2D Wavelets
Topics

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• Fast 2D DWT
• Lifting steps scheme
• JPEG2000

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• Human Visual System

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Separable Wavelet bases

To any wavelet orthonormal basis \( \{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2} \) of \( L^2(\mathbb{R}) \), one can associate a separable wavelet orthonormal basis of \( L^2(\mathbb{R}^2) \):

\[
\left\{ \psi_{j_1,n_1}(x_1) \psi_{j_2,n_2}(x_2) \right\}_{(j_1,j_2,n_1,n_2)\in\mathbb{Z}^4}
\]

- The functions \( \psi_{j_1,n_1}(x_1) \) and \( \psi_{j_1,n_1}(x_1) \) mix informations at two different scales along \( x_1 \) and \( x_2 \), which is something that we could want to avoid.

- Separable multiresolutions lead to another construction of separable wavelet bases with wavelets that are products of functions dilated at the same scale.
Separable multiresolutions

- The notion of resolution is formalized with orthogonal projections in spaces of various sizes.

- The approximation of an image $f(x_1, x_2)$ at the resolution $2^{-j}$ is defined as the orthogonal projection of $f$ on a space $V_{2^j}$ that is included in $L^2(\mathbb{R}^2)$.

- The space $V_{2^j}$ is the set of all approximations at the resolution $2^j$.
  - When the resolution decreases, the size of $V_{2^j}$ decreases as well.

- The formal definition of a multiresolution approximation $\{V_{2^j}\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ is a straightforward extension of Definition 7.1 that specifies multiresolutions of $L^2(\mathbb{R})$.
  - The same causality, completeness, and scaling properties must be satisfied.
Separable spaces and bases

- Tensor product
  - Used to extend spaces of 1D signals to spaces of multi-dimensional signals
  - A tensor product \( x_1 \otimes x_2 \) between vectors of two Hilbert spaces \( H_1 \) and \( H_2 \) satisfies the following properties

  **Linearity**
  \[ \forall \lambda \in C, \lambda (x_1 \otimes x_2) = (\lambda x_1) \otimes x_2 = x_1 \otimes (\lambda x_2) \]

  **Distributivity**
  \[ (x_1 + y_1) \otimes (x_2 + y_2) = (x_1 \otimes x_2) + (x_1 \otimes y_2) + (y_1 \otimes x_2) + (y_1 \otimes y_2) + \]

  - This tensor product yields a new Hilbert space \( H = H_1 \otimes H_2 \) including all the vectors of the form \( x_1 \otimes x_2 \) where \( x_1 \in H_1 \) and \( x_2 \in H_2 \) as well as a linear combination of such vectors
  - An inner product for \( H \) is derived as \( \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle_{H_1} \langle x_2, y_2 \rangle_{H_2} \)
Separable bases

• Theorem A.3 Let $H = H_1 \otimes H_2$. If $\left\{e_n^1\right\}_{n \in N}$ and $\left\{e_n^2\right\}_{n \in N}$ are Riesz bases of $H_1$ and $H_2$, respectively, then $\left\{e_n^1 \otimes e_m^2\right\}_{n,m \in N^2}$ is a Riesz basis for $H$. If the two bases are orthonormal then the tensor product basis is also orthonormal.

→ To any wavelet orthonormal basis one can associate a separable wavelet orthonormal basis of $L^2(\mathbb{R}^2)$  

$\left\{\psi_{j,n}(x), \psi_{l,m}(y)\right\}_{(j,n,l,m) \in \mathbb{Z}^4}$

However, wavelets $\psi_{j,n}(x)$ and $\psi_{l,m}(x)$ mix the information at two different scales along $x$ and $y$, which often we want to avoid.
Separable Wavelet bases

• Separable multiresolutions lead to another construction of separable wavelet bases whose elements are products of functions dilated at the same scale.

• We consider the particular case of separable multiresolutions

• A separable 2D multiresolution is composed of the tensor product spaces

\[ V^2_j = V_j \otimes V_j \]

• \( V^2_j \) is the space of finite energy functions \( f(x,y) \) that are linear expansions of separable functions

\[ f(x,y) = \sum_n a[n] f_n(x) g_n(y) \quad f_n \in V_j \quad g_n \in V_j \]

• If \( \{V_j\}_{j \in \mathbb{Z}} \) is a multiresolution approximation of \( L^2(\mathbb{R}) \), then \( \{V^2_j\}_{j \in \mathbb{Z}} \) is a multiresolution approximation of \( L^2(\mathbb{R}^2) \).
It is possible to prove (Theorem A.3) that
\[
\begin{bmatrix}
\varphi_{j,n,m}(x,y) = \varphi_{j,n}(x)\varphi_{j,m}(y) = \frac{1}{2^j} \varphi\left(\frac{x - 2^j n}{2^j}\right)\varphi\left(\frac{y - 2^j m}{2^j}\right)
\end{bmatrix}_{(n,m) \in \mathbb{Z}^2}
\]
is an orthonormal basis of $V_j^2$.

A 2D wavelet basis is constructed with separable products of a scaling function and a wavelet.
Examples

EXAMPLE 7.13: Piecewise Constant Approximation

Let $\mathbf{v}_j$ be the approximation space of functions that are constant on $[2^j m, 2^j (m + 1)]$ for any $m \in \mathbb{Z}$. The tensor product defines a two-dimensional piecewise constant approximation. The space $\mathbf{v}_j^2$ is the set of functions that are constant on any square $[2^j n_1, 2^j (n_1 + 1)] \times [2^j n_2, 2^j (n_2 + 1)]$, for $(n_1, n_2) \in \mathbb{Z}^2$. The two-dimensional scaling function is

$$\phi^2(x) = \phi(x_1) \phi(x_2) = \begin{cases} 1 & \text{if } 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 7.14: Shannon Approximation

Let $\mathbf{v}_j$ be the space of functions with Fourier transforms that have a support included in $[-2^{-j} \pi, 2^{-j} \pi]$. Space $\mathbf{v}_j^2$ is the set of functions the two-dimensional Fourier transforms of which have a support included in the low-frequency square $[-2^{-j} \pi, 2^{-j} \pi] \times [-2^{-j} \pi, 2^{-j} \pi]$. The two-dimensional scaling function is a perfect two-dimensional low-pass filter the Fourier transform of which is

$$\hat{\phi}(\omega_1) \hat{\phi}(\omega_2) = \begin{cases} 1 & \text{if } |\omega_1| \leq 2^{-j} \pi \text{ and } |\omega_2| \leq 2^{-j} \pi \\ 0 & \text{otherwise.} \end{cases}$$
Separable wavelet bases

- A separable wavelet orthonormal basis of $L^2(R^2)$ is constructed with separable products of a scaling function and a wavelet.
- The scaling function is associated to a one-dimensional multiresolution approximation $\{V_j\}_{j \in \mathbb{Z}}$.
- Let $\{V^2_j\}_{j \in \mathbb{Z}}$ be the separable two-dimensional multiresolution defined by
  \[
  V^2_j = V_j \otimes V_j
  \]
- Let $W^2_j$ be the detail space equal to the orthogonal complement of the lower-resolution approximation space $V^2_j$ in $V^2_{j-1}$:
  \[
  V^2_{j-1} = V^2_j \oplus W^2_j
  \]
- To construct a wavelet orthonormal basis of $L^2(R^2)$, Theorem 7.25 builds a wavelet basis of each detail space $W^2_j$. 

Theorem 7.25

Let \( \phi \) be a scaling function and \( \psi \) be the corresponding wavelet generating an orthonormal basis of \( L^2(\mathbb{R}) \). We define three wavelets

\[
\psi^1(x, y) = \phi(x)\psi(y) \\
\psi^2(x, y) = \psi(x)\phi(y) \\
\psi^3(x, y) = \psi(x)\psi(y)
\]

and denote for \( 1 \leq k \leq 3 \)

\[
\psi_{j,n,m}^k(x, y) = \frac{1}{2^j} \psi^k\left(\frac{x - 2^j n}{2^j}, \frac{y - 2^j m}{2^j}\right)
\]

The wavelet family

\[
\left\{\psi_{j,n,m}^1(x, y), \psi_{j,n,m}^2(x, y), \psi_{j,n,m}^3(x, y)\right\}_{(n,m) \in \mathbb{Z}^2}
\]

is an orthonormal basis of \( W_j^2 \) and

\[
\left\{\psi_{j,n,m}^1(x, y), \psi_{j,n,m}^2(x, y), \psi_{j,n,m}^3(x, y)\right\}_{(j,n,m) \in \mathbb{Z}^3}
\]

is an orthonormal basis of \( L^2(\mathbb{R}^2) \)

On the same line, one can define **biorthogonal** 2D bases.
Separable wavelet bases

- The three wavelets extract image details at different scales and in different directions.
- Over positive frequencies, \( \hat{\phi}(\omega) \) and \( \hat{\psi}(\omega) \) have an energy mainly concentrated, respectively, on \([0, \pi]\) and \([\pi, 2\pi]\).
- The separable wavelet expressions imply that

\[
\begin{align*}
\hat{\psi}^1(\omega_x, \omega_y) &= \hat{\phi}(\omega_x)\hat{\psi}(\omega_y) \\
\hat{\psi}^2(\omega_x, \omega_y) &= \hat{\psi}(\omega_x)\hat{\phi}(\omega_y) \\
\hat{\psi}^3(\omega_x, \omega_y) &= \hat{\psi}(\omega_x)\hat{\psi}(\omega_y)
\end{align*}
\]
Bi-dimensional wavelets

\[ \varphi(x, y) = \varphi(x)\varphi(y) \]
\[ \psi^1(x, y) = \varphi(x)\psi(y) \]
\[ \psi^2(x, y) = \psi(x)\varphi(y) \]
\[ \psi^3(x, y) = \psi(x)\psi(y) \]

\[ \frac{1}{\sqrt{a_1 a_2}} \psi \left( \frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2} \right) \text{ where } (x = (x_1 x_2) \in \mathbb{R}^2) \]
Example: Shannon wavelets

Example 7.16

For a Shannon multiresolution approximation, the resulting two-dimensional wavelet basis paves the two-dimensional Fourier plane \((\omega_1, \omega_2)\) with dilated rectangles. The Fourier transforms \(\hat{\phi}\) and \(\hat{\psi}\) are the indicator functions of \([-\pi, \pi]\) and \([-2\pi, -\pi] \cup [\pi, 2\pi]\), respectively. The separable space \(V^2_f\) contains functions with a two-dimensional Fourier transform support included in the low-frequency square \([-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]\). This corresponds to the support of \(\hat{\phi}^2_{j,n}\) indicated in Figure 7.23.

The detail space \(W^2_f\) is the orthogonal complement of \(V^2_f\) in \(V^1_{f-1}\) and thus includes functions with Fourier transforms supported in the frequency annulus between the two squares \([-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]\) and \([-2^{-j+1}\pi, 2^{-j+1}\pi] \times [-2^{-j+1}\pi, 2^{-j+1}\pi]\).
<table>
<thead>
<tr>
<th>$a_{L+3}$</th>
<th>$d_{L+3}^2$</th>
<th>$d_{L+2}^2$</th>
<th>$d_{L+1}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{L+3}^1$</td>
<td>$d_{L+3}^3$</td>
<td>$d_{L+2}^3$</td>
<td>$d_{L+1}^3$</td>
</tr>
</tbody>
</table>

$\text{Diagram with squares and lines}$
Biorthogonal separable wavelets

Let $\phi, \psi, \tilde{\phi}$ and $\tilde{\psi}$ be a two dual pairs of scaling functions and wavelets that generate a biorthogonal wavelet basis of $L^2(\mathbb{R})$.

The dual wavelets of $\psi^1, \psi^2$ and $\psi^3$ are

$\psi^1(x, y) = \tilde{\phi}(x) \tilde{\psi}(y)$
$\psi^2(x, y) = \tilde{\psi}(x) \tilde{\phi}(y)$
$\psi^3(x, y) = \tilde{\psi}(x) \tilde{\psi}(y)$

One can verify that

$\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{j,n \in \mathbb{Z}^3}$

and

$\{\tilde{\psi}_{j,n}^1, \tilde{\psi}_{j,n}^2, \tilde{\psi}_{j,n}^3\}_{j,n \in \mathbb{Z}^3}$

are biorthogonal Riesz basis of $L^2(\mathbb{R}^2)$.
Fast 2D Wavelet Transform

\[ a_j[n,m] = \langle f, \varphi_{j,n,m} \rangle \quad \text{Approximation at scale } j \]

\[ d^k_j[n,m] = \langle f, \psi^k_{j,n,m} \rangle \quad \text{Details at scale } j \]

\[ k = 1, 2, 3 \]

\[ [a_J, \{d^1_j, d^2_j, d^3_j\}_{1 \leq j \leq J}] \quad \text{Wavelet representation} \]

Analysis

\[ a_{j+1}[n,m] = a_j * h \overline{h}[2n,2m] \]

\[ d^1_{j+1}[n,m] = a_j * \overline{h}g[2n,2m] \]

\[ d^2_{j+1}[n,m] = a_j * \overline{g}h[2n,2m] \]

\[ d^3_{j+1}[n,m] = a_j * \overline{g}g[2n,2m] \]

Synthesis

\[ a_j[n,m] = \tilde{a}_{j+1} * hh[n,m] + \tilde{d}^1_{j+1} * hg[n,m] + \tilde{d}^2_{j+1} * gh[n,m] + \tilde{d}^3_{j+1} * gg[n,m] \]
Fast 2D DWT

(a)

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Finite images and complexity

• When aL is a finite image of $N=N_1 \times N_2$ pixels, we face boundary problems when computing the convolutions
  – A suitable processing at boundaries must be chosen

• For square images with $N_1N_2$, the resulting images $a_j$ and $dk\,j$ have $22j$ samples. Thus, the images of the wavelet representation include a total of $N$ samples.
  – If $h$ and $g$ have size $K$, one can verify that $2K2^{-2(j-1)}$ multiplications and additions are needed to compute the four convolutions
  – Thus, the wavelet representation is calculated with fewer than $8/3\,KN$ operations.
  – The reconstruction of $a_L$ by factoring the reconstruction equation requires the same number of operations.
Matlab notations

Decomposition Step

where

- $\begin{bmatrix} 2 \downarrow & 1 \end{bmatrix}$: Downsample columns: keep the even indexed columns.
- $\begin{bmatrix} 1 \downarrow & 2 \end{bmatrix}$: Downsample rows: keep the even indexed rows.
- $\begin{bmatrix} X \end{bmatrix}$: Convolve with filter $X$ the rows of the entry.
- $\begin{bmatrix} X \end{bmatrix}$: Convolve with filter $X$ the columns of the entry.

Initialization

$CA_0 = s$ for the decomposition initialization.
Matlab notations

Reconstruction Step

Two-Dimensional IDWT

$cA_{j+1}$

$cD_{j+1}^{(h)}$

$cD_{j+1}^{(v)}$

$cD_{j+1}^{(d)}$

where

- **Upsample columns**: insert zeros at odd-indexed columns.
- **Upsample rows**: insert zeros at odd-indexed rows.
- **Convolve with filter X the rows of the entry**.
- **Convolve with filter X the columns of the entry**.
Example
Example
Subband structure for images

$S \rightarrow cA_1 \rightarrow cA_2 \rightarrow cD_{1(h)} \rightarrow cD_{2(h)} \rightarrow cD_{2(v)} \rightarrow cD_{2(d)} \rightarrow cD_{1(h)} \rightarrow cD_{1(v)} \rightarrow cD_{1(d)}$