University of Verona
Master's Degree in Mathematics
a.y. 2014/15

Second partial test in Optimization

Verona, 2nd February 2015

Solve obligatorily Exercise 3 and ***one between*** *Exercises 1 and 2.*

Exercise 1. For $i = 1, 2, 3$, consider the following problem: minimize the functional

$$J(x(\cdot)) := \int_0^1 \left(4t^2 x'(t) + 5x'(t)^2 + x(t)^2 x'(t) + 6x(t)^2\right) dt, \qquad x(\cdot) \in \mathscr{C}_i,$$

where

$$\begin{aligned} \mathscr{C}_1 :=& C^2(]0,1[) \cap C^0([0,1]), \\ \mathscr{C}_2 :=& \{x(\cdot) \in \mathscr{C}_1 : x(0) = 0,\, x(1) = 1\}, \\ \mathscr{C}_3 :=& \left\{x(\cdot) \in \mathscr{C}_2 : \int_0^1 x^2(t)\, dt = 1\right\}. \end{aligned}$$

In each case establish wheater the infimum is attained or not, and for $\mathscr{C}_1$ and $\mathscr{C}_2$ in the positive case find explicitly the point of minimum (not required for $\mathscr{C}_3$).

Exercise 2. Consider the following control system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1(t) = 3x_1(t) - x_2(t); \\ \\ \dot{x}_2(t) = u(t) + 2x_1(t) - x_2(t). \end{cases}$$

where $u(\cdot) \in \mathscr{U} := \{u : \mathbb{R} \to U := [-1, 1] \text{ measurable}\}$.

Consider the problem $\max\limits_{u \in \mathscr{U}} x_1(T)$.

(1) Write the adjoint system with terminal conditions and solve it explicitly.
(2) Apply Pontryagin's Maximum Principle to isolate the candidates and establish if they are optimal.

Exercise 3.

(1) Write down the statement of Pontryagin's Maximum Principle.

(2) Formulate mathematically Dido's problem, and solve it analytically.

(3) Write down the statement of the parametric contraction's principle.

(4) Let Ω be an open bounded nonempty subset of $\mathbb{R}^d$ with C^∞ boundary. Let $\nu(x)$ be the external unit normal to Ω at $x \in \partial\Omega$. Consider the control system $\dot{x}(t) = f(x(t), u(t))$ where $f \in C^{1,1}(\mathbb{R}^d \times \mathbb{R}^m)$ is bounded, an measurable admissible controls $u : [0, +\infty[\to U$, where U is a compact subset of $\mathbb{R}^m$. Prove that if there exists $\mu > 0$ such that $f(x, u) \cdot \nu(x) < -\mu$ for every $x \in \partial\Omega$, then for every $x_0 \in \Omega$ there exists trajectories starting from x_0 and remaining in Ω per ogni $t > 0$.

(5) State and prove a result concerning the sum rule for the subdifferential of convex analysis.

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Written test in Optimization

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Exercise 1. Let Ω be an open bounded subset of $\mathbb{R}^2$. Consider the problem:

$$\inf_{u \in H_0^1(\Omega)} \int_\Omega \left(|\nabla u(x_1, x_2)|^2 + \partial_{x_2} u(x_1, x_2)\, \partial_{x_1} u(x_1, x_2) + (\cos(x_2)\, u(x_1, x_2) - 2)^2 \right) dx_1\, dx_2.$$

(1) Prove that the problem admits a unique solution.
(2) Formulate the problem as $\mathscr{F}(u) = F(u) + G \circ \Lambda(u)$, where $F : X \to]-\infty, +\infty]$, $G : Y \to]-\infty, +\infty]$ and $\Lambda : X \to Y$, carefully precising the function spaces X, Y and discussing the regularity of F, G, Λ.
(3) Write the dual problem and the extremality relations. Establish if the dual problem admits a unique solution.
(4) Use the above results to write a partial differential equation whose solution is the minimum.

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