UNIVERSITÀ di **VERONA** | Masters Programme in **Mathematics**

Written test in Optimization

Verona, 14th November 2017

Name and surname: ______________________ ID number: | | | | | | | | |

Exercise 1. Let Ω be a bounded open subset of $\mathbb{R}^2$. Consider the problem:

$$\inf_{u\in H_0^1(\Omega)}\int_\Omega \Big(5\,|\nabla u\,(x_1,x_2)|^2 - 2\partial_{x_2}u\,(x_1,x_2)\,\partial_{x_1}u\,(x_1,x_2) + \big((x_1^4+3x_2^2)\,u\,(x_1,x_2)-2\big)^2 + $$
$$+4[\partial_{x_1}u(x_1,x_2)]^2 + [\partial_{x_2}u(x_1,x_2)]^2\Big)\,dx_1\,dx_2.$$

(1) Prove that the problem admits a unique solution.
(2) State the problem in the form $\mathscr{F}(u)=F(u)+G\circ\Lambda(u)$, where $F:X\to]-\infty,+\infty]$, $G:Y\to]-\infty,+\infty]$ and $\Lambda:X\to Y$, carefully precising the function spaces X,Y and discussing the regularity properties of F,G,Λ.
(3) Write the dual problem and the extremality conditions, establish whether the dual problems admits a unique solution.
(4) Use the previous results to write a partial differential equations satisfied by the minimum.

Exercise 2. Let Ω be an open bounded subset of $\mathbb{R}^d$, $q\in H_0^1(\Omega;\mathbb{R})$ be fixed. Set:

$$\mathscr{C}:=\{v\in H_0^1(\Omega;\mathbb{R}):\,\|\nabla v-\nabla q\|_{L^2(\Omega;\mathbb{R}^d)}\le 1\}.$$

Consider the problem

$$\inf_{u\in\mathscr{C}}\int_\Omega \frac{|u(x)|^2}{2}\,dx.$$

(1) Prove that the problem admits a unique solution.
(2) Formulate the problem in the whole space in the form $\mathscr{F}(u)=F(u)+G\circ\Lambda(u)$, where $F:X\to]-\infty,+\infty]$, $G:Y\to]-\infty,+\infty]$, and $\Lambda:X\to Y$, carefully precising the functional spaces X,Y and discuting the regularity of F,G,Λ.
(3) Write the dual problem and the extremality relations. Establish if the dual problem admits an unique solution.

Exercise 3.

(1) Prove that the two marginals of a convex functions $\Phi:X\times Y\to\mathbb{R}\cup]-\infty,+\infty]$ are convex.
(2) Let Ω_1,Ω_2 be nonempty convex subsets of a Banach space X. We say that Ω_1,Ω_2 are an *extremal system* if for every $\varepsilon>0$ there exists $a\in X$, $\|a\|\le\varepsilon$ such that $(\Omega_1+a)\cap\Omega_2=\emptyset$. Prove that Ω_1,Ω_2 are an extremal system if and only if $0\notin\mathrm{int}(\Omega_1-\Omega_2)$ where $\Omega_1-\Omega_2:=\{x_1-x_2:\,x_i\in\Omega_i=1,2\}$.
(3) Let $f:\mathbb{R}^2\to\mathbb{R}\cup\{+\infty\}$ be defined as $f(x_1,x_2)=(3x_1+4x_2)^3$ if $3x_1+4x_2>0$ and $f(x_1,x_2)=+\infty$ if $3x_1+4x_2\le 0$. Prove that f is convex and compute f^* and f^{**}.
(4) Let C be a closed nonempty convex subset of $\mathbb{R}^d$ with $\mathrm{int}\,C\neq\emptyset$. Prove that $C=\overline{\mathrm{int}\,C}$.
(5) Discuss the continuity properties of convex functions defined on a Banach space, proving some relevant results.

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