2D Fourier Transform
Overview

• Signals as functions (1D, 2D)
  – Tools

• 1D Fourier Transform
  – Summary of definition and properties in the different cases
    • CTFT, CTFS, DTFS, DTFT
    • DFT

• 2D Fourier Transforms
  – Generalities and intuition
  – Examples
  – A bit of theory

• Discrete Fourier Transform (DFT)
• Discrete Cosine Transform (DCT)
Signals as functions

1. Continuous functions of real independent variables
   - 1D: $f=f(x)$
   - 2D: $f=f(x,y)$ $x,y$
   - Real world signals (audio, ECG, images)

2. Real valued functions of discrete variables
   - 1D: $f=f[k]$
   - 2D: $f=f[i,j]$
   - *Sampled* signals

3. Discrete functions of discrete variables
   - 1D: $y=y[k]$
   - 2D: $y=y[i,j]$
   - *Sampled and quantized* signals
   - For ease of notations, we will use the same notations for 2 and 3
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of \((x,y)\) values for which \(f(x,y)\) is defined: 2D lattice \([i,j]\) defining the pixel locations
  - Set of values taken by the function: gray levels

- Digital images can be seen as functions defined over a discrete domain \(#\{i,j: 0<i<I, 0<j<J\}\#
  - \(I,J\): number of rows (columns) of the matrix corresponding to the image
  - \(f=f[i,j]\): gray level in position \([i,j]\)
Example 1: $\delta$ function

$$\delta[i, j] = \begin{cases} 1 & i = j = 0 \\ 0 & i, j \neq 0; i \neq j \end{cases}$$

$$\delta[i, j - J] = \begin{cases} 1 & i = 0; j = J \\ 0 & \text{otherwise} \end{cases}$$
Example 2: Gaussian

Continuous function

\[ f(x, y) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{x^2 + y^2}{2\sigma^2}} \]

Discrete version

\[ f[i, j] = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{i^2 + j^2}{2\sigma^2}} \]
Example 3: Natural image
Example 3: Natural image
Fourier Transform

• Different formulations for the different classes of signals
  – Summary table: Fourier transforms with various combinations of continuous/discrete time and frequency variables.
  – Notations:
    • CTFT: continuous time FT
    • DTFT: Discrete Time FT
    • CTFS: CT Fourier Series (summation synthesis)
    • DTFS: DT Fourier Series (summation synthesis)
    • P: periodical signals
    • T: sampling period
    • $\omega_s$: sampling frequency ($\omega_s = 2\pi/T$)
    • For DTFT: $T=1 \rightarrow \omega_s = 2\pi$
1D FT: basics

• Define frequency
  \[ = \frac{1}{T} \]
  cycles per unit time
  cycles per unit distance

• Here \( f = 1 \)
Fourier Transform: Concept

- A signal can be represented as a weighted sum of sinusoids.
- Fourier Transform is a change of basis, where the basis functions consist of sines and cosines (complex exponentials).

**FIGURE 4.1** The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.
Fourier Transform

- Cosine/sine signals are easy to define and interpret.
- However, it turns out that the analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.

- A complex number has real and imaginary parts: $z = x + jy$

- A complex exponential signal:

$$r e^{j\alpha} = r \left( \cos \alpha + j \sin \alpha \right)$$
## Overview

<table>
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<th>Transform</th>
<th>Time</th>
<th>Frequency</th>
<th>Analysis/Synthesis</th>
<th>Duality</th>
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<tr>
<td>(Continuous Time) Fourier Transform (CTFT)</td>
<td>C</td>
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<td>$F(\omega) = \int f(t)e^{-j\omega t} , dt$</td>
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Dualities

SIGNAL DOMAIN

Sampling
Periodicity
Sampling + Periodicity

FOURIER DOMAIN

Periodicity
Sampling
Sampling + Periodicity

DTFT
CTFS
DTFS/DFT
**Discrete time signals**

- Sequences of samples
- $f[k]$: sample values
- Assumes a unitary spacing among samples ($T_s=1$)
- Normalized frequency $\Omega$
- Transform
  - DTFT for NON periodic sequences
  - CTFS for periodic sequences
  - DFT for *periodized* sequences
- All transforms are $2\pi$ periodic

---

- **Sampled** signals
- $f(kT_s)$: sample values
- The sampling interval (or period) is $T_s$
- Non normalized frequency $\omega$
- Transform
  - DTFT
  - CSTF
  - DFT
  - BUT accounting for the fact that the sequence values have been generated by sampling a real signal $\rightarrow f_k = f(kT_s)$
- All transforms are periodic with period $\omega_s$

$$\Omega = \omega T_s$$
CTFT

- Continuous Time Fourier Transform
- Continuous time \textit{a-periodic} signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency $\omega$ is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

\[
F(\omega) = \int f(t)e^{-j\omega t} dt \quad \text{analysis}
\]

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega \quad \text{synthesis}
\]
CTFT: change of notations

- Fourier Transform of a 1D continuous signal
  \[ F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} \, dx \]
  “Euler’s formula”
  \[ e^{-j\omega x} = \cos(\omega x) - j\sin(\omega x) \]
- Inverse Fourier Transform
  \[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega x} \, d\omega \]

Change of notations:
\[
\begin{align*}
\omega &\rightarrow 2\pi u \\
\omega_x &\rightarrow 2\pi u \\
\omega_y &\rightarrow 2\pi v
\end{align*}
\]
Then CTFT becomes

- Fourier Transform of a 1D continuous signal

\[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \, dx \]

“Euler's formula” \( e^{-j2\pi ux} = \cos(2\pi ux) - j \sin(2\pi ux) \)

- Inverse Fourier Transform

\[ f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} \, du \]
Continuous Time Fourier Series

Continuous time periodic signals
- The signal is periodic with period $T_0$
- The transform is “sampled” (it is a series)

Our notations

\[ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_0}(t)e^{-j\omega_0 t} \, dt \]

\[ f_{T_0}(t) = \sum_n D_n e^{j\omega_0 t} \]

\[ \omega_0 = \frac{2\pi}{T_0} \]

Table notations

\[ F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-j2\pi kt/T} \, dt \]

\[ f(t) = \sum_k F[k]e^{j2\pi kt/T} \]

Fundamental frequency

\[ T_0 \leftrightarrow T \]

\[ D_n \leftrightarrow F[k] \]
CTFS

• Representation of a continuous time signal as a sum of orthogonal components in a complete orthogonal signal space
  – The exponentials are the basis functions

• Fourier series are periodic with period equal to the fundamental in the set $(2\pi/T_0)$

• Properties
  – even symmetry $\rightarrow$ only cosinusoidal components
  – odd symmetry $\rightarrow$ only sinusoidal components
CTFS: example 1

![Graph showing CTFS example 1 with different curves for f(t), k=5, and k=15. The x-axis represents time in seconds (t/s), and the y-axis represents a value from 0 to 5. The graph shows periodic oscillations with different amplitudes for each case.]
CTFS: example 2
From sequences to discrete time signals

- Looking at the sequence as to a set of samples obtained by sampling a real signal with frequency $\omega_s$ we can still use the formulas for calculating the transforms as derived for the sequences by
  - Stretching the time axis (and thus squeezing the frequency axis if $T_s > 1$)
  - Enclosing the sampling interval $T_s$ in the value of the sequence samples (DFT)

\[
\Omega = \omega T_s
\]

\[
2\pi \rightarrow \omega_s = \frac{2\pi}{T_s}
\]

- Enclosing the sampling interval $T_s$ in the value of the sequence samples (DFT)

\[
f_k = T_s f\left(kT_s\right)
\]
DTFT

- Discrete Time Fourier Transform
- *Discrete time a-periodic* signal
- The transform is periodic and continuous with period $\Omega_0 = 2\pi$

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<td>$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k] e^{-j\Omega k}$</td>
<td>$F\left(e^{j\omega}\right) = \sum f[n] e^{-j2\pi n\omega_s/\omega_s}$</td>
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<td>$f[k] = \frac{1}{2\pi} \int F(\Omega) e^{j\Omega k} d\Omega$</td>
<td>$f[n] = \frac{1}{\omega_s/2} \int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega}\right) e^{j2\pi n\omega_s/\omega_s} d\omega$</td>
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Normalized frequency

$F(\Omega) = \bar{F}_c \left( \frac{\Omega}{T_s} \right)$

$\Omega = \omega T_s$

$T_s = 2\pi / \omega_s$

Non normalized frequency

$T_s = 2\pi / \omega_s$
Discrete Time Fourier Transform (DTFT)

- $F(\Omega)$ can be obtained from $F_c(\omega)$ by replacing $\omega$ with $\Omega/T_s$. Thus $F(\Omega)$ is identical to $F_c(\omega)$ frequency scaled by a factor $1/T_s$
  - $T_s$ is the sampling interval in time domain

- Notations

\[
F(\Omega) = \overline{F_c}\left(\frac{\Omega}{T_s}\right)
\]

\[
\omega_s = \frac{2\pi}{T_s} \rightarrow T_s = \frac{2\pi}{\omega_s}
\]

periodicity of the spectrum

\[
\omega = \frac{\Omega}{T_s} \rightarrow \Omega = \omega T_s
\]

normalized frequency (the spectrum is $2\pi$-periodic)

\[
F(\Omega) \rightarrow F(\omega T_s) = F\left(\frac{2\pi \omega}{\omega_s}\right)
\]

\[
F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-i\Omega k} \rightarrow F(\omega T_s) = F(\omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j2k\pi\omega/\omega_s}
\]
\textbf{DTFT: unitary frequency}

\[ \Omega = 2\pi u \quad (\omega = 2\pi f) \]

\[ F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \rightarrow F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku} \]

\[ f[k] = \frac{1}{2\pi} \int_{2\pi}^{2\pi} F(\Omega)e^{j\Omega k} d\Omega \rightarrow f[k] = \frac{1}{2} \int_{1}^{1} F(u)e^{j2\pi ku} du = \frac{1}{2} \int_{-\frac{1}{2}}^{-\frac{1}{2}} F(u)e^{j2\pi ku} du \]

\[ \begin{cases} 
  F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku} \\
  f[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi ku} du 
\end{cases} \]

\text{NOTE: when } T_s = 1, \Omega = \omega \text{ and the spectrum is } 2\pi\text{-periodic. The unitary frequency } u = 2\pi/\Omega \text{ corresponds to the signal frequency } f = 2\pi/\omega. \text{ This could give a better intuition of the transform properties.}
Connection DTFT-CTFT

sampling

\[ f(t) \]

\[ f(kT_s) \]

\[ f[k] \]

periodization \( F_c(\omega) \)

\[ F(\Omega) \]

\[ 0 \]

\[ 2\pi/Ts \]

\[ 0 \]

\[ 2\pi \]
Differences DTFT-CTFT

- The DTFT is periodic with period $\Omega_s = 2\pi$ (or $\omega_s = 2\pi/T_s$)
- The discrete-time exponential $e^{j\Omega k}$ has a unique waveform only for values of $\Omega$ in a continuous interval of $2\pi$
- *Numerical computations can be conveniently performed with the Discrete Fourier Transform (DFT)*
DTFS

- Discrete Time Fourier Series
- **Discrete time periodic** sequences of period $N_0$
  - Fundamental frequency

\[ \Omega_0 = \frac{2\pi}{N_0} \]

our notations
\[
D_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}
\]
\[
f[k] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 k}
\]

table notations
\[
F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/T}
\]
\[
f[k] = \sum_{n=0}^{N-1} F[k] e^{j2\pi kn/T}
\]
Discrete Fourier Transform (DFT)

\[
F_r = \sum_{k=0}^{N_0-1} f_k e^{-j\Omega_0 k} = \sum_{k=0}^{N_0-1} f_k e^{-j\frac{2\pi}{N_0} r k}
\]

\[
f_k = \frac{1}{N_0} \sum_{k=0}^{N_0-1} F_r e^{j\Omega_0 k} = \frac{1}{N_0} \sum_{k=0}^{N_0-1} F_r e^{j\frac{2\pi}{N_0} k}
\]

\[
\Omega_0 = \frac{2\pi}{N_0}
\]

• The DFT transforms \(N_0\) samples of a discrete-time signal to the same number of discrete frequency samples

• The DFT and IDFT are a self-contained, one-to-one transform pair for a length-\(N_0\) discrete-time signal (that is, the DFT is not merely an approximation to the DTFT as discussed next)

• However, the DFT is very often used as a practical approximation to the DTFT
DFT

zero padding

\[ F(\Omega) \]

\[ k \]

\[ 0, N_0 \]

\[ r \]

\[ 0, 2\pi/N_0, 2\pi, 4\pi \]
Discrete Cosine Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A discrete cosine transform (DCT) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, of which four are common
- Strong connection with the Karunen-Loeve transform
  - VERY important for signal compression
DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an even periodic extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at both the left and right boundaries of the domain
  - Second, one has to specify around what point the function is even or odd
    - In particular, consider a sequence $abcd$ of four equally spaced data points, and say that we specify an even left boundary. There are two sensible possibilities: either the data is even about the sample $a$, in which case the even extension is $dcbabcd$, or the data is even about the point halfway between $a$ and the previous point, in which case the even extension is $dcbaabcd$ ($a$ is repeated).
Symmetries

DCT-I:

DCT-II:

DCT-III:

DCT-IV:

N = 11
DCT

\[ X_k = \sum_{n=0}^{N_0-1} x_n \cos \left( \frac{\pi}{N_0} \left( n + \frac{1}{2} \right) k \right) \quad k = 0, \ldots, N_0 - 1 \]

\[ x_n = \frac{2}{N_0} \left\{ \frac{1}{2} X_0 + \sum_{k=0}^{N_0-1} X_k \cos \left( \frac{\pi k}{N_0} \left( k + \frac{1}{2} \right) \right) \right\} \]

- **Warning**: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by \((2/N_0)^{1/2}\) so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of \(\sqrt{2}\) (see above), this can be used to make the transform matrix orthogonal.
Sinusoids

- Frequency domain characterization of signals

\[ F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} dt \]

Signal domain

Frequency domain
Gaussian
rect

$sinc$ function
# Images vs Signals

## 1D
- **Signals**
- **Frequency**
  - Temporal
  - Spatial
- **Time (space) frequency characterization of signals**
- **Reference space for**
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ....

## 2D
- **Images**
- **Frequency**
  - Spatial
- **Space/frequency characterization of 2D signals**
- **Reference space for**
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ....
2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies
2D Frequency domain

- Large vertical frequencies correspond to horizontal lines.
- Small horizontal and vertical frequencies correspond to smooth grayscale changes in both directions.
- Large horizontal and vertical frequencies correspond sharp grayscale changes in both directions.
- Large horizontal frequencies correspond to vertical lines.
Vertical grating
Double grating
Smooth rings
Margherita Hack

log amplitude of the spectrum
Einstein

log amplitude of the spectrum
What we are going to analyze

- 2D Fourier Transform of continuous signals (2D-CTFT)

\[ F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} \, dt, \quad f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t} \, d\omega \]

- 2D Fourier Transform of discrete space signals (2D-DTFT)

\[ F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}, \quad f[k] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\Omega)e^{j\Omega k} \, d\Omega \]

- 2D Discrete Fourier Transform (2D-DFT)

\[ F_r = \sum_{k=0}^{N_0-1} f[r]\Omega_0^{-jkr}k, \quad f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}, \Omega_0 = \frac{2\pi}{N_0} \]
2D Continuous Fourier Transform

• Continuous case \((x \text{ and } y \text{ are real})\) – 2D-CTFT (notation 1)

\[
\hat{f}(\omega_x, \omega_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-j(\omega_x x + \omega_y y)} \, dx \, dy
\]

\[
f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{f}(\omega_x, \omega_y) e^{j(\omega_x x + \omega_y y)} \, d\omega_x \, d\omega_y
\]

\[
\iint f(x, y) g^*(x, y) \, dx \, dy = \frac{1}{4\pi^2} \iint \hat{f}(\omega_x, \omega_y) \hat{g}^*(\omega_x, \omega_y) \, d\omega_x \, d\omega_y \quad \text{Parseval formula}
\]

\[
f = g \rightarrow \iint |f(x, y)|^2 \, dx \, dy = \frac{1}{4\pi^2} \iint |\hat{f}(\omega_x, \omega_y)|^2 \, d\omega_x \, d\omega_y \quad \text{Plancherel equality}
\]
2D Continuous Fourier Transform

- Continuous case ($x$ and $y$ are real) – 2D-CTFT

$$
\omega_x = 2\pi u \\
\omega_y = 2\pi v
$$

$$
\hat{f}(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-j2\pi(ux + vy)} dx dy
$$

$$
f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{f}(u, v) e^{j2\pi(ux + vy)} (2\pi)^2 dudv =
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{f}(u, v) e^{j2\pi(ux + vy)} (2\pi)^2 dudv
$$
2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

\[
\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy
\]

\[
f(x,y) = \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} \, du \, dv =
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(u,v)|^2 \, du \, dv
\]

Plancherel’s equality
The independent variable \((t,x,y)\) is discrete.

\[
F_r = \sum_{k=0}^{N_0-1} f[k] e^{-j r \Omega_0 k}
\]

\[
f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{j r \Omega_0 k}
\]

\[
\Omega_0 = \frac{2\pi}{N_0}
\]

\[
F[u,v] = \sum_{i=0}^{N_0-1} \sum_{k=0}^{N_0-1} f[i,k] e^{-j \Omega_0 (ui+vk)}
\]

\[
f_{N_0}[i,k] = \frac{1}{N_0^2} \sum_{u=0}^{N_0-1} \sum_{v=0}^{N_0-1} F[u,v] e^{j \Omega_0 (ui+vk)}
\]

\[
\Omega_0 = \frac{2\pi}{N_0}
\]
Delta

• Sampling property of the 2D-delta function (Dirac's delta)

\[
\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0) f(x, y) \, dx \, dy = f(x_0, y_0)
\]

• Transform of the delta function

\[
F\{\delta(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy = 1
\]

\[
F\{\delta(x-x_0, y-y_0)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-x_0, y-y_0)e^{-j2\pi(ux+vy)} \, dx \, dy = e^{-j2\pi(ux_0+vy_0)} \quad \text{shifting property}
\]
Constant functions

- Inverse transform of the impulse function

\[
F^{-1}\left\{\delta(u,v)\right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u,v)e^{j2\pi(ux+vy)}
dudv = e^{j2\pi(0x+v0)} = 1
\]

- Fourier Transform of the constant (=1 for all x and y)

\[
k(x, y) = 1 \quad \forall x, y
\]

\[
F\{k\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)}
dxdy = \delta(u,v)
\]
Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

\[ s(x, y) = \cos(2\pi fx) \]

\[
F \{\cos(2\pi fx)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(2\pi fx)e^{-j2\pi(ux+vy)} \, dxdy =
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{e^{j2\pi (fx)} + e^{-j2\pi (fx)}}{2} \right] e^{-j2\pi(ux+vy)} \, dxdy
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ e^{-j2\pi (u-f)x} + e^{-j2\pi (u+f)x} \right] e^{-j2\pi vy} \, dxdy =
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi vy} \, dy \int_{-\infty}^{\infty} \left[ e^{-j2\pi (u-f)x} + e^{-j2\pi (u+f)x} \right] dx = \frac{1}{2} \int_{-\infty}^{\infty} \left[ e^{-j2\pi (u-f)x} + e^{-j2\pi (u+f)x} \right] dx =
\]

\[
= \frac{1}{2} \left[ \delta(u - f) + \delta(u + f) \right]
\]
Vertical grating
Ex. 1
Ex. 3

Magnitudes
Examples
Properties

- **Linearity**: \( af(x, y) + bg(x, y) \Leftrightarrow aF(u, v) + bG(u, v) \)

- **Shifting**: \( f(x - x_0, y - y_0) \Leftrightarrow e^{-j2\pi(ux_0 + vy_0)}F(u, v) \)

- **Modulation**: \( e^{j2\pi(u_0x + v_0y)}f(x, y) \Leftrightarrow F(u - u_0, v - v_0) \)

- **Convolution**: \( f(x, y) * g(x, y) \Leftrightarrow F(u, v)G(u, v) \)

- **Multiplication**: \( f(x, y)g(x, y) \Leftrightarrow F(u, v) * G(u, v) \)

- **Separability**: \( f(x, y) = f(x)f(y) \Leftrightarrow F(u, v) = F(u)F(v) \)
1. Separability of the 2D Fourier transform
   - 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

\[
F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy =
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} e^{-j2\pi vy} \, dx \, dy = \int_{-\infty}^{\infty} e^{-j2\pi vy} \, dy \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} \, dx =
\]

\[
= \int_{-\infty}^{\infty} F(u,y) e^{-j2\pi vy} \, dy = F(u,v)
\]

2D DFT $\rightarrow$ 1D DFT along the rows $\rightarrow$ 1D DFT along the cols
Separability

• Separable functions can be written as \( f(x, y) = f(x)g(y) \)

2. The FT of a separable function is the product of the FTs of the two functions

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy = \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy} \, dx \, dy = \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy} \, dy \int_{-\infty}^{\infty} h(x)e^{-j2\pi ux} \, dx = \\
= H(u)G(v)
\]

\( f(x, y) = h(x)g(y) \Rightarrow F(u, v) = H(u)G(v) \)
2D Fourier Transform of a Discrete function

- Fourier Transform of a 2D *aperiodic* signal defined over a 2D discrete grid
  - The grid can be thought of as a 2D brush used for sampling the continuous signal with a given spatial resolution \((T_x, T_y)\)

1D

\[
F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}, \quad f[k] = \frac{1}{2\pi} \int F(\Omega)e^{j\Omega k} \, dt
\]

2D

\[
F(\Omega_x, \Omega_y) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1, k_2]e^{-j(k_1\Omega_x + k_2\Omega_y)}
\]

\[
f[k] = \frac{1}{4\pi^2} \int \int F(\Omega_x, \Omega_y)e^{j(k_1\Omega_x + k_2\Omega_y)} \, d\Omega_x \, d\Omega_y
\]

\(\Omega_x, \Omega_y\): normalized frequency
Unitary frequency notations

\[
\begin{align*}
\Omega_x &= 2\pi u \\
\Omega_y &= 2\pi v
\end{align*}
\]

\[
F(u, v) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1, k_2] e^{-j2\pi(k_1u+k_2v)}
\]

\[
f[k_1, k_2] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u, v) e^{-j2\pi(k_1u+k_2v)} dudv
\]

- The integration interval for the inverse transform has width=1 instead of 2\pi
  - It is quite common to choose

\[
-\frac{1}{2} \leq u, v < \frac{1}{2}
\]
**Properties**

- **Periodicity**: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - The period is 1 for the unitary frequency notations and 2π for normalized frequency notations.
  - Proof (referring to the firsts case)

\[
F(u+k, v+l) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi((u+k)m+(v+l)n)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}e^{-j2\pi km}e^{-j2\pi ln} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)} = F(u,v)
\]
Properties

• Linearity
• shifting
• modulation
• convolution
• multiplication
• separability

• energy conservation properties also exist for the 2D Fourier Transform of discrete signals.

• NOTE: in what follows, \((k_1, k_2)\) is replaced by \((m, n)\)
2D DTFT Properties

- **Linearity** \( af[m, n] + bg[m, n] \iff aF(u, v) + bG(u, v) \)
- **Shifting** \( f[m - m_0, n - n_0] \iff e^{-j2\pi(um_0 + vn_0)} F(u, v) \)
- **Modulation** \( e^{j2\pi(u_0m + v_0n)} f[m, n] \iff F(u - u_0, v - v_0) \)
- **Convolution** \( f[m, n] * g[m, n] \iff F(u, v)G(u, v) \)
- **Multiplication** \( f[m, n]g[m, n] \iff F(u, v) * G(u, v) \)
- **Separable functions** \( f[m, n] = f[m]f[n] \iff F(u, v) = F(u)F(v) \)
- **Energy conservation** \( \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f[m, n]|^2 = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |F(u, v)|^2 \, du \, dv \)
Impulse Train

- Define a \textit{comb} function (impulse train) as follows

\[
\text{comb}_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m - kM, n - lN]
\]

where \( M \) and \( N \) are integers

\[
\text{comb}_2[n]
\]
Appendix
2D-DTFT: delta

- Define Kronecker delta function

\[ \delta[m, n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases} \]

- DT Fourier Transform of the Kronecker delta function

\[ F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \delta[m, n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1 \]
2D DT Fourier Transform: constant

- Fourier Transform of 1

\[ f[k, l] = 1, \forall k, l \]

\[ F[u, v] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[ 1 e^{-j2\pi(uk+vl)} \right] = \]

\[ = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(u-k, v-l) \]

periodic with period 1 along \( u \) and \( v \)

To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.