C^n \arrow[r] \arrow[d, "d_C^n"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "f^{n+1}"] \arrow[d, "d_A^{n+1}"'] & B^{n+1} \arrow[r, "g^{n+1}"] \arrow[d, "d_B^{n+1}"] & C^{n+1} \arrow[r] \arrow[d, "d_C^{n+1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with $\mathrm{Im}\, d_A{}^{n-1} \subset \mathrm{Ker}\, d_A{}^n$ for all $n \in \mathbb{Z}$, and similarly for B and C, induces diagrams

$$\begin{tikzcd}
 & A^n/\mathrm{Im}\, d_A{}^{n-1} \arrow[r, "\overline{f^n}"] \arrow[d, "\overline{d_A{}^n}"] & B^n/\mathrm{Im}\, d_B{}^{n-1} \arrow[r, "\overline{g^n}"] \arrow[d, "\overline{d_B{}^n}"] & C^n/\mathrm{Im}\, d_C{}^{n-1} \arrow[r] \arrow[d, "\overline{d_C{}^n}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}\, d_A{}^{n+1} \arrow[r, "\overline{f^{n+1}}"] & \mathrm{Ker}\, d_B{}^{n+1} \arrow[r, "\overline{g^{n+1}}"] & \mathrm{Ker}\, d_C{}^{n+1} &
\end{tikzcd}$$

The kernels and cokernels of the vertical maps are respectively

$$\mathrm{Ker}\, \overline{d_A{}^n} = \mathrm{Ker}\, d_A{}^n / \mathrm{Im}\, d_A{}^{n-1} = H^n(A^{\cdot})$$
$$\mathrm{Coker}\, \overline{d_A{}^n} = \mathrm{Ker}\, d_A{}^{n+1} / \mathrm{Im}\, d_A{}^n = H^{n+1}(A^{\cdot})$$

and similarly for B and C. Now apply the Snake Lemma to get

$$\delta^n : H^n(C^{\cdot}) = \mathrm{Ker}\, \overline{d_C{}^n} \to \mathrm{Coker}\, \overline{d_A{}^n} = H^{n+1}(A^{\cdot}).$$

$\square$

8.4 The functors Ext^n

Theorem 8.4.1. *Let A, B be two R-modules, and let the complex*

$$P_\bullet : \qquad \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to 0 \cdots$$

be a projective resolution of A. Consider the abelian group complex

$$\mathrm{Hom}_R(P_\bullet, B) : 0 \to \mathrm{Hom}_R(P_0, B) \stackrel{\mathrm{Hom}_R(p_1,B)}{\longrightarrow} \mathrm{Hom}_R(P_1, B) \stackrel{\mathrm{Hom}_R(p_2,B)}{\longrightarrow} \mathrm{Hom}_R(P_2, B) \to \cdots$$

Then the cohomology groups $H^n(\mathrm{Hom}_R(P_\bullet, B))$ do not depend from the choice of $P_\bullet$, and

$$\mathrm{Hom}_R(A, B) \cong H^0(\mathrm{Hom}_R(P_\bullet, B))$$

$$\mathrm{Ext}^1_R(A, B) \cong H^1(\mathrm{Hom}_R(P_\bullet, B))$$

Definition. For $n \in \mathbb{N}$ we set

$$\mathrm{Ext}^n_R(A, B) = H^n(\mathrm{Hom}_R(P_\bullet, B))$$

called the n-th *extension group.* We thus obtain additive covariant (respectively, contravariant) functors

$$\mathrm{Ext}^n_R(A, -) : R\ \mathrm{Mod} \to \mathrm{Ab},$$

$$\mathrm{Ext}^n_R(-, B) : R\ \mathrm{Mod} \to \mathrm{Ab}.$$

The Ext-functors “repair” the non-exactness of the Hom-functors as follows.

Lemma 8.4.2. *Let $\mathfrak{E} : 0 \to B \stackrel{\beta}{\to} B' \stackrel{\beta'}{\to} B'' \to 0$ be a short exact sequence in R Mod, and A an R-module. Then there is a long exact sequence*

$$0 \to \mathrm{Hom}_R(A, B) \xrightarrow{\mathrm{Hom}_R(A,\beta)} \mathrm{Hom}_R(A, B') \xrightarrow{\mathrm{Hom}_R(A,\beta')} \mathrm{Hom}_R(A, B'') \xrightarrow{\delta}$$

$$\mathrm{Ext}^1_R(A, B) \xrightarrow{\mathrm{Ext}^1_R(A,\beta)} \mathrm{Ext}^1_R(A, B') \xrightarrow{\mathrm{Ext}^1_R(A,\beta')} \mathrm{Ext}^1_R(A, B'') \longrightarrow \cdots$$

Here $\delta = \delta(A, \mathfrak{E})$ is given by $\delta(f) = [\mathfrak{E} f]$.
The dual statement for the contravariant functors $\mathrm{Hom}(-, B)$, $\mathrm{Ext}^1_R(-, B)$ also holds true.

Note that, since every short exact sequence starting at an injective module is split exact, we have that a module I is injective if and only if $\mathrm{Ext}^1_R(A, I) = 0$ for all modules A. Similarly, a module P is projective if and only if $\mathrm{Ext}^1_R(P, B) = 0$ for all module B. As a consequence, we obtain the following description of Ext^1.

Proposition 8.4.3. *Let A, B be left R-modules.*
If $0 \to B \to I \stackrel{\pi}{\to} C \to 0$ is a short exact sequence where I is injective, then

$$\mathrm{Ext}^1_R(A, B) \cong \mathrm{Coker}\,\mathrm{Hom}_R(A, \pi)$$

Similarly, if $0 \to K \stackrel{\iota}{\to} P \to A \to 0$ is a short exact sequence where P is projective, then

$$\mathrm{Ext}^1_R(A, B) \cong \mathrm{Coker}\,\mathrm{Hom}_R(\iota, B)$$

8.5 Homological dimensions

Proposition 8.5.1. *The following statements are equivalent for a module A.*

1. *A has a projective resolution $0 \to P_n \to \ldots \to P_1 \to P_0 \to A \to 0$*
2. *$\mathrm{Ext}_R^{n+1}(A, B) = 0$ for all modules B*
3. *$\mathrm{Ext}_R^m(A, B) = 0$ for all module B and all $m > n$.*

If n is the minimum integer for which the conditions above are satisfied, then A is said to have *projective dimension* n, and we set $\mathrm{pdim}\, A = n$. If there is no such n, then $\mathrm{pdim}\, A = \infty$. Dually, one defines the *injective dimension* $\mathrm{idim}\, A$ of a module A.

The supremum of the projective dimensions attained on R Mod coincides with the supremum of the injective dimensions attained on R Mod and is called the *(left) global dimension* of R. It is denoted by $\mathrm{gldim}\, R$. If R is a right and left noetherian ring, e.g. a finite dimensional algebra, then this number coincides with the right global dimension, that is, with the supremum of the projective (or injective) dimensions attained on right modules.

Theorem 8.5.2. (Auslander) *For any ring R the global dimension is attained on finitely generated modules:*

$$\mathrm{gldim}\, R = \sup\{\, \mathrm{pdim}\,(R/I) \mid I \text{ left ideal of } R\}.$$

In particular, if R is a finite dimensional algebra, then

$$\mathrm{gldim}\, R = \max\{\, \mathrm{pdim}\,(S) \mid S \text{ simple left module over } R\}.$$

Proof. Let $n = \sup\{\, \mathrm{pdim}\,(R/I) \mid I \text{ left ideal of } R\}$. In order to verify that $\mathrm{gldim}\, R = n$, we prove that every module has injective dimension bounded by n. So, let A be an arbitrary left R-module with injective coresolution

$$0 \to A \to E_0 \to E_1 \to \ldots \to E_{n-1} \to C_n \to 0.$$

We have to show that C_n is injective. We use Baer's Lemma stating that C_n is injective if and only if for every left ideal I of R with embedding $I \overset{i}{\hookrightarrow} R$ and for every homomorphism $f \in \mathrm{Hom}_R(I, C_n)$ there is $f' \in \mathrm{Hom}_R(R, C_n)$ making the following diagram commutative:

$$\begin{tikzcd} I \arrow[rr, "i"] \arrow[dr, "f"'] & & R \arrow[dl, "f'"] \\ & C_n & \end{tikzcd}$$

Observe that this means that the map $\mathrm{Hom}_R(i, C_n) : \mathrm{Hom}_R(R, C_n) \to \mathrm{Hom}_R(I, C_n)$ is surjective. Now consider the short exact sequence

$$0 \to I \overset{i}{\hookrightarrow} R \to R/I \to 0$$

and recall from Proposition 8.4.3 that $\operatorname{Coker}\operatorname{Hom}_R(i, C_n) \cong \operatorname{Ext}^1_R(R/I, C_n)$. By dimension shifting $\operatorname{Ext}^1_R(R/I, C_n) \cong \operatorname{Ext}^{n+1}_R(R/I, A)$ which is zero since $\operatorname{pdim} R/I \leq n$ by assumption. This completes the proof.
For the additional statement, recall that over a finite dimensional algebra every finitely generated module M has finite length and is therefore a finite extension of the simple modules $S_1, \ldots, S_n$. Moreover, it follows easily from Lemma 8.4.2 that in a short exact sequence $0 \to A \to B \to C \to 0$ the projective dimension of B is bounded by the maximum of the projective dimensions of A and C. Hence the projective dimension of M is bounded by $\max\{\operatorname{pdim} S_i \mid 1 \leq i \leq n\}$. □

A ring R has global dimension zero if and only if all R-modules are projective, or equivalently, all R-modules are semisimple. This condition is symmetric, that is, all left R-modules are semisimple if and only if so are all right R-modules. Rings with this property are called *semisimple* and are described by the following result. For details we refer to [20, Chapter 1] [27, p. 115], [15, Chapter 2], [17, 2.2], or [23, Chapter 3].

Theorem 8.5.3. (Wedderburn-Artin) *A ring R is semisimple if and only if it is isomorphic to a product of finitely many matrix rings over division rings*

$$R \cong M_{n_1}(D_1) \times \ldots M_{n_r}(D_r).$$

The rings of global dimension one are precisely the hereditary non-semisimple rings.

Theorem 8.5.4. *The following statements are equivalent for a ring R.*

(1) Every left ideal of R is projective.

(2) Every submodule of a projective left R-module is projective.

(3) Every factor module of an injective left R-module is injective.

(4) $\operatorname{gldim} R \leq 1$.

If R is a finite dimensional algebra, then (1) - (4) are also equivalent to

(5) The Jacobson radical J is a projective left R-module.

A ring R satisfying the equivalent conditions above is said to be left hereditary.

Proof. For the implication (1)⇒(2) one needs the following result:

Theorem 8.5.5. (Kaplansky) *Let R be a ring such that every left ideal of R is projective. Then every submodule of a free module is isomorphic to a sum of left ideals.*

For finitely generated modules over a finite dimensional algebra Λ, we can also proceed as follows. Take a finitely generated submodule $M \subset P$ of a projective module P. In order to show that M is projective, we can assume w.l.o.g. that M is indecomposable. P is a direct

summand of a free module $\Lambda^{(I)} = \bigoplus_{i=1}^{n} \Lambda e_i^{(I)}$. Choose i such that the composition $M \subset P \subset \bigoplus_{i=1}^{n} \Lambda e_i^{(I)} \xrightarrow{\mathrm{pr}} \Lambda e_i$ is non-zero. The image of this map is contained in $\Lambda e_i \subset \Lambda$ and is therefore a left ideal of Λ, which by assumption must be projective. So the indecomposable module M has a non-zero projective factor module and is thus projective.
(2)⇒ (4) follows immediately from the definition of global dimension.
(4)⇒ (2): Take a submodule $M \subset P$ of a projective module P, and consider the short exact sequence $0 \to M \to P \to P/M \to 0$. For any $N \in R\,\mathrm{Mod}$ we have a long exact sequence

$$\ldots \to \mathrm{Ext}^1_R(P, N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^2(P/M, N) \to \ldots$$

where $\mathrm{Ext}^1_R(P, N) = 0$ as P is projective, and $\mathrm{Ext}^2(P/M, N) = 0$ as all modules have projective dimension bounded by one. Thus $\mathrm{Ext}^1_R(M, N) = 0$ for all $N \in R\,\mathrm{Mod}$, proving that M is projective.
(3)⇔ (4) is proven dually, and (2)⇒ (1),(5) is trivial.
It remains to show (5)⇒ (4): The hypothesis (5) states the left module R/J has projective dimension one. Now recall that every simple module is a direct summand of R/J and use Theorem 8.5.2. □

From Theorem 8.5.4 we deduce some important properties of hereditary rings.

Corollary 8.5.6. *Let R be left hereditary, $M \in R\,\mathrm{Mod}$. Then there is a non-zero R-homomorphism $f : M \to P$ with P projective if and only if M has a non-zero projective direct summand. Moreover, if M is indecomposable, then every non-zero R-homomorphism $f : M \to P$ with P projective is a monomorphism.*

Let now Λ be a hereditary finite dimensional algebra. Then the following hold true.

(1) If P is an indecomposable projective Λ-module, then $\mathrm{End}_\Lambda P$ is a division ring.

(2) If $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then $\mathrm{Hom}_\Lambda(M, P) = 0$ for all projective modules ${}_\Lambda P$.

(3) Tr induces a duality $\Lambda\,\mathrm{mod}_{\mathcal{P}} \to \mathrm{mod}\,\Lambda_{\mathcal{P}}$ which is isomorphic to the functor $\mathrm{Ext}^1_\Lambda(-, \Lambda)$, and τ induces an equivalence $\tau : \Lambda\,\mathrm{mod}_{\mathcal{P}} \longrightarrow \Lambda\,\mathrm{mod}_{\mathcal{I}}$ with inverse τ^-.

Proof. We sketch the argument for (3). By (2) we have $P(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, and similarly, $I(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{I}}$. Moreover, if $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then a minimal projective presentation $0 \to P_1 \to P_0 \to M \to 0$ yields a long exact sequence $0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Ext}^1_\Lambda(M, \Lambda) \to 0$ where $M^* = 0$, so $\mathrm{Ext}^1_\Lambda(M, \Lambda) \cong \mathrm{Tr}\,M$. □

Here are some examples of rings of global dimension one.

Example 8.5.7. (1) Principal ideal domains and, more generally, Dedekind domains are (left and right) hereditary.

(2) The upper triangular matrix ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} = \{\begin{pmatrix} z & q \\ 0 & q' \end{pmatrix} \mid z \in \mathbb{Z}, q, q' \in \mathbb{Q}\}$ (viewed as a subring of $M_2(\mathbb{Q})$) is right hereditary but not left hereditary.
(3) A finite dimensional algebra Λ over an algebraically closed field is hereditary if and only if it is isomorphic to the path algebra of some finite acyclic quiver Q (that is, the ideal of relations $\mathcal{I} = 0$).
Indeed, the if-part follows from the fact that $J(kQ)$ is projective, see Example 4.2.6.
Assume now that Λ is hereditary. By construction, there is an arrow $i \to j$ if and only if $d_{ji} \neq 0$, which implies the existence of a proper monomorphism $\Lambda e_j \to \Lambda e_i$, and hence yields that the length of Λe_j is strictly smaller than the length of Λe_i. This shows that Q has no oriented cycles.
Let us verify that $\mathcal{I} = 0$. We know from Example 4.2.6 that ${}_{kQ}J(kQ)$ is projective. Then the sequence of $kQ/\mathcal{I}$-modules $0 \longrightarrow \mathcal{I}/\mathcal{I}\,J(kQ) \longrightarrow J(kQ)/\mathcal{I}\,J(kQ) \longrightarrow J(kQ)/\mathcal{I} \longrightarrow 0$ is a projective cover of $J(kQ)/\mathcal{I}$. But by assumption, $J(kQ)/\mathcal{I} \subseteq kQ/\mathcal{I} \cong \Lambda$ is a projective module, so $\mathcal{I} = \mathcal{I}\,J(kQ)$, and by Nakayama's Lemma $\mathcal{I} = 0$.

For a more detailed treatment on hereditary rings we refer e.g. to [22, 1.2], [27, p. 120], [15, 3.7], or [17, 5.5].

8.6 The tensor product

Definition. Given a right R-module A and a left R-module B, their *tensor product* $A \otimes_R B$ is an abelian group equipped with a map $\tau : A \times B \to A \otimes_R B$ satisfying the conditions

(i) $\tau(a + a', b) = \tau(a, b) + \tau(a', b)$

(ii) $\tau(a, b + b') = \tau(a, b) + \tau(a, b')$

(iii) $\tau(ar, b) = \tau(a, rb)$

for all $a, a' \in A, b, b' \in B, r \in R$, and having the following universal property:
for any map $\tilde{\tau} : A \times B \to C$ into an abelian group C satisfying the conditions (i)-(iii) there is a unique group homomorphism $f : A \otimes_R B \to C$ making the following diagram commutative

$$\begin{tikzcd} A \times B \arrow[rr, "\tau"] \arrow[dr, "\tilde{\tau}"'] & & A \otimes_R B \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Construction. By the universal property, the tensor product of two modules A and B is uniquely determined up to isomorphism. Its existence is proven by giving the following explicit construction (which obviously verifies the universal property above):

$$A \otimes_R B = F/K$$

where

F is the free abelian group with basis $A \times B$, that is, every element of F can be written in a unique way as a finite linear combination of elements of the form $(a, b) \in A \times B$ with coefficients in $\mathbb{Z}$, and

K is the subgroup of F generated by all elements of the form

$$(a + a', b) - (a, b) - (a', b)$$
$$(a, b + b') - (a, b) - (a, b')$$
$$(ar, b) - (a, rb)$$

for some $a, a' \in A, b, b' \in B, r \in R$.

The elements of $A \otimes_R B$ are then the images of elements of F via the canonical epimorphism $F \to F/K$ and are thus of the form

$$\sum_{i=1}^{n} a_i \otimes b_i$$

for some $n \in \mathbb{N}$ and $a_i \in A$, $b_i \in B$
(but this representation is not unique! For example $0 \otimes b = a \otimes 0 = 0$ for all $a \in A, b \in B$).

Of course, the following rules hold true for all $a, a' \in A, b, b' \in B, r \in R$:

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
$$a \otimes (b + b') = a \otimes b + a \otimes b'$$
$$ar \otimes b = a \otimes rb$$

Observe that the tensor product of non-zero modules need not be non-zero.

Example 8.6.1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. Indeed, if $a \otimes b \in \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, then

$$a \otimes b = a \cdot (3 - 2) \otimes b = a \cdot 3 \otimes b - a \cdot 2 \otimes b = a \otimes 3 \cdot b - a \cdot 2 \otimes b = a \otimes 0 - 0 \otimes b = 0.$$

Homomorphisms. Given a right R-module homomorphism $f : A \to A'$ and a left R-module homomorphism $g : B \to B'$, there is a unique abelian group homomorphism

$$f \otimes g : A \otimes_R B \to A' \otimes_R B'$$

such that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ for all $a \in A$ and $b \in B$ (use the universal property!).

In general the tensor product of modules is just an abelian group. When starting with bimodules, however, it becomes a module.

Module structure. If S is a ring and ${}_SA_R$ is an S-R-bimodule, then $A \otimes_R B$ is a left S-module via
$$s \cdot a \otimes b = sa \otimes b$$
Moreover, given $f \in \mathrm{Hom}_R(B, B')$, the map
$$A \otimes_R f = \mathrm{id}_A \otimes f : A \otimes_R B \to A \otimes_R B', \sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i \otimes f(b_i)$$
is an S-module homomorphism.

The analogous statements hold true if ${}_RB_S$ is a bimodule.

Theorem 8.6.2. (Adjointness of Hom **and** $\otimes$**)** *Let R, S be rings, ${}_SA_R$ be an S-R-bimodule, B a left R-module and C a left S-module. Then there is a natural group homomorphism*
$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C)).$$

Proof. (Sketch) The isomorphism
$$\varphi : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$
is given by mapping $f \in \mathrm{Hom}_S(A \otimes_R B, C)$ to the R-homomorphism $\varphi(f) : B \to \mathrm{Hom}_S(A, C)$ where $\varphi(f)(b) : A \to C$, $a \mapsto f(a \otimes b)$.
The inverse map
$$\psi : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$
is given by mapping $g \in \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$ to the S-homomorphism $\psi(g) : A \otimes_R B \to C$ where $\psi(g)(a \otimes b) = g(b)(a)$. $\square$

Corollary 8.6.3. *Let Λ be a finite dimensional algebra over a field k with standard duality $D = \mathrm{Hom}(-, k)$. Then*
$$D(A \otimes B) \cong \mathrm{Hom}_\Lambda(B, D(A))$$
for all right Λ-modules A and left Λ-modules B.

Corollary 8.6.4. *Let R, S be rings, ${}_SA_R$ be an S-R-bimodule. Then*
$$A \otimes_R - : R\ \mathrm{Mod} \to S\,\mathrm{Mod}$$
is an additive, covariant, right exact functor.

The following result will be very useful.

Lemma 8.6.5. *Let $M, P \in R$ Mod, and let P be finitely generated projective. Then there is a natural group homomorphism*
$$\mathrm{Hom}_R(P, M) \cong P^* \otimes_R M.$$

Remark 8.6.6. (1) If V, W are finite dimensional vector spaces over a field k, then $V \otimes_k W$ is isomorphic to the vector space $\mathrm{Bil}(V^* \times W^*, K)$ of all bilinear maps $V^* \times W^* \to K$. Under this bijection an element $v \otimes w$ corresponds to the bilinear map $(\varphi, \psi) \mapsto \varphi(v)\psi(w)$. Indeed, $V^{**} \cong V$, so by Lemma 8.6.5 we have $V \otimes_k W \cong \mathrm{Hom}_k(V^*, W) \cong \mathrm{Hom}_k(V^*, W^{**})$. Further, $\mathrm{Hom}_k(V^*, W^{**} \cong \mathrm{Bil}(V^* \times W^*, K)$ via $g \mapsto \sigma_g$, where σ_g is the bilinear map given by $\sigma_g(\varphi, \psi) = g(\varphi)(\psi)$.
(2) Let B be a left R-module with projective resolution $P^{\cdot}$, and A a right R-module. The homology groups of the complex $A \otimes_R P^{\cdot} : \ldots A \otimes_R P_1 \to A \otimes_R P_0 \to 0$ define the Tor-functors:

$$A \otimes_R B = H^0(A \otimes_R P^{\cdot}),$$

$$\mathrm{Tor}_n^R(A, B) = H^n(A \otimes_R P^{\cdot}) \quad \text{for } n \geq 1.$$

8.7 Exercises - Part 5

(Published on December 13, **solutions to be submitted on January 10, 2017**.)

Exercise 17. Let K be a field and Q be the Kronecker quiver $1 \rightrightarrows 2$.

(a) Let $\lambda \in K \setminus \{0\}$ and let M_λ be the representation $K \underset{1}{\overset{\lambda}{\rightrightarrows}} K$. Show that there is a short exact sequence $\varepsilon_\lambda \colon 0 \to S(2) \to M_\lambda \to S(1) \to 0$.

(b) Let $\lambda, \mu \in K \setminus \{0\}$. Show that ε_λ and ε_μ are equivalent if and only if $\lambda = \mu$.

Exercise 18. Let A, B and B' be R-modules. Let $\beta \in \operatorname{Hom}_R(B, B')$. Show that the map $\operatorname{Ext}^1_R(A, \beta) \colon \operatorname{Ext}^1_R(A, B) \to \operatorname{Ext}^1_R(A, B')$, where the assignment $[\varepsilon] \mapsto [\beta\varepsilon]$ is given by sending the short exact sequence $\varepsilon \colon 0 \to B \to E \to A \to 0$ to the short exact sequence $\beta\varepsilon$ via the pushout,

$$\begin{tikzcd}
\varepsilon\colon 0 \arrow[r] & B \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\beta\varepsilon\colon 0 \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is well defined.

Exercise 19. (a) Show that an R-module P is projective if and only if $\operatorname{Ext}^n_R(P, B) = 0$ for all R-modules B and for all $n > 0$.

(b) Let $P_\bullet \colon \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \longrightarrow 0$ be a projective resolution of an R-module A and $K_n = \ker p_n$ for each $n \geq 0$. Show that $\operatorname{Ext}^1_R(K_n, B) \cong \operatorname{Ext}^{n+2}_R(A, B)$ for all $n \geq 0$.

(c) Given $A \in R$ Mod, show that if $\operatorname{Ext}^{n+1}_R(A, B) = 0$ for all R-modules B, then there is a projective resolution of A of the form $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A$.

(d) Given $B \in R$ Mod, show that if $\operatorname{Ext}^{n+1}_R(A, B) = 0$ for all R-modules A, then there is an injective (co)resolution of B of the form $0 \to B \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$.

(e) Conclude that $\sup\{\operatorname{proj\,dim} A \mid A \in R \text{ Mod}\} = \sup\{\operatorname{inj\,dim} B \mid B \in R \text{ Mod}\}$.

Exercise 20. Let K be a field and Q the quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"]
\end{tikzcd}$$

(a) Let $\Lambda_1 = KQ/\mathcal{I}_1$, where $\mathcal{I}_1 = (\alpha\gamma)$.

(i) Determine all indecomposable projective representations of Λ_1.

(ii) Compute the global dimension of Λ_1.

(b) Let $\Lambda_2 = KQ/\mathcal{I}_2$, where $\mathcal{I}_2 = (\alpha\gamma, \gamma\beta)$. Compute the global dimension of Λ_2.

(c) Let $\Lambda_3 = KQ/\mathcal{I}_3$, where $\mathcal{I}_3 = (\alpha\gamma, \gamma\beta, \beta\alpha)$. Compute the global dimension of Λ_3.

9 AUSLANDER-REITEN THEORY

Let now Λ be again a finite dimensional algebra as in Section 7. As we have seen above, over hereditary algebras the functor $\operatorname{Ext}^1_\Lambda(-,\Lambda)$ is isomorphic to the transpose. In general, we have the following result.

Lemma 9.0.1. *Let $\mathfrak{E} : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence, and let $A \in \Lambda mod_{\mathcal{P}}$. Then there is a natural homomorphism $\delta = \delta(A, \mathfrak{E})$ such that the sequence $0 \to \operatorname{Hom}_\Lambda(A,X) \to \operatorname{Hom}_\Lambda(A,Y) \to \operatorname{Hom}_\Lambda(A,Z) \xrightarrow{\delta} \operatorname{Tr} A \otimes_\Lambda X \to \operatorname{Tr} A \otimes_\Lambda Y \to \operatorname{Tr} A \otimes_\Lambda Z \to 0$ is exact.*

Proof. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \to 0$ be a minimal projective presentation of A. Since the P_i, $i = 0,1$, are finitely generated projective, we know from 7.2.1 that $\operatorname{Hom}_\Lambda(P_i, M) \cong P_i^* \otimes_\Lambda M$ for any $M \in \Lambda\mathrm{Mod}$. So the cokernel of $\operatorname{Hom}(p_1, M) : \operatorname{Hom}_\Lambda(P_0, M) \longrightarrow \operatorname{Hom}_\Lambda(P_1, M)$ is isomorphic to $\operatorname{Tr} A \otimes_\Lambda M$. Hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \operatorname{Hom}_\Lambda(A,X) & \longrightarrow & \operatorname{Hom}_\Lambda(A,Y) & \longrightarrow & \operatorname{Hom}_\Lambda(A,Z) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \operatorname{Hom}_\Lambda(P_0,X) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0,Y) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0,Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \operatorname{Hom}_\Lambda(P_1,X) & \longrightarrow & \operatorname{Hom}_\Lambda(P_1,Y) & \longrightarrow & \operatorname{Hom}_\Lambda(P_1,Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Tr} A \otimes X & \longrightarrow & \operatorname{Tr} A \otimes Y & \longrightarrow & \operatorname{Tr} A \otimes Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and by the snake-lemma [27, 6.5] we obtain the claim. □

9.1 The Auslander-Reiten formula

Before proving the main result of this section, we need the following lemma.

Lemma 9.1.1. *Let $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ be a short exact sequence, $A \in \Lambda\, mod_{\mathcal{P}}$. Then there is a k-isomorphism $\operatorname{Coker}\operatorname{Hom}_\Lambda(i, \tau A) \cong D\operatorname{Coker}\operatorname{Hom}_\Lambda(A, \pi)$.*

Theorem 9.1.2 (Auslander-Reiten 1975)**.** *Let A, C be Λ-modules with $A \in \Lambda\, mod_{\mathcal{P}}$. Then there are natural k-isomorphisms*

$$\begin{array}{rl}
\text{(I)} & \overline{\operatorname{Hom}}_\Lambda(C, \tau A) \cong D\operatorname{Ext}^1_\Lambda(A, C) \\
\text{(II)} & D\,\underline{\operatorname{Hom}}_\Lambda(A, C) \cong \operatorname{Ext}^1_\Lambda(C, \tau A)
\end{array}$$

These formulae were first proved in [11], see also [22]. A more general version of (II), valid for arbitrary rings, is proved in [7, I, 3.4], cf.[18].
If Λ is hereditary, the Auslander-Reiten-formulas simplify as follows.

Corollary 9.1.3. *Let A, C be Λ-modules with $A \in \Lambda\, mod_{\mathcal{P}}$.*

1. *If* $\mathrm{pdim} A \leq 1$, *then* $\mathrm{Hom}_\Lambda(C, \tau A) \cong D\,\mathrm{Ext}^1_\Lambda(A, C)$.
2. *If* $\mathrm{idim} \tau A \leq 1$, *then* $D\,\mathrm{Hom}_\Lambda(A, C) \cong \mathrm{Ext}^1_\Lambda(C, \tau A)$.

Here is a first application.

Example 9.1.4. *If $\Lambda = kA_3$ is the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$, then every short exact sequence $0 \to P_2 \to E \to S_2 \to 0$ splits. Indeed, we know from 9.3 that $\tau S_2 \cong S_3$, so $\mathrm{Ext}^1_\Lambda(S_2, P_2) \cong \mathrm{Hom}_\Lambda(P_2, S_3) = 0$.*

9.2 Almost split sequences

Throughout this section we consider the category Λ-mod of finitely generated left Λ-modules, where, as before Λ is a finite-dimensional algebra.

Definition. We will require the following definitions.

(1) A homomorphism $g\colon B \to C$ in Λ-mod is called *right almost split* if

 (a) g is not a split epimorphism, and

 (b) if $h\colon X \to C$ is not a split epimorphism, then h factors through g, i.e. there exists $h'\colon X \to B$ such that $h = gh'$.

$$\begin{tikzcd} & X \arrow[dl, dotted, "\exists h'"'] \arrow[d, "h"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

(2) A homomorphism $g\colon B \to C$ is *right minimal* if each $h\colon B \to B$ with $gh = g$ is an isomorphism.

(3) A homomorphism $g\colon B \to C$ is *minimal right almost split* if it is right minimal and right almost split.

There are obvious dual definitions of *left almost split*, *left minimal* and *minimal left almost split*.

Lemma 9.2.1. *If $C \in \Lambda$-mod is indecomposable non-projective, then $P(C, C) \subset J(\mathrm{End}\, C)$.*

Proof. Since $C \in \Lambda$-mod is indecomposable, we have that $\mathrm{End}\, C$ is local so that $J(\mathrm{End}\, C)$ is the unique maximal ideal of $\mathrm{End}\, C$. Since 1_C does not factor through a projective unless C is projective, we have that $P(C, C)$ is a proper ideal of $\mathrm{End}\, C$, giving $P(C, C) \subset J(\mathrm{End}\, C)$. □

The following list of equivalent conditions will enable us to define an important class of short exact sequences.

Proposition 9.2.2. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in Λ-mod. The following statements are equivalent.*

(1) f is left almost split and g is right almost split.

(2) C is indecomposable and f is left almost split.

(3) A is indecomposable and g is right almost split.

(4) f is minimal left almost split.

(5) g is minimal right almost split.

Definition. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in Λ-mod is called *almost split* (or an *Auslander-Reiten sequence*, or an *AR sequence*) if it satisfies one of the equivalent conditions above.

Theorem 9.2.3 (Auslander–Reiten, 1975)**.** *Let Λ be a finite-dimensional algebra.*

(1) If $C \in \Lambda\text{-mod}$ is indecomposable and non-projective then there is an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\Lambda\text{-mod}$ with $A \cong \tau C$.

(2) If $A \in \Lambda\text{-mod}$ is indecomposable and non-injective then there is an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\Lambda\text{-mod}$ with $C \cong \tau^{-1} A$.

Proof. The first statement was proved in the lectures. We prove the second statement here, using the first statement.
Assume that $A \in \Lambda\text{-mod}$ is indecomposable and non-injective. Then it follows that DA is indecomposable and non-projective, so that the first statement asserts the existence of an almost split sequence $0 \to \tau DA \xrightarrow{f} B \xrightarrow{g} DA \to 0$. Applying the duality D to this sequence gives another short exact sequence

$$0 \to D^2 A \xrightarrow{Dg} DB \xrightarrow{Df} D\tau DA \to 0.$$

By results in Section 7, $D^2 A \cong A$ and $D\tau DA = D^2 \operatorname{Tr} DA = D^2 \tau^{-1} A \cong \tau^{-1} A$.
We claim that $Dg\colon D^2 A \to DB$ is left almost split. Suppose $h\colon D^2 A \to X$ is not a split monomorphism. Then $Dh\colon DX \to D^3 A \cong DA$ is not a split epimorphism (see the discussions in Section 5). Therefore, since g is right almost split, $Dh = gh'$ for some $h'\colon DX \to B$. Applying D again gives $h = D^2 h = Dh' Dg$, as required. □

Theorem 9.2.3 was originally proved in [11]. Another proof, using functorial arguments, is given in [8]. For generalizations of this result to arbitrary rings see [7, 6, 29, 30].

Corollary 9.2.4. *Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ are almost split sequences in $\Lambda\text{-mod}$. Then the following conditions are equivalent.*

(1) $A \cong A'$.

(2) $C \cong C'$.

(3) There is a commutative diagram in which the vertical maps are isomorphisms.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "a", "\sim"'] & B \arrow[r, "g"] \arrow[d, "b", "\sim"'] & C \arrow[r] \arrow[d, "c", "\sim"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

9.3 The Auslander-Reiten quiver

We now use almost split maps to get a 'combinatorial picture' of Λ-mod. First, we take care of the indecomposable projective and the indecomposable injective modules.

Proposition 9.3.1.

(1) If P is indecomposable projective, then the embedding $g\colon \operatorname{Rad} P \hookrightarrow P$ is minimal right almost split in ΛMod.

(2) If I is indecomposable injective, then the natural surjection $f\colon I \to I/\operatorname{Soc} I$ is minimal left almost split in ΛMod.

Proof. (1) Note that $\operatorname{Rad} P = JP$ and P/JP is simple [13, I,3.5 and 4.4], so $\operatorname{Rad} P$ is the unique maximal submodule of P. Thus, if $h\colon X \to P$ is not a split epimorphism, then it is not an epimorphism and therefore $\mathrm{Im} h$ is contained in $\operatorname{Rad} P$. Hence g is right almost split. Moreover, g is right minimal since every $t \in \operatorname{End} \operatorname{Rad} P$ with $gt = g$ has to be a monomorphism, hence an isomorphism.
(2) is proved using dual arguments. $\square$

Definition. A morphism $f\colon M \to N$ is *irreducible* if

(a) f is neither a split monomorphism nor a split epimorphism; and,

(b) if there exist morphisms $g\colon M \to X$ and $h\colon X \to N$ such that $f = hg$ then either g is a split monomorphism or h is a split epimorphism.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

Irreducible morphisms can also be described in terms of the following notion, which is treated in detail in [13, V.7].

Definition. For two modules $M, N \in \Lambda$-mod, we define the *radical* of $\operatorname{Hom}_\Lambda(M,N)$ by

$$r(M,N) = \{\, f \in \operatorname{Hom}_\Lambda(M,N) \mid \text{for each indecomposable module } Z \in \Lambda\,\text{mod}, \text{ every composition of the form } Z \to M \xrightarrow{f} N \to Z \text{ is a non-isomorphism}\}$$

For $n \in \mathbb{N}$ set

$$r^n(M,N) = \{f \in \operatorname{Hom}_\Lambda(M,N) \mid f = gh \text{ with } h \in r(M,X), g \in r^{n-1}(X,N), X \in \Lambda\,\text{mod}\}$$

Proposition 9.3.2. *Let $M, N, X \in \Lambda$-mod and assume M and N are indecomposable.*

(1) $r(M,X) = \{f\colon M \to X \mid f$ is not a split monomorphism$\}$.

(2) $r(X,N) = \{f\colon X \to N \mid f$ is not a split epimorphism$\}$.

(3) $r(M,N) = \{f\colon M \to N \mid f$ is not an isomorphism$\}$, whence $r(M,M) = J(\operatorname{End} M)$.

(4) $f \in \operatorname{Hom}_\Lambda(M,N)$ is irreducible if and only if $f \in r(M,N) \setminus r^2(M,N)$.

Proof. We prove (1) and (4) above; (2) and (3) are proved as in (1).
For statement (1), suppose first that $f \notin r(M,X)$, which means that there is some indecomposable module Z and a composition $Z \xrightarrow{h} M \xrightarrow{f} X \xrightarrow{g} Z$ such that $gfh\colon Z \to Z$ is an isomorphism. Since gfh is an isomorphism, $h\colon Z \to M$ is a split monomorphism, whence the indecomposability of M implies that h is an isomorphism. It now follows that f is a split monomorphism. Conversely, if f is a split monomorphism with left inverse $f'\colon X \to M$ then the composition $M \xrightarrow{1_M} M \xrightarrow{f} X \xrightarrow{f'} M$ is an isomorphism.
For statement (4), if $f\colon M \to N$ is irreducible, then f is not an isomorphism, which means $f \in r(M,N)$. Suppose $f = hg$ with $g\colon M \to X$ and $h\colon X \to N$. Since f is irreducible, either g is a split monomorphism or h is a split epimorphism. If g is a split monomorphism then by (1) $g \notin r(M,X)$ so that $hg \notin r^2(M,N)$. Similarly, if h is a split epimorphism then by (2) we also get $hg \notin r^2(M,N)$. Thus, $f \in r(M,N) \setminus r^2(M,N)$.
Conversely, suppose $f \in r(M,N) \setminus r^2(M,N)$. Therefore, f is neither a split monomorphism nor a split epimorphism. Suppose $f = hg$ for some g and h as above. Then either $g \notin r(M,X)$ and g is a split monomorphism or else $h \notin r(X,N)$ and h is a split epimorphism. Thus, f is irreducible. □

Since the irreducible morphisms arise as components of minimal right almost split maps and minimal left almost split maps, we obtain the following result.

Proposition 9.3.3. *Let M, N be indecomposable modules with an irreducible map $M \to N$. Let $g\colon B \to N$ be a minimal right almost split map, and $f\colon M \to B'$ a minimal left almost split map. Then there are integers $a, b > 0$ and modules $X, Y \in \Lambda\,\mathrm{mod}$ such that*

(1) $B \cong M^a \oplus X$ and M is not isomorphic to a direct summand of X,

(2) $B' \cong N^b \oplus Y$ and N is not isomorphic to a direct summand of Y.

Moreover,

$$a = \dim r(M,N)/r^2(M,N)_{\operatorname{End} M/J(\operatorname{End} M)}$$

$$b = \dim_{\operatorname{End} N/J(\operatorname{End} N)} r(M,N)/r^2(M,N)$$

Thus $a = b$ provided that k is an algebraically closed field.

Proof. The $\operatorname{End} N$-$\operatorname{End} M$-bimodule structure on $\operatorname{Hom}_\Lambda(M,N)$ induces an $\operatorname{End} N/J(\operatorname{End} N)$-$\operatorname{End} M/J(\operatorname{End} M)$-bimodule structure on $r(M,N)/r^2(M,N)$. Now $\operatorname{End} N/J(\operatorname{End} N)$ and $\operatorname{End} M/J(\operatorname{End} M)$ are skew fields. Consider the minimal right

almost split map $g : B \longrightarrow N$. If $g_1, \dots, g_a : M \to N$ are the different components of $g \mid_{M^a}$, then $\overline{g_1}, \dots, \overline{g_a}$ is the desired $\operatorname{End} M/J(\operatorname{End} M)$-basis. Dual considerations yield an $\operatorname{End} N/J(\operatorname{End} N)$-basis of $r(M,N)/r^2(M,N)$. For details, we refer to [13, VII.1].
Finally, since $\operatorname{End} N/J(\operatorname{End} N)$ and $\operatorname{End} M/J(\operatorname{End} M)$ are finite dimensional skew field extensions of k, we conclude that $a = b$ provided that k is an algebraically closed field. □

Definition. The *Auslander-Reiten quiver (AR-quiver)* $\Gamma = \Gamma(\Lambda)$ of Λ is constructed as follows. The set of vertices Γ_0 consists of the isomorphism classes $[M]$ of finitely generated indecomposable Λ-modules. The set of arrows Γ_1 is given by the following rule: set an arrow

$$[M] \xrightarrow{(a,b)} [N]$$

if there is an irreducible map $M \to N$ with (a, b) as above in Proposition 9.3.3.

Observe that Γ is a locally finite quiver (i.e. there exist only finitely many arrows starting or ending at each vertex) with the simple projectives as sources and the simple injectives as sinks. Moreover, if k is an algebraically closed field, we can drop the valuation by drawing multiple arrows.

Proposition 9.3.4. *Consider an arrow from* Γ

$$[M] \xrightarrow{(a,b)} [N]$$

(1) Translation of arrows*:*
If M, N *are indecomposable non-projective modules, then in* Γ *there is also an arrow*

$$[\tau M] \xrightarrow{(a,b)} [\tau N]$$

(2) Meshes*:*
If N *is an indecomposable non-projective module, then in* Γ *there is also an arrow*

$$[\tau N] \xrightarrow{(b,a)} [M]$$

Proof. (1) can be proven by exploiting the properties of the equivalence $\tau = D\operatorname{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 7.4.1. In fact, the following is shown in [12, 2.2]: If N is an indecomposable non-projective module with a minimal right almost split map $g : B \longrightarrow N$, and $B = P \oplus B'$ where P is projective and $B' \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ has non non-zero projective summand, then there are an injective module $I \in \Lambda\,\mathrm{mod}$ and a minimal right almost split map $g' : I \oplus \tau B' \longrightarrow \tau N$ such that $\tau(\underline{g}) = \overline{g'}$. Now the claim follows easily.
(2) From the almost split sequence $0 \longrightarrow \tau N \longrightarrow M^a \oplus X \longrightarrow N \longrightarrow 0$ we immediately infer that there is an arrow $[\tau N] \xrightarrow{(b',a)} [M]$ in Γ. So we only have to check $b' = b$. We know from 9.3.3 that $b' = \dim r(\tau N, M)/r^2(\tau N, M)_{\operatorname{End}\tau N/J(\operatorname{End}\tau N)}$. Now, the equivalence $\tau = D\operatorname{Tr}\colon\ \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 7.4.1 defines an isomorphism $\underline{\mathrm{End}}_\Lambda N \cong \overline{\mathrm{End}}_\Lambda \tau N$, which

in turn induces an isomorphism $\operatorname{End} N/J(\operatorname{End} N) \cong \operatorname{End} \tau N/J(\operatorname{End} \tau N)$. Moreover, using 9.3.3 and denoting by ℓ the length of a module over the ring k, it is not difficult to verify that $b' \cdot \ell(\operatorname{End} \tau N/J(\operatorname{End} \tau N)) = a \cdot \ell(\operatorname{End} M/J(\operatorname{End} M)) = \ell(r(M,N)/r^2(M,N)) = b \cdot \ell(\operatorname{End} N/J(\operatorname{End} N))$, which implies $b' = b$. $\square$

Remark 9.3.5. If Q is a finite connected acyclic quiver and $\Lambda = kQ$, then the number of arrows $[\Lambda e_j] \to [\Lambda e_i]$ in Γ coincides with the number of arrows $i \to j$ in Q, and with the number of arrows $[I_j] \to [I_i]$ in Γ.

Example: Let $\Lambda = K\mathbb{A}_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$.
Λ is a serial algebra. The module $I_3 \cong P_1$ has the composition series $P_1 \supset P_2 \supset P_3 \supset 0$. Furthermore, $I_3/\operatorname{Soc} I_3 \cong I_2$, and $I_2/\operatorname{Soc} I_2 \cong I_1$. So, there are only three almost split sequences, namely $0 \to P_3 \to P_2 \to S_2 \to 0$, and $0 \to P_2 \to S_2 \oplus P_1 \to I_2 \to 0$, and $0 \to S_2 \to I_2 \to I_1 \to 0$. Hence $\Gamma(\Lambda)$ has the form

$$\begin{tikzcd}
& & P_1 = I_3 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I_2 \arrow[dr] & \\
P_3 = S_3 \arrow[ur] \arrow[rr, dashed, no head] & & S_2 \arrow[ur] \arrow[rr, dashed, no head] & & I_1 = S_1
\end{tikzcd}$$

9.4 Knitting preprojective components

For simplicity, in this subsection we shall assume that Λ is a finite-dimensional algebra over an algebraically closed field.

Definition. Suppose Λ has n non-isomorphic simple modules S_i for $1 \leq i \leq n$. For $A \in \Lambda$-mod, we define $\underline{\dim}\, A = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, the *dimension vector* of A, where m_i is the number of composition factors isomorphic to S_i in a composition series for A.

Proposition 9.4.1 (Additivity of dimension vectors). *For each exact sequence* $0 \to A' \to A \to A'' \to 0$ *in* Λ-mod *we have* $\underline{\dim}\, A = \underline{\dim}\, A' + \underline{\dim}\, A''$.

Proof. See, for example, [13, I.1.1]. $\square$

Remark 9.4.2. We remark on further properties of the dimension vector.

(1) If $\underline{\dim} A = (m_1, \ldots, m_n)$, then $l(A) = \sum_{i=1}^{n} m_i$.

(2) Consider the Grothendieck group $K_0(\Lambda)$ defined as the group generated by the isomorphism classes $[A]$ of Λ-mod with the relations $[A'] + [A''] = [A]$ whenever $0 \to A' \to A \to A'' \to 0$ is exact in Λ-mod. Note that $K_0(\Lambda)$ is a free abelian group with basis $[S_1], \ldots, [S_n]$, see [13, I,1.7]. The assignment $[A] \mapsto \underline{\dim} A$ defines an isomorphism between $K_0(\Lambda)$ and $\mathbb{Z}^n$.

Definition. Let $\mathcal{C}$ be a connected component of the Auslander–Reiten quiver $\Gamma = \Gamma(\Lambda)$ of Λ. The component $\mathcal{C}$ is called *preprojective* if the following hold.

1. $\mathcal{C}$ contains no oriented cycles $[X_1] \to [X_2] \to \cdots [X_n] \to [X_1]$.
2. For any indecomposable module M in $\mathcal{C}$ there exists an integer $t \geq 0$ such that $M \cong \tau^{-t}P$ for some indecomposable projective module P.

An indecomposable module $M \in \Lambda$-mod is *preprojective* if it occurs in a preprojective component of $\Gamma(\Lambda)$. An arbitrary $M \in \Lambda$-mod is *preprojective* if it is a direct sum of indecomposable preprojective modules.
There are obvious dual definitions of *preinjective components* of $\Gamma(\Lambda)$ and *(indecomposable) preinjective modules.*

Remark 9.4.3. If Λ is a hereditary finite-dimensional algebra then the following hold.

- An indecomposable module M is preprojective if there exists an integer $t \geq 0$ such that $M \cong \tau^{-t}P$ for some indecomposable projective module P.
- The preprojective components of $\Gamma(\Lambda)$ contain all indecomposable projective modules.
- An Indecomposable module M is preinjective if there exists an integer $t \geq 0$ such that $M \cong \tau^{t}I$ for some indecomposable injective I.
- The preinjective components of $\Gamma(\Lambda)$ contain all indecomposable injective modules.

We note that some authors refer to preprojective components and modules as *postprojective* components and modules.

The following proposition says that preprojective or preinjective indecomposable modules are uniquely determined by their dimension vectors.

Proposition 9.4.4. *If M and N are preprojective or preinjective indecomposable modules such that $\underline{\dim}\, M = \underline{\dim}\, N$ then $M \cong N$.*

Proof. See, for example, [2, IX.1.1 and IX.3.1]. □

Knitting algorithm for preprojective components (dimension vector version).
When working over finite-dimensional hereditary algebras, Proposition 9.4.4 tells us that we can identify an indecomposable preprojective or preinjective module with its dimension vector. This facilitates a nice algorithm for computing the preprojective and preinjective components of the Auslander–Reiten quivers of finite-dimensional hereditary algebras.
The algorithm can be applied more generally, but outside preprojective or preinjective components one needs to take care when using dimension vectors, because non-preprojective and non-preinjective modules are not necessarily uniquely determined by their dimension vectors.

There is a dual version of the following algorithm to obtain preinjective components starting with simple injective modules; we leave it as an exercise to write down the dual version of the algorithm.

Preparations.

(1) Compute the dimension vectors of the indecomposable projective modules P_i.

(2) Write $\operatorname{Rad} P_i = \bigoplus_j R_{ij}^{(r_{ij})}$, where each R_{ij} is indecomposable, r_{ij} is the multiplicity of R_{ij} as a direct summand of $\operatorname{Rad} P_i$ and $R_{ij} \cong R_{ik}$ if and only if $j = k$.

(3) Compute the dimension vectors of the indecomposable injective modules I_i.

We will now construct a sequence of subquivers of the AR quiver $\Gamma = \Gamma(\Lambda)$, Δ_n, Δ'_n and Δ''_n for $n \geq 0$.

Base step.

(a) **Define** Δ_0 to be the quiver (without arrows) whose vertices are $[\underline{\dim}\, S]$ for S simple projective.

(b) **Add projectives:** for each $[\underline{\dim}\, S] \in (\Delta_0)_0$, if $S \cong R_{ij}$ for some i and j then add a vertex $[\underline{\dim}\, P_i]$ and r_{ij} arrows $[\underline{\dim}\, S] \to [\underline{\dim}\, P_i]$. Call the new quiver Δ'_0.

(c) **Translate non-injectives:** for each $[\underline{\dim}\, S] \in (\Delta_0)_0$ with S non-injective add a new vertex $[\underline{\dim}\, \tau^{-1} S]$ to Δ'_0. For each arrow $[\underline{\dim}\, S] \to [\underline{\dim}\, Y]$ constructed so far, add an arrow $[\underline{\dim}\, Y] \to [\underline{\dim}\, \tau^{-1} S]$ to Δ'_0. Call the new quiver Δ''_0.

For the inductive step we introduce some terminology. If y is a vertex of a quiver Q, then the *direct predecessors* of y are the vertices $x \in Q_0$ such that there is an arrow $x \to y$ in Q_1.

Inductive step.

(a) **Define** Δ_n**:** let Δ_n be the full subquiver of Δ''_{n-1} such that all direct predecessors of $[\underline{\dim}\, X] \in (\Delta''_{n-1})_0$ are contained in Δ_{n-1}. If $X \cong P_i$ we impose the additional requirement that $[\underline{\dim}\, R_{ij}] \in (\Delta_{n-1})_0$ for all j.

(b) **Add projectives:** for each $[\underline{\dim}\, X] \in (\Delta_n)_0$ if $X \cong R_{ij}$ for some i, j then (if not added already) add the vertex $[\underline{\dim}\, P_i]$ to Δ''_{n-1} and r_{ij} arrows $[\underline{\dim}\, X] \to [\underline{\dim}\, P_i]$. Call the new quiver Δ'_n.

(c) **Translate non-injectives:** for each $[\underline{\dim}\, X] \in (\Delta_n)_0 \setminus (\Delta_{n-1})_0$ with X non-injective, add the vertex $[\underline{\dim}\, \tau^{-1} X]$ to Δ'_n. For each arrow $[\underline{\dim}\, X] \to [\underline{\dim}\, Y]$ constructed so far, add an arrow $[\underline{\dim}\, Y] \to [\underline{\dim}\, \tau^{-1} X]$ to Δ'_n. Call the new quiver Δ''_n.

Three things can happen when knitting:

- We have to stop because we are not able to translate non-injectives (for example, if we have not obtained all arrows ending at a given indecomposable non-injective module), or cannot add a projective because summands of its radical are never constructed. Indeed, without a simple projective module, we cannot even begin knitting.

- The algorithm terminates, in which case by a result in the next chapter, we have computed the whole AR quiver.
- The algorithm never terminates, we will see examples of this when looking at preprojective components for tame hereditary algebras in the next lectures.

10 ALGEBRAS OF FINITE REPRESENTATION TYPE

10.1 Characterisations of finite-representation type

Definition. A finite-dimensional algebra Λ is said to be *of finite representation type* (or *representation-finite*) if there are only finitely many isomorphism classes of finitely generated indecomposable left Λ-modules. This is equivalent to the fact that there are only finitely many isomorphism classes of finitely generated indecomposable right Λ-modules.

Finite-dimensional algebras of finite representation type are completely described by their AR-quiver.

Theorem 10.1.1 (Auslander 1974, Ringel-Tachikawa 1973)**.** *Let Λ be an finite dimensional algebra of finite-representation type.*

1. *Every module is a direct sum of finitely generated indecomposable modules.*
2. *Every non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y is a sum of compositions of irreducible maps between indecomposable modules.*

Proof. For the proof of the second statement, we require the following lemma; see [13, VI.1.3] for a proof.

Theorem 10.1.2 (Harada-Sai Lemma)**.** *Let Λ be a finite-dimensional algebra. Any composition of $2^n - 1$ non-isomorphisms between indecomposable modules of length at most n is zero.*

Take a non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y. If $g : B \to Y$ is minimal right almost split, and $B = \bigoplus_{i=1}^n B_i$ with indecomposable modules B_i, then we can factor f as follows:

$$\begin{tikzcd} B_i \arrow[r, hook] & B \arrow[r, "g"'] & Y \\ & X \arrow[ul, dotted, "h_i"] \arrow[u, dotted, "h"'] \arrow[ur, "f"'] & \end{tikzcd} \qquad f = gh = \sum_{i=1}^{n} g|_{B_i} \circ h_i \quad \text{with irreducible maps } g|_{B_i}.$$

Moreover, if h_i is not an isomorphism, we can repeat the argument. But this procedure will stop eventually, because we know from the assumption and the Harada-Sai Lemma that there is a bound on the length of nonzero compositions of non-isomorphisms between indecomposable modules So after a finite number of steps we see that f has the desired shape. $\square$

Remark 10.1.3. In [5], Auslander also proved the converse of the first statement in Theorem 10.1.1. Combining this with a result of Zimmermann-Huisgen we obtain that an finite dimensional algebra is of finite representation type if and only if every left module is a direct sum of indecomposable left modules. The question whether the same holds true for any left artinian ring is known as the *Pure-Semisimple Conjecture.*

Observe that for the proof of the second statement in Theorem 10.1.1, actually, we only need a bound on the length of the modules involved. In fact, the following was proved in [4].

Theorem 10.1.4 (Auslander 1974)**.** *Let Λ be an indecomposable finite dimensional algebra with AR-quiver Γ. Assume that Γ has a connected component $\mathcal{C}$ such that the lengths of the modules in $\mathcal{C}$ are bounded. Then Λ is of finite representation type, and $\Gamma = \mathcal{C}$.*

In particular, of course, this applies to the case where Γ has a finite component. We sketch Yamagata's proof of Theorem 10.1.4, see also [13, VI.1.4].

Remark 10.1.5. For $A, B \in \Lambda\,\mathrm{mod}$ the descending chain $\mathrm{Hom}_\Lambda(A,B) \supset r(A,B) \supset r^2(A,B) \supset \cdots$ of k-subspaces of $\mathrm{Hom}_\Lambda(A,B)$ is stationary.

Proof of Theorem 10.1.4. The proof proceeds in three steps.
Step 1: The preceding remark, together with the Lemma of Harada and Sai, yields an integer n such that every $A \in \mathcal{C}$ satisfies

$$r^n(A,B) = 0 = r^n(B,A) \quad \text{for every} \quad B \in \Lambda\,\mathrm{mod}.$$

Step 2: If $A \in \mathcal{C}$, and $B \in \Lambda\,\mathrm{mod}$ is an indecomposable module with $\mathrm{Hom}_\Lambda(A,B) \neq 0$ or $\mathrm{Hom}_\Lambda(B,A) \neq 0$, then $B \in \mathcal{C}$. In fact, by similar arguments as in the proof of Theorem 10.1.1, every non-zero map $f \in \mathrm{Hom}_\Lambda(A,B)$ can be written as

$$0 \neq f = \sum g_1 \dots g_{m-1} h$$

where $g_1, \dots, g_{m-1}$ are irreducible maps between indecomposable modules, and by the above considerations, eventually in one of the summands the map h has to be an isomorphism. So, we find a path $A \xrightarrow{g_r} \dots \xrightarrow{g_1} B$ in $\mathcal{C}$ such that, moreover, the composition $g_1 \dots g_r \neq 0$.
Step 3: In particular, if $A \in \mathcal{C}$, we infer that any indecomposable projective module P with $\mathrm{Hom}_\Lambda(P,A) \neq 0$ belongs to $\mathcal{C}$. Since Λ is indecomposable, this shows that all indecomposable projectives are in $\mathcal{C}$. Furthermore, every indecomposable module $X \in \Lambda\,\mathrm{mod}$ satisfies $\mathrm{Hom}(P,X) \neq 0$ for some indecomposable projective P and hence belongs to $\mathcal{C}$ as well. But this means $\Gamma = \mathcal{C}$. Moreover, since there are only finitely many indecomposable projectives and there is a bound on the length of non-zero paths in $\mathcal{C}$, we conclude that $\Gamma = \mathcal{C}$ is finite. □

We have the following corollary of Theorem 10.1.4.

Corollary 10.1.6 (First Brauer-Thrall-Conjecture)**.** *A finite dimensional algebra is of finite representation type if and only if the lengths of the indecomposable finitely generated modules are bounded.*

The following conjecture is verified for finite-dimensional algebras over perfect fields, for example algebraically closed fields, but is open in general.

Conjecture 10.1.7 (Second Brauer-Thrall-Conjecture)**.** *If Λ is a finite dimensional k-algebra where k is an infinite field, and Λ is not of finite representation type, then there are infinitely many $n_1, n_2, n_3, \cdots \in \mathbb{N}$ and for each n_k there are infinitely many isomorphism classes of indecomposable Λ-modules of length n_k.*

11 TAME AND WILD ALGEBRAS

11.1 The Cartan matrix and the Coxeter transformation

Throughout this section we assume that k is an algebraically closed field.
We first set up some notation:

$$\underline{e_i} = (0, \dots, 1, 0, \dots, 0) = \underline{\dim} S_i$$

$$\underline{p_i} = \underline{\dim}\,\Lambda e_i = \underline{\dim} P_i$$

$$\underline{q_i} = \underline{\dim}\,D(e_i\Lambda) = \underline{\dim} I_i$$

Lemma 11.1.1. *Let Λ be a finite dimensional hereditary algebra. Then the matrix*

$$C = \begin{pmatrix} \underline{p_1} \\ \vdots \\ \underline{p_n} \end{pmatrix} \in \mathbb{Z}^{n\times n}$$

is invertible in $\mathbb{Z}^{n\times n}$.

Proof. We give an argument from [25, p. 70]. Take $1 \leq i \leq n$ and a projective resolution $0 \to Je_i \to \Lambda e_i \to S_i \to 0$ of S_i. Then $Je_i = \oplus_{k=1}^{n} \Lambda e_k{}^{r_{ik}}$ with multiplicities $r_{ik} \in \mathbb{Z}$, and by Proposition 9.4.1, we see that $\underline{e_i} = \underline{p_i} - \sum r_{ik}\underline{p_k}$ can be written as a linear combination of $\underline{p_1}, \dots, \underline{p_n}$ with coefficients in $\mathbb{Z}$. This shows that there is a matrix $R \in \mathbb{Z}^{n\times n}$ such that $R \cdot C = E_n$. □

Definition. Let Λ be a finite-dimensional hereditary algebra. The matrix C defined above is called the *Cartan matrix* of Λ. It defines the *Coxeter transformation*

$$c\colon \mathbb{Z}^n \to \mathbb{Z}^n\,, \qquad \underline{z} \mapsto -\,\underline{z}\;C^{-1}\,C^t$$

We are now going to see how the Coxeter transformation can be used to compute τ.

Proposition 11.1.2. *Let Λ be a finite-dimensional hereditary algebra over a field k.*

(1) For each $1 \leq i \leq n$ we have $c(\underline{p_i}) = -\underline{q_i}$.

(2) If $A \in \Lambda\,\mathrm{mod}$ is indecomposable non-projective, then $c(\underline{\dim} A) = \underline{\dim}\,\tau A$.

(3) An indecomposable module $A \in \Lambda\,\mathrm{mod}$ is projective if and only if $c(\underline{\dim} A)$ is negative.

Proof. (1) First of all, note that $\underline{\dim} A = (\dim_k e_1 A, \dots, \dim_k e_n A)$. In particular

$$\underline{p_i} = (\dim_k e_1\Lambda e_i, \dots, \dim_k e_n \Lambda e_i)$$

$$\underline{q_i} = (\dim_k e_i\Lambda e_1, \dots, \dim_k e_i \Lambda e_n) = \underline{\dim} e_i\Lambda.$$

This shows that $C^t = \begin{pmatrix} \underline{q_1} \\ \vdots \\ \underline{q_n} \end{pmatrix}$ and therefore $c(\underline{p_i}) = c(\underline{e_i}\,C) = -\underline{e_i}\,C^t = -\underline{q_i}$.

(2) Consider a minimal projective resolution $0 \longrightarrow Q \longrightarrow P \longrightarrow A \longrightarrow 0$. Then $c(\underline{\dim} A) = c(\underline{\dim} P) - c(\underline{\dim} Q)$. Applying the functor ${}^* = \mathrm{Hom}_\Lambda(-,\Lambda)$ and using that $\mathrm{Hom}_\Lambda(A,\Lambda) = 0$ by Remark 8.5.6(2), we obtain a minimal projective resolution $0 \longrightarrow P^* \longrightarrow Q^* \longrightarrow \mathrm{Tr}\, A \longrightarrow 0$ and therefore a short exact sequence $0 \longrightarrow \tau A \longrightarrow DQ^* \longrightarrow DP^* \longrightarrow 0$. Thus $\underline{\dim}\tau A = \underline{\dim} DQ^* - \underline{\dim} DP^*$, and the claim follows from (1).
(3) follows immediately from (1) and (2). □

11.2 Gabriel's classification of hereditary algebras

The Cartan matrix is also used to define the Tits form, which plays an essential role in Gabriel's classification of tame hereditary algebras.

Definition. Let Λ be a finite-dimensional hereditary algebra.

(1) Consider the (usually non-symmetric) bilinear form

$$B\colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q},\ (\underline{x}, \underline{y}) \mapsto \underline{x}\, C^{-1} \underline{y}^t$$

and the corresponding quadratic form

$$\chi\colon \mathbb{Q}^n \to \mathbb{Q}\,, \qquad \underline{x} \mapsto B(\underline{x}, \underline{x})$$

χ is called the *Tits form* of Λ.

(2) A vector $\underline{x} \in \mathbb{Z}^n$ is called a *root* of χ provided $\chi(\underline{x}) = 1$.

(3) A vector $\underline{x} \in \mathbb{Q}^n$ is called a *radical vector* provided $\chi(\underline{x}) = 0$. The radical vectors form a subspace of $\mathbb{Q}^n$ which we denote by

$$N = \{\underline{x} \in \mathbb{Q}^n \mid \chi(\underline{x}) = 0\}$$

(4) Finally, we say that a vector $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ is *positive* if all components $x_i \geq 0$.

The Tits form can be interpreted as follows, see [25, p. 71].

Proposition 11.2.1. *Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. For two vertices $i, j \in Q_0$ denote by d_{ji} the number of arrows $i \to j \in Q_1$. Then*

(1) Homological interpretation of χ *(Euler form): For $X, Y \in \Lambda\, mod$*

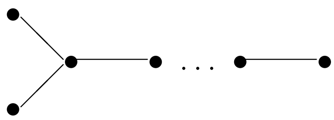
$$B(\underline{\dim} X, \underline{\dim} Y) = \dim_k \mathrm{Hom}_\Lambda(X, Y) - \dim_k \mathrm{Ext}^1_\Lambda(X, Y)$$

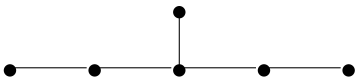
(2) Combinatorial interpretation of χ *(Ringel form): For $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$*

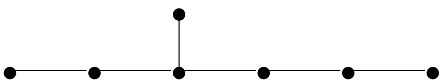
$$\chi(\underline{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{i \to j \in Q_1} d_{ji} x_i x_j$$

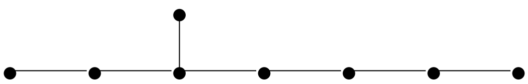
Definition. A graph Δ is called *simply-laced Dynkin* if it occurs on the following list.

$\mathbb{A}_n$: 

$\mathbb{D}_n$: 

$\mathbb{E}_6$: 

$\mathbb{E}_7$: 

$\mathbb{E}_8$: 

Theorem 11.2.2 (Gabriel 1972)**.** *Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. The following statements are equivalent.*

(a) Λ is of finite representations type.

(b) χ is positive definite, i.e. $\chi(x) > 0$ for all $\underline{x} \in \mathbb{Q}^n \setminus \{0\}$.

(c) The underlying graph of Q is simply-laced Dynkin.

If (a) - (c) are satisfied, the assignment $A \mapsto \underline{\dim} A$ defines a bijection between the isomorphism classes of indecomposable finite dimensional Λ-modules and the positive roots of χ. In particular, the finite dimensional indecomposable modules are uniquely determined by their dimension vector.

The following example shows that, even in the representation-finite case, the property of an indecomposable module being uniquely determined by its dimension vector is not generally true outside the hereditary case. For a more detailed discussion of when indecomposable modules are determined by their dimension vector see [2, Ch. IX].

Example 11.2.3. Consider the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and let $\Lambda = kQ/(\alpha\beta)$. Observe that $\operatorname{gldim} \Lambda = 2$, so that Λ is not a hereditary algebra. However, Λ is representation-finite. It has five indecomposable modules, listed below.

$$P_1 = I_1\colon\ \begin{tikzcd} k^2 \arrow[r, bend left, "{[1\ 0]}"] & k \arrow[l, bend left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \end{tikzcd} \quad P_2\colon\ \begin{tikzcd} k \arrow[r, bend left, "{[0]}"] & k \arrow[l, bend left, "{[1]}"] \end{tikzcd} \quad I_2\colon\ \begin{tikzcd} k \arrow[r, bend left, "{[1]}"] & k \arrow[l, bend left, "{[0]}"] \end{tikzcd}$$

$$S_1\colon\ \begin{tikzcd} k \arrow[r, bend left, "{[0]}"] & 0 \arrow[l, bend left, "{[0]}"] \end{tikzcd} \quad S_2\colon\ \begin{tikzcd} 0 \arrow[r, bend left, "{[0]}"] & k \arrow[l, bend left, "{[0]}"] \end{tikzcd}$$

In particular, the dimension vectors are:

$$\underline{\dim}\, P_1 = (2,1), \quad \underline{\dim}\, P_2 = (1,1), \quad \underline{\dim}\, I_2 = (1,1), \quad \underline{\dim}\, S_1 = (1,0), \quad \underline{\dim}\, S_2 = (0,2).$$

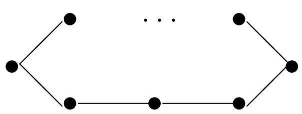
The Auslander–Reiten quiver of Λ is shown below. Note that it cannot be computed by the knitting algorithm: there are no simple projective or simple injective modules so we cannot even start.

$$\begin{tikzcd}[column sep=small]
& & P_1 = I_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I_2 \arrow[dr] & \\
S_1 \arrow[ur] \arrow[rr, dashed, no head] & & S_2 \arrow[ur] \arrow[rr, dashed, no head] & & S_1
\end{tikzcd}$$

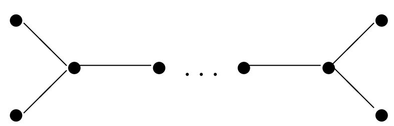
Note that we have written the simple module S_1 twice: both instances should be identified meaning that there is an oriented cycle in the AR quiver of Λ. In addition, observe that $\tau S_1 = S_2$ and $\tau S_2 = S_1$.

Moreover, Λ provides an example of a finite-dimensional algebra of finite representation type whose Auslander–Reiten quiver consists of a single finite component which is neither preprojective nor preinjective. Indeed, the simple modules are never equal to an AR translate of an injective module or an inverse AR translate of a projective module. This algebra belongs to a wider family of algebras called *Nakayama* algebras; see [2, Ch. V] and [13, IV.2].

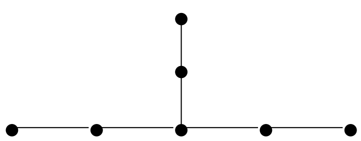
Definition. A graph Δ is called *Euclidean* or *extended Dynkin* if it occurs on the following list. $\widetilde{\mathbb{A}}_n$:



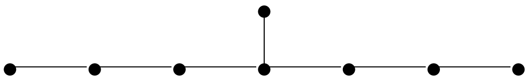
$\widetilde{\mathbb{D}}_n$:



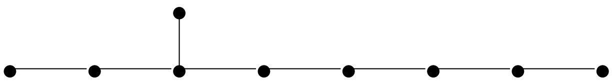
$\widetilde{\mathbb{E}}_6$:



$\widetilde{\mathbb{E}_7}$:



$\widetilde{\mathbb{E}_8}$:



Theorem 11.2.4 (Gabriel 1972)**.** *Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. The following statements are equivalent.*

(a) χ is positive semidefinite, i.e. $\chi(x) \geq 0$ for all $\underline{x} \in \mathbb{Q}^n \setminus \{0\}$, and there are non-trivial radical vectors.

(b) Q is of Euclidean *type, that is, its underlying graph belongs to the following list.*

If (a) and (b) are satisfied, then Λ is said to be tame of infinite representation type.

We will see in the next section that also in the latter case the isomorphism classes of indecomposable finite dimensional modules, though infinite in number, can be classified.

Remark 11.2.5. There is a general definiton of tameness for arbitrary finite dimensional algebras. A finite-dimensional k-algebra Λ over an algebraically closed field k is called *tame* if, for each dimension d, there are finitely many Λ-$k[x]$-bimodules $M_1, \cdots, M_n$ which are free of rank d as right $k[x]$-modules, such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq n$ and $\lambda \in k$. In other words, Λ is tame iff for each dimension d there is a finite number of one-parameter families of indecomposable d-dimensional modules such that all indecomposable modules of dimension d belong (up to isomorphism) to one of these families.

Moreover, Λ is said to be *of wild representation type* if there is a representation embedding from $k < x, y >$mod into $\Lambda\,$mod, where $k < x, y >$ denotes the free associative algebra in two non-commuting variables. Observe that in this case there is a representation embedding AMod $\to \Lambda\,$Mod for any finite dimensional k-algebra A, and furthermore, any finite dimensional k-algebra A occurs as the endomorphism ring of some Λ-module.

A celebrated theorem of Drozd [14] states that every finite dimensional algebra Λ over an algebraically closed field k is either tame or wild.

11.3 The AR-quiver of a hereditary algebra

Let Λ be a finite-dimensional hereditary algebra and let Γ be its AR quiver. We denote by $\mathbf{p}$ the preprojective component of Γ and by $\mathbf{q}$ the preinjective component of Γ. The remaining components of Γ will be called *regular components*.
We denote by $\mathbb{N}Q^{\mathrm{op}}$ and $-\mathbb{N}Q^{\mathrm{op}}$ the quivers obtained from Q by drawing the opposite quiver Q^{op}, and applying the "Knitting Procedure/Algorithm" described in 9.3.4 and 9.4.

Theorem 11.3.1 (Gabriel-Riedtmann 1979, [16]). *Let Λ and Q be as above.*

(1) If Q is a Dynkin quiver, then $\Gamma = \mathbf{p} = \mathbf{q}$ is a full finite subquiver of $\mathbb{N}Q^{\mathrm{op}}$.

(2) If Q is not a Dynkin quiver, then $\mathbf{p} = \mathbb{N}Q^{\mathrm{op}}$, and $\mathbf{q} = -\mathbb{N}Q^{\mathrm{op}}$, and the modules in $\mathbf{p}$ and $\mathbf{q}$ are uniquely determined by their dimension vectors. Moreover $\mathbf{p} \cap \mathbf{q} = \emptyset$, and $\mathbf{p} \cup \mathbf{q} \subsetneqq \Gamma$.

Thus, regular components only occur when Λ is of infinite representation type. They have a rather simple shape, as shown independently in [9] and [24]. For a proof, we refer to [13, VIII.4].

Theorem 11.3.2 (Auslander-Bautista-Platzeck-Reiten-Smalø; Ringel 1979). *Let Λ be of infinite representation type. Let $\mathcal{C}$ be a regular component of Γ. For each $[M]$ in $\mathcal{C}$ there are at most two arrows ending in $[M]$.*

Construction of the regular component $\mathcal{C}$.

For each $M \in \mathcal{C}$ we consider a minimal right almost split map $g : B \longrightarrow M$, and we denote by $\alpha(M)$ the number of summands in an indecomposable decomposition $B = B_1 \oplus \ldots \oplus B_{\alpha(M)}$ of B. We have stated the Theorem in a weak form; actually, it is even known that $\alpha(M) \leq 2$.

In order to construct $\mathcal{C}$, let us start with a module $C_0 \in \mathcal{C}$ of minimal length. Such a module is called *quasi-simple* (or *simple regular*).

Note that $\alpha(C_0) = 1$. Otherwise there is an almost split sequence of the form $0 \to \tau C_0 \to X_1 \oplus X_2 \xrightarrow{(g_1,g_2)} C_0 \to 0$ with non-zero modules X_1, X_2, and one can check that g_i cannot be both epimorphisms. But then $l(X_i) < l(C_0)$ for some i, a contradiction.

Now $\alpha(C_0) = 1$ implies that in Γ there is a unique arrow $[X] \xrightarrow{(1,1)} [C_0]$ ending in C_0, and therefore by 9.3.4(3), also a unique arrow starting in $[C_0]$ with valuation $(1,1)$. So we have an almost split sequence $0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{g_0} \tau^- C_0 \longrightarrow 0$ with C_1 being indecomposable. Moreover, we have an almost split sequence $0 \longrightarrow \tau C_1 \longrightarrow C_0 \oplus Y \xrightarrow{(f_0,h)} C_1 \longrightarrow 0$ where $Y \neq 0$ because f_0 is an irreducible monomorphism. Hence $\alpha(C_1) = 2$ and Y is indecomposable. Furthermore, one checks that h must be an irreducible epimorphism.

Setting $C_2 = \tau^- Y$ and $g_1 = \tau^- h$, we obtain an almost split sequence $0 \longrightarrow C_1 \xrightarrow{(f_1,g_0)^t} C_2 \oplus \tau^- C_0 \xrightarrow{(g_1,\tau^- f_0)} \tau^- C_1 \longrightarrow 0$ where g_0, g_1 are irreducible epimorphisms and $f_1, \tau^- f_0$ are irreducible monomorphisms.

Proceeding in this manner, we obtain a chain of irreducible monomorphisms $C_0 \hookrightarrow C_1 \hookrightarrow C_2 \ldots$ with almost split sequences $0 \longrightarrow C_i \longrightarrow C_{i+1} \oplus \tau^- C_{i-1} \longrightarrow \tau^- C_i \longrightarrow 0$ for all i.

The component $\mathcal{C}$ thus has the shape

$$\begin{tikzcd}
 & & & \vdots & & \vdots & & \vdots \\
 & & & Y = \tau C_2 \arrow[dr, "h"] & & C_2 \arrow[dr, "g_1"] & & \tau^- C_2 \\
 & \cdots & \tau C_1 \arrow[ur] \arrow[dr] & & C_1 \arrow[ur, "f_1"] \arrow[dr, "g_0"] & & \tau^- C_1 \arrow[ur] & \cdots \\
\cdots & \tau C_0 \arrow[ur] & & C_0 \arrow[ur, "f_0"] & & \tau^- C_0 \arrow[ur, "\tau^- f_0"] & \cdots &
\end{tikzcd}$$

and every module in $\mathcal{C}$ has the form $\tau^r C_i$ for some i and some $r \in \mathbb{Z}$.
Observe that if $\tau^r C_i \cong C_i$ for some i and r, then $\tau^r C \cong C$ for all C in $\mathcal{C}$.

Corollary 11.3.3. *Let $\mathbb{A}_\infty$ be the infinite quiver $\bullet \to \bullet \to \bullet \to \bullet \cdots$ Then $\mathcal{C}$ has either the form $\mathbb{Z}A_\infty$ or it has the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ where $n = \min\{r \in \mathbb{N} \mid \tau^r C \cong C$ for some $C \in \mathcal{C}\}$.*

Definition. We call $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ a *(stable) tube*, and we call it *homogeneous* if $n = 1$.

Stable tubes do not occur in the wild case. In the tame case, the regular components form a family of tubes $\mathbf{t} = \bigcup \mathbf{t}_\lambda$ indexed over the projective line $\mathbb{P}_1 k$, and all but at most three $\mathbf{t}_\lambda$ are homogeneous.

11.4 The Tame Hereditary Case

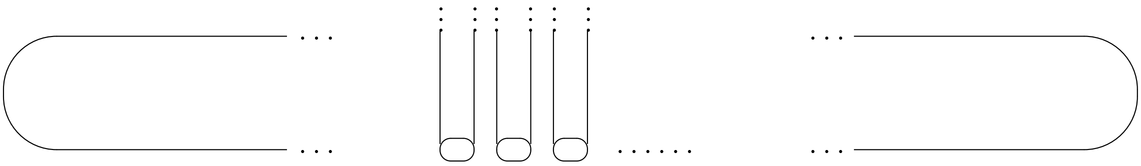
Let k be an algebraically closed field, and let Λ be a finite dimensional hereditary k-algebra with Gabriel-quiver Q of Euclidean type. The following properties are shown, for example, in [25].

(1) The $\mathbb{Q}$-subspace $N = \{\underline{x} \in \mathbb{Q}^n \mid \chi(\underline{x}) = 0\}$ formed by the radical vectors is one-dimensional and can be generated by a vector $\underline{v} = (v_1, \dots, v_n) \in \mathbb{N}^n$ with at least one component $v_i = 1$.

(2) There is a $\mathbb{Q}$-linear map $\delta\colon \mathbb{Q}^n \to \mathbb{Q}$ which is invariant under c, that is, $\delta(c\underline{x}) = \delta(\underline{x})$ for all $\underline{x} \in \mathbb{Q}^n$, and moreover satisfies $\delta(\underline{p_i}) \in \mathbb{Z}$ for each $1 \le i \le n$ and $\delta(\underline{p_i}) = -1$ for at least one i.

    The map δ is called the *defect*, and an indecomposable projective module $P = \Lambda e_i$ with defect -1 is called *peg*.

(3) As we have seen in the last section, the AR-quiver Γ has the shape



where $\mathbf{t} = \bigcup \mathbf{t}_\lambda$ and $\mathbf{t}_\lambda$ are tubes of rank n_λ with almost all $n_\lambda = 1$.

(4) The categories $\mathbf{p}, \mathbf{q}, \mathbf{t}$ are *numerically determined*:

If X is an indecomposable Λ-module, then

$$X \text{ belongs to } \mathbf{p} \text{ if and only if } \delta(\underline{\dim} X) < 0$$
$$X \text{ belongs to } \mathbf{q} \text{ if and only if } \delta(\underline{\dim} X) > 0$$
$$X \text{ belongs to } \mathbf{t} \text{ if and only if } \delta(\underline{\dim} X) = 0$$

(5) The dimension vectors $\underline{\dim} X$ of the indecomposable Λ-modules X are either positive roots of χ or positive radical vectors of χ. The assignment $X \mapsto \underline{\dim} X$ defines bijections

$$\{\text{isomorphism classes of } \mathbf{p}\} \longrightarrow \{\text{positive roots of } \chi \text{ with negative defect}\}$$
$$\{\text{isomorphism classes of } \mathbf{q}\} \longrightarrow \{\text{positive roots of } \chi \text{ with positive defect}\}$$

For any positive radical vector $\underline{x} \in \mathbb{Z}^n$ of χ there is a whole $\mathbb{P}_1 k$-family of isomorphism classes of $\mathbf{t}$ having dimension vector $\underline{x}$.

(6) $\mathbf{p}$ is *closed under predecessors*: If $X \in \Lambda\,\mathrm{Mod}$ is an indecomposable module with $\mathrm{Hom}(X, P) \neq 0$ for some $P \in \mathbf{p}$, then $X \in \mathbf{p}$.

In fact, $\mathbf{p}$ inherits "closure properties" from the projective modules. This can be proven employing the notion of preprojective partition together with the existence of almost split sequences in $\Lambda\,\mathrm{Mod}$. For finitely generated X there is also an easier argument: Since by Proposition 7.4.1 the functor $\tau\colon\ \mathrm{mod}\,\Lambda_{\mathcal{P}} \to \Lambda\,\mathrm{mod}_{\mathcal{I}}$ is an equivalence, $\mathrm{Hom}(X, P) \neq 0$ implies that either X is projective or $\mathrm{Hom}(\tau X, \tau P) \neq 0$. Continuing in this way and using that $\tau^n P$ is projective for some n, we infer that there exists an $m \leq n$ such that $\tau^m X$ is projective, which proves $X \in \mathbf{p}$.

(7) $\mathbf{q}$ is *closed under successors*: If $X \in \Lambda\,\mathrm{Mod}$ is an indecomposable module with $\mathrm{Hom}(Q, X) \neq 0$ for some $Q \in \mathbf{q}$, then $X \in \mathbf{q}$.

This is shown with dual arguments.

(8) The additive closure $\mathrm{add}\mathbf{t}$ of $\mathbf{t}$ is an *exact abelian serial subcategory of* Λmod: Each object is a direct sum of indecomposable objects, and each indecomposable object X has a unique chain of submodules in $\mathrm{add}\mathbf{t}$

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1 \supset X_0 = 0$$

such that the consecutive factors are simple objects of add $\mathbf{t}$. The simple objects of add $\mathbf{t}$ are precisely the quasi-simple modules introduced in 11.3.2. Their endomorphism rings are skew fields.

(9) The tubular family $\mathbf{t}$ is *separating*, that is:

(a) $\mathrm{Hom}(\mathbf{q},\mathbf{p}) = \mathrm{Hom}(\mathbf{q},\mathbf{t}) = \mathrm{Hom}(\mathbf{t},\mathbf{p}) = 0$

(b) Any map from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ factors through any $\mathbf{t}_\lambda$.

So, between the components of the AR-quiver, there are only maps from left to right. Actually, even inside $\mathbf{p}$ and $\mathbf{q}$ there are only maps from left to right.

(10) $\mathbf{t}$ is *stable*, i.e. it does not contain indecomposable modules that are projective or injective, and it is *sincere*, i.e. every simple module occurs as the composition factor of at least one module from $\mathbf{t}$.

Let us illustrate the above properties with an example.

11.5 The Kronecker Algebra

Consider the quiver

$$Q = \widetilde{\mathbb{A}_1} : \qquad \overset{1}{\bullet} \rightrightarrows \overset{2}{\bullet}$$

The algebra $\Lambda = kQ$ is called the Kronecker algebra, cf. [19].

(1) The Coxeter transformation and the Tits form:

$$\left.\begin{aligned} \underline{p_1} &= \underline{\dim}\Lambda e_1 = (1,2) \\ \underline{p_2} &= \underline{\dim}\Lambda e_2 = (0,1) \end{aligned}\right\} \text{ hence } \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

So we have

$$c(\underline{x}) = -\underline{x}\, C^{-1}\, C^t = \underline{x} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

$$\chi(\underline{x}) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2$$

$$N = \left\{\underline{x} \in \mathbb{Q}^2 \,|\, x_1 = x_2\right\} \quad \text{is generated by } \underline{v} = (1,1)\,.$$

We can then write

$$c(\underline{x}) = \underline{x}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right) = \underline{x} + 2(x_1 - x_2)\underline{v}$$

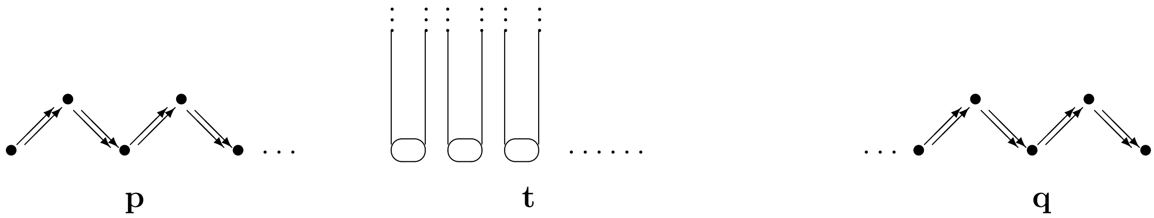
and since $c(\underline{v}) = \underline{v}$, we have

$$c^m \underline{x} = \underline{x} + 2m(x_1 - x_2)\underline{v} \quad \text{for each } m.$$

(2) Take $\delta\colon \mathbb{Q}^2 \to \mathbb{Q}$, $\underline{x} \mapsto B(\underline{v}, \underline{x}) = x_1 - x_2$. The $\mathbb{Q}$-linear map δ is the *defect.*

Then $\delta(\underline{p_1}) = -1 = \delta(\underline{p_2})$, so $P_1 = \Lambda e_1$ and $P_2 = \Lambda e_2$ are *pegs.*

(3) The AR-quiver Γ:



The shape of $\mathbf{t}$ is explained below. For $\mathbf{p}$ and $\mathbf{q}$ we refer to Theorem 11.3.1.

We can now compute the dimension vectors. For example, from the first two arrows on the left we deduce that there is an almost split sequence $0 \longrightarrow P_2 \longrightarrow P_1 \oplus P_1 \longrightarrow C \longrightarrow 0$ and $\underline{\dim} C = (1,2) + (1,2) - (0,1) = (2,3)$. In this way we observe

(4) $\mathbf{p}$ consists of the modules X with $\underline{\dim} X = (m, m+1)$, so $\delta(\underline{\dim} X) = -1$.

$\mathbf{q}$ consists of the modules X with $\underline{\dim} X = (m+1, m)$, so $\delta(\underline{\dim} X) = 1$.

The modules in $\mathbf{t}$ are precisely the modules X with $\underline{\dim} X = (m, m)$, so $\delta(\underline{\dim} X) = 0$.

Let us check the last statement. Let $X \in \mathbf{t}$ and $\underline{\dim} X = (l, m)$. If $l < m$, then

$$c^m(\underline{\dim} X) = (l,m)\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + m\begin{pmatrix}2 & 2\\ -2 & -2\end{pmatrix}\right) = (l,m) + 2m(l-m, l-m)$$

is negative. By 11.1.2 we have $c^m(\underline{\dim} X) = c(\underline{\dim} \tau^{m-1} X)$, thus $\tau^{m-1}X$ is projective, and $X \in \mathbf{p}$. Dually, $l > m$ implies $X \in \mathbf{q}$. Hence we conclude $l = m$.

(5) Let us now compute $\mathbf{t}$. First of all, the quasi-simple modules, that is, the indecomposable regular modules of minimal length, are precisely the modules X with $\underline{\dim} X = \underline{v} = (1,1)$. A complete irredundant set of quasi-simples is then given by

$$V_\lambda : K \underset{\lambda}{\overset{1}{\rightrightarrows}} K,\ \lambda \in K, \quad \text{and} \quad V_\infty : K \underset{\lambda}{\overset{0}{\rightrightarrows}} K$$

Note that each V_λ is sincere with composition factors S_1, S_2.

Furthermore, applying $\operatorname{Hom}(-, V_\mu)$ on the projective resolution $0 \to \Lambda e_2 \to \Lambda e_1 \to V_\lambda \to 0$ we see that V_λ, V_ν are "perpendicular":

$$\dim_k \operatorname{Hom}_\Lambda(V_\lambda, V_\mu) = \dim_k \operatorname{Ext} 1_\Lambda(V_\lambda, V_\mu) = \begin{cases} 1 & \mu = \lambda \\ 0 & \text{else} \end{cases}$$

Next, we check that each V_λ defines a homogeneous tube $\mathbf{t}_\lambda$.
In fact, $\tau V_\lambda \cong V_\lambda$ for all $\lambda \in K \cup \{\infty\}$:

$$\underline{\dim}\tau V_\lambda = c(\underline{\dim}V_\lambda) = (1,1)\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = (1,1)\,,$$

$$\text{hence} \quad \tau V_\lambda \cong V_\mu \text{ with } \operatorname{Ext}^1(V_\lambda, V_\mu) \neq 0\,, \text{ so } \mu = \lambda\,.$$

So, for each $\lambda \in K \cup \{\infty\}$ there is a chain of irreducible monomorphisms

$$V_\lambda = V_{\lambda,1} \hookrightarrow V_{\lambda,2} \hookrightarrow \dots$$

that gives rise to a homogeneous tube $\mathbf{t}_\lambda \cong \mathbb{Z}A_\infty \setminus \langle\tau\rangle$ consisting of modules $V_{\lambda,j}$ with $\tau V_{\lambda,j} \cong V_{\lambda,j}$, $\underline{\dim}V_{\lambda,j} = (j,j)$, $\delta(\underline{\dim}V_{\lambda,j}) = 0$, and $V_{\lambda,j+1}/V_{\lambda,j} \cong V_\lambda$.

Moreover, there are neither nonzero maps nor extensions between different tubes $\mathbf{t}_\lambda$.

Finally, let us indicate how to show that every indecomposable regular module X is contained in some tube $\mathbf{t}_\lambda$. We already know that X has the form $X : K^m \underset{\beta}{\overset{\alpha}{\rightrightarrows}} K^m$. Now, suppose that α is a isomorphism. Then, since k is algebraically closed, $\alpha^{-1}\beta$ has an eigenvalue λ, and, as explained in [13, VIII.7.3], it is possible to embed $V_\lambda \subset X$. This proves that X belongs to $\mathbf{t}_\lambda$. Similarly, if $\operatorname{Ker}\alpha \neq 0$, it is possible to embed $V_\infty \subset X$, which proves that X belongs to t_∞.

(6) To show that $\mathbf{t}$ is separating, we check that every $f\colon P \to Q$ with $P \in \mathbf{p}$, and $Q \in \mathbf{q}$, factors through any $\mathbf{t}_\lambda$. The argument is taken from [25, p.126].

Let $\lambda \in K \cup \{\infty\}$ be arbitrary, and let $\underline{\dim}P = (l, l+1)$ and $\underline{\dim}Q = (m+1, m)$. Choose an integer $j \geq l + m + 1$. We are going to show that f factors through $V_{\lambda,j}$.

Note that $\operatorname{Ext}^1_\Lambda(P, V_{\lambda,j}) = 0$. So, using the homological interpretation of B in Proposition 11.2.1 we obtain $\dim_k \operatorname{Hom}_\Lambda(P, V_{\lambda,j}) = \dim_k \operatorname{Hom}_\Lambda(P, V_{\lambda,j}) - \dim_k \operatorname{Ext}^1_\Lambda(P, V_{\lambda,j}) = B(\underline{\dim}P, \underline{\dim}V_{\lambda,j}) = (l, l+1)\left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} j \\ j \end{smallmatrix}\right) = j$.

So, the k-spaces $\operatorname{Hom}_\Lambda(P, V_{\lambda,j})$, $j \geq 0$, form a strictly increasing chain. Hence there exists a map $g\colon P \to V_{\lambda,j}$ such that $\operatorname{Im} g \not\subset V_{\lambda,j-1}$, and by length arguments we infer that $\operatorname{Im} g$ is a proper submodule of $V_{\lambda,j}$. Thus $\operatorname{Im} g$ is not regular. Then it must contain a preprojective summand P', and we conclude that g is a monomorphism. Consider the exact sequence

$$0 \longrightarrow P \xrightarrow{g} V_{\lambda,j} \longrightarrow Q' \longrightarrow 0$$

The module Q' cannot have regular summands, so it is a direct sum of preinjective modules and satisfies

$$\delta(\underline{\dim}Q') = \delta(\underline{\dim}V_{\lambda,j}) - \delta(\underline{\dim}P) = 1$$

This shows $Q' \in q$. Furthermore, $\underline{\dim} Q' = (s+1, s)$ with $s = j - (l+1) \geq m$, which proves $\mathrm{Ext}^1_\Lambda(Q', Q) = 0$. Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[rr, "g"] \arrow[dr, "f"'] & & V_{\lambda,j} \arrow[r] \arrow[dl, dotted] & Q' \arrow[r] & 0 \\ & & Q & & & \end{tikzcd}$$

and the claim is proved.

11.6 Exercises - Part 6

(Published on January 3, **solutions to be submitted January 19, 2017**.)

Exercise 21. Let Λ be a finite-dimensional algebra over an algebraically closed field. Let M and N be Λ-modules and $f\colon M \to N$ be a minimal right almost split map.

(a) Show that N is indecomposable.

(b) Show that f is not a split monomorphism.

(c) Suppose there exists a module L and morphisms $f_1\colon M \to L$ and $f_2\colon L \to N$ such that $f = f_2 f_1$ and f_2 is not a split epimorphism. Show that f_1 is a split monomorphism.

(d) Conclude that f is an irreducible morphism.

Exercise 22. Using the knitting algorithm on dimension vectors, compute the following.

(a) The AR quiver of the path algebra of the quiver
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \\ & & 6 & & \end{tikzcd}.$$

(b) The AR quiver of the first 4 inverse AR translates of the indecomposable projective modules for the path algebra of the quiver
$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}.$$

Exercise 23. Let $K_0(\Lambda)$ be the Grothendieck group of Λ, i.e. the free abelian group on isomorphism classes $[M]$ of modules $M \in \Lambda$-mod subject to the relations $[L] = [M] + [N]$ for each short exact sequence $0 \to M \to L \to N \to 0$ in Λ-mod.

(a) Show the set $\{[S_1], \ldots, [S_n]\}$, where the S_i are the simple left Λ-modules, generates $K_0(\Lambda)$. (Hint: For any $M \in \Lambda$-mod consider a composition series and use the additivity of the dimension vector on short exact sequences.)

(b) Show that the set $\{[S_1], \ldots, [S_n]\}$ is $\mathbb{Z}$-linearly independent in $K_0(\Lambda)$ and deduce that $\underline{\dim}\colon K_0(A) \to \mathbb{Z}^n$ defines an isomorphism of abelian groups.

Exercise 24. Let Q be a quiver without oriented cycles and $\chi\colon \mathbb{Z}^n \to \mathbb{Z}$ be the quadratic form given by $\chi(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$. Let $M \in \Lambda$-mod have dimension vector $\underline{\dim}\, M = \mathbf{d}$. Show that

$$\chi(\mathbf{d}) = \dim \operatorname{Hom}_\Lambda(M, M) - \dim \operatorname{Ext}^1_\Lambda(M, M).$$

(You may assume that any such M has a projective resolution of the form

$$0 \to \bigoplus_{\alpha \in Q_1} P(t(\alpha))^{(d_{s(\alpha)})} \to \bigoplus_{i \in Q_0} P(i)^{(d_i)} \to M \to 0.$$

Hints: Use the long exact Hom-Ext sequence; recall $\dim \operatorname{Hom}_\Lambda(P(i), M) = d_i$.)

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