BOOLEAN METHODS

Stanford University

Boolean methods

- Exploit Boolean properties.
  - *Don't care* conditions.

- Minimization of the local functions.

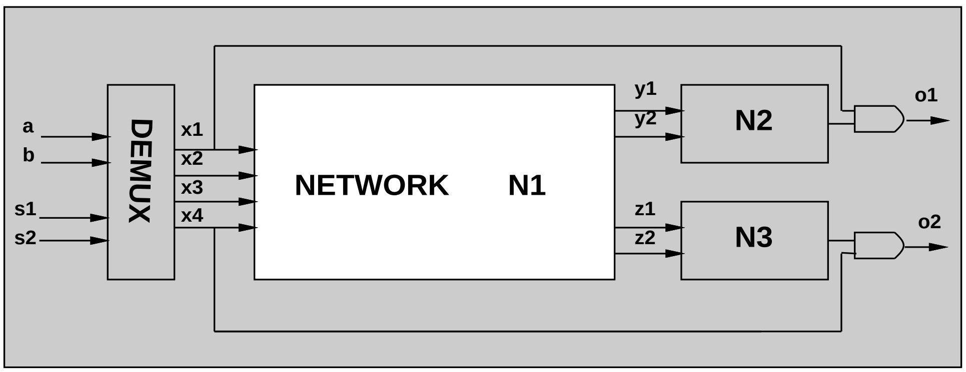
- Slower algorithms, better quality results.

External *don't care* conditions

- *Controllability don't care* set CDC_{in}:
  - Input patterns never produced by the environment at the network's input.

- *Observability don't care* set ODC_{out}:
  - Input patterns representing conditions when an output is not observed by the environment.
  - Relative to each output.
  - Vector notation used: $\mathbf{ODC}_{out}$.

Example



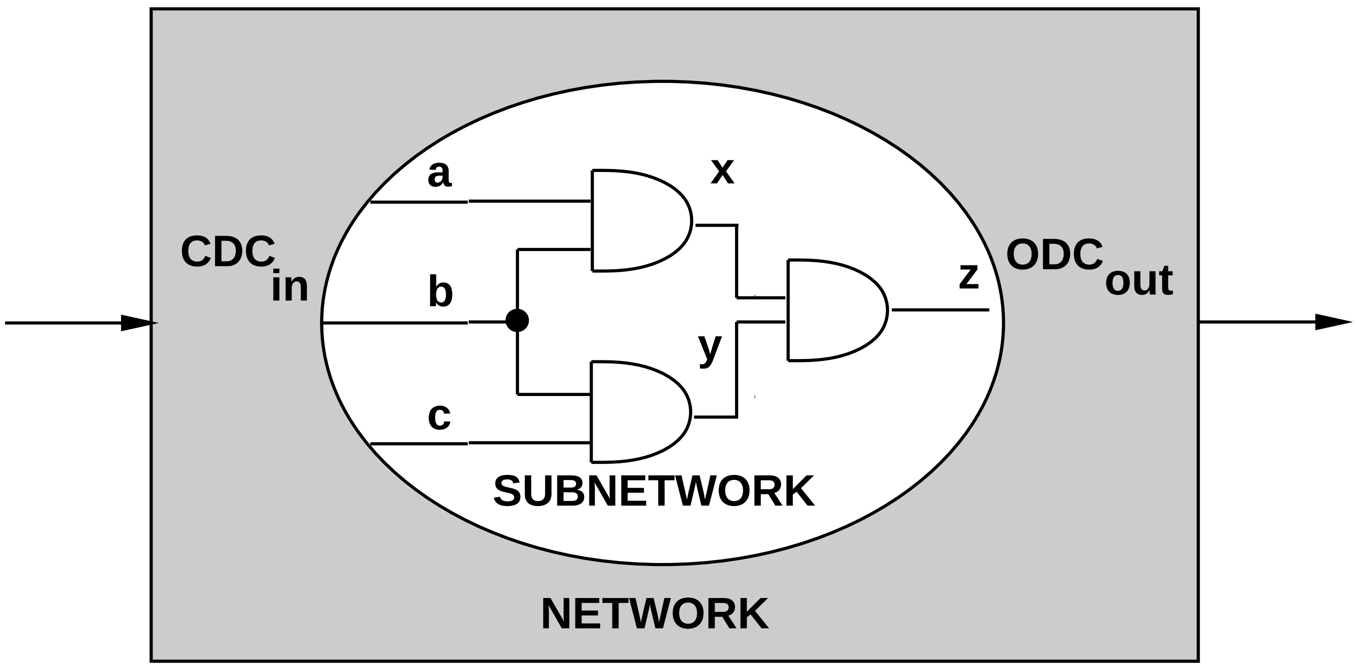
- Inputs driven by a de-multiplexer.
- $CDC_{in} = x_1'x_2'x_3'x_4' + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.
- Outputs observed when $\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{1}$

$$\mathbf{ODC}_{out} = \begin{bmatrix} x_1' \\ x_1' \\ x_4' \\ x_4' \end{bmatrix}$$

Example
overall external *don't care* set

$$\mathbf{DC}_{ext} = \mathbf{CDC}_{in} + \mathbf{ODC}_{out} = \begin{bmatrix} x_1' + x_2 + x_3 + x_4 \\ x_1' + x_2 + x_3 + x_4 \\ x_4' + x_2 + x_3 + x_1 \\ x_4' + x_2 + x_3 + x_1 \end{bmatrix}$$

Internal *don't care* **conditions**



Internal *don't care* **conditions**

- Induced by the network structure.

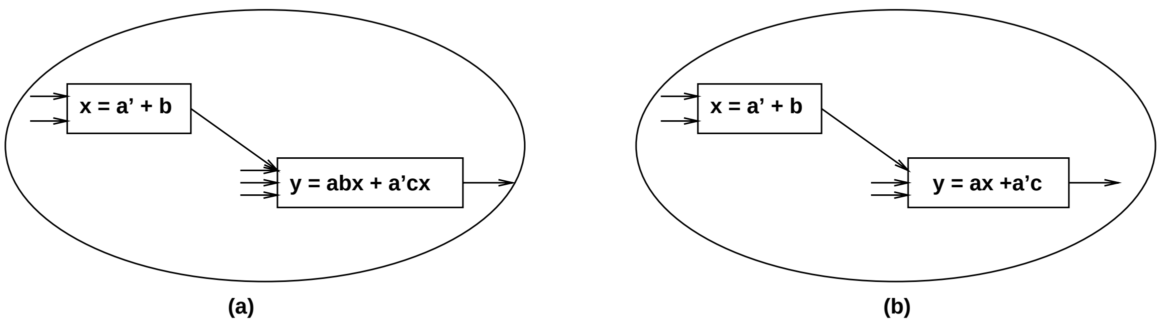
- *Controllability don't care* conditions:

  – Patterns never produced at the inputs of a subnetwork.

- *Observability don't care* conditions:

  – Patterns such that the outputs of a subnetwork are not observed.

Example



- CDC of v_y includes $ab'x + a'x'$.

- Minimize f_y to obtain: $\widetilde{f_y} = ax + a'c$.

Satisfiability *don't care* **conditions**

- Invariant of the network:
  - $x = f_x \rightarrow x \neq f_x \subseteq SDC$.

- $SDC \ = \ \displaystyle\sum_{v_x \in V^G} \ x \oplus f_x$

- Useful to compute controllability *don't cares* .

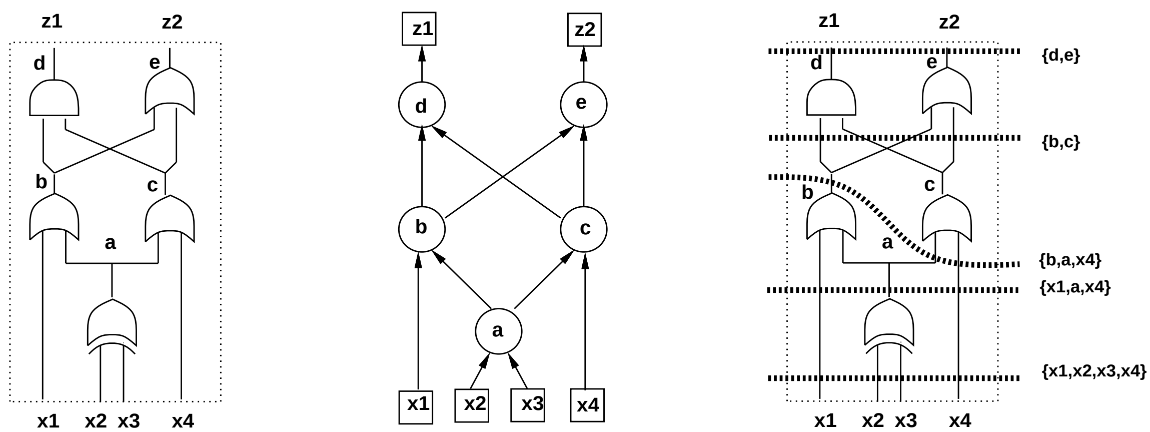
CDC computation

- Network traversal algorithm:
  - Consider different *cuts* moving from input to output.
- Initial CDC is CDC_{in}.
- Move *cut* forward.
  - Consider SDC contributions of predecessors.
  - Remove unneded variables by *consensus*.

CDC computation

CONTROLLABILITY($G_n(V, E)$, CDC_{in}) {
    $C = V^I$;
    $CDC_{cut} = CDC_{in}$;
    foreach vertex $v_x \in V$ in topological order {
        $C = C \cup v_x$;
        $CDC_{cut} = CDC_{cut} + f_x \oplus x$;
        $D = \{v \in C$ s.t. all dir. succ. of v are in $C\}$
        foreach vertex $v_y \in D$
            $CDC_{cut} = \mathcal{C}_y(CDC_{cut})$;
        $C = C \; - \; D$;
    };
    $CDC_{out} = CDC_{cut}$;
}

Example


z1
z2
d
e
b
c
a
x1
x2
x3
x4
z1
z2
d
e
b
c
a
x1
x2
x3
x4
z1
z2
d
e
b
c
a
x1
x2
x3
x4
{d,e}
{b,c}
{b,a,x4}
{x1,a,x4}
{x1,x2,x3,x4}

(a) (b) (c)

Example

- Assume $CDC_{in} = x_1' x_4'$.
- Select vertex v_a:
  - Contribution to CDC_{cut}: $a \oplus (x_2 \oplus x_3)$.
  - Drop variables $D = \{x_2, x_3\}$ by consensus:
  - $CDC_{cut} = x_1' x_4'$.
- Select vertex v_b:
  - Contribution to CDC_{cut}: $b \oplus (x_1 + a)$.
    * $CDC_{cut} = x_1' x_4' + b \oplus (x_1 + a)$.
  - Drop variable x_1 by consensus:
    * $CDC_{cut} = b' x_4' + b' a$.
- ...
- $CDC_{out} = e' = z_2'$.

CDC computation
by image computation

- Network behavior at *cut*: **f**.

- CDC_{cut} is just the complement of the *image* of $(CDC_{in})'$ with respect to **f**.

- CDC_{cut} is just the complement of the *range* of **f** when $CDC_{in} = \emptyset$.

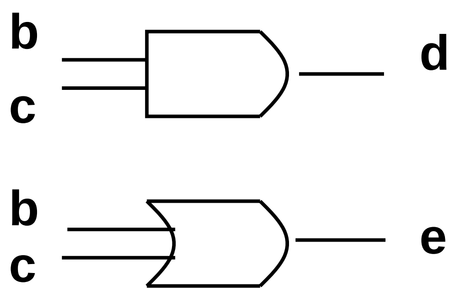
- Range can be computed recursively.

  – Terminal case: scalar function.

    * Range of $y = f(\mathbf{x})$ is $y + y'$ (any value) unless f (or f') is a tautology and the range is y (or y').

Example

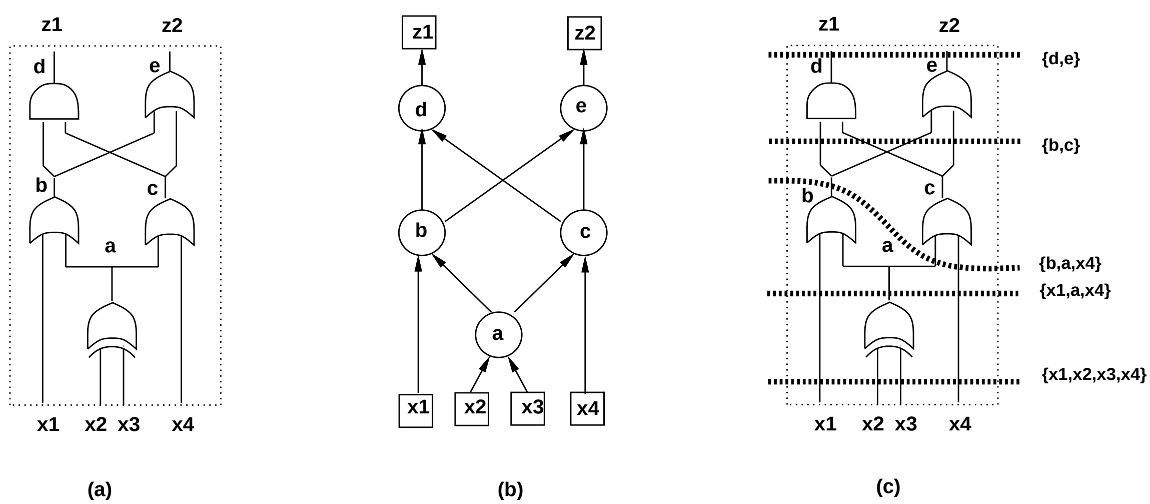
RANGE VECTORS



d	0	0	1
e	0	1	1

- $range(\mathbf{f}) = d\ range((b+c)|_{d=bc=1}) +$
  $+ d'\ range((b+c)|_{d=bc=0})$
- When $d = 1$, then $bc = 1 \rightarrow b + c = 1$ is TAUTOLOGY.
- If I choose 1 as top entry in output vector:
  - the bottom entry is also 1.
  - $\begin{bmatrix} 1 \\ ? \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- When $d = 0$, then $bc = 0 \rightarrow b + c = \{0, 1\}$.
- If I choose 0 as top entry in output vector:
  - the bottom entry can be 0 or 1.
- $range(\mathbf{f}) = de + d'(e + e') = de + d' = d' + e$

Example



$$\mathbf{f} = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix} = \begin{bmatrix} (x_1 + a)(x_4 + a) \\ (x_1 + a) + (x_4 + a) \end{bmatrix} = \begin{bmatrix} x_1 x_4 + a \\ x_1 + x_4 + a \end{bmatrix}$$

Example

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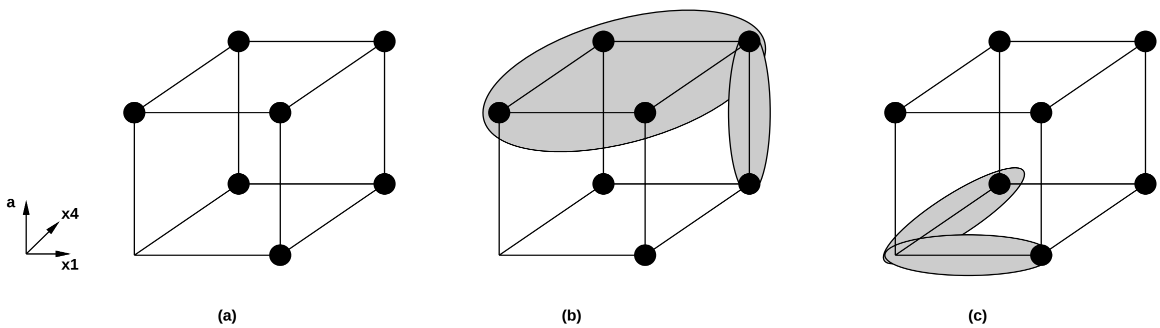

$$
\begin{aligned}
range(\mathbf{f}) &= \\
&= d\ range(f^2|_{(x_1x_4+a)=1}) + \\
&\quad d'\ range(f^2|_{(x_1x_4+a)=0}) \\
&= d\ range(x_1 + x_4 + a|_{(x_1x_4+a)=1}) + \\
&\quad d'\ range(x_1 + x_4 + a|_{(x_1x_4+a)=0}) \\
&= d\ range(1) + d'\ range(a'(x_1 \oplus x_4)) \\
&= de + d'(e + e') \\
&= e + d'
\end{aligned}
$$

- $CDC_{out} = (e + d')' = de' = z_1 z_2'$.

Perturbation method

- Modify network by adding an extra input δ.
- Extra input can flip polarity of a signal x.
- Replace local function f_x by $f_x \oplus \delta$.
- Perturbed terminal behavior: $\mathbf{f}^x(\delta)$.

Example

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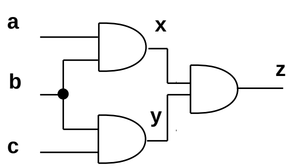
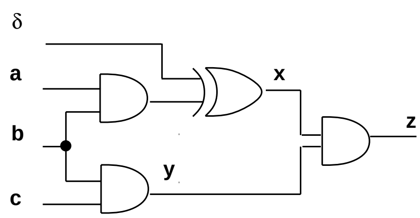
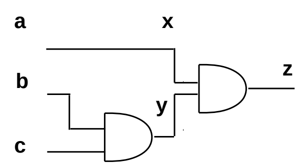


(a)



(b)



(c)

Observability *don't care* **conditions**

- Conditions under which a change in polarity of a signal x is not perceived at the outputs.

- Complement of the Boolean difference:

  – $\partial f / \partial x \; = \; f|_{x=1} \oplus f|_{x=0}$.

- Equivalence of perturbed function: $\mathbf{f}^x(0) \, \overline{\oplus} \, \mathbf{f}^x(1)$.

Observability *don't care* **computation**

- Problem:
  - Outputs are not expressed as function of all variables.
  - If network is flattened to obtain **f**, it may explode in size.

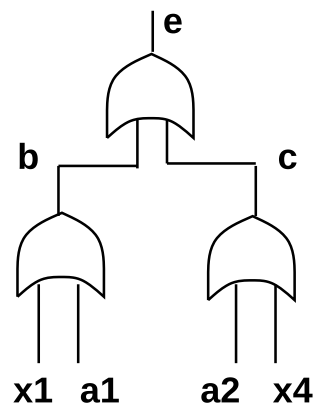
- Requirement:
  - Local rules for ODC computation.
  - Network traversal.

Single-output network with tree structure

- Traverse network tree.
- At root:
  - ODC_{out} is given.
- At internal vertices:
  - $ODC_x = (\partial f_y / \partial x)' + ODC_y$

Example

$$
\begin{aligned}
e &= b + c \\
b &= x_1 + a_1 \\
c &= x_4 + a_2
\end{aligned}
$$



- Assume $ODC_{out} = ODC_e = 0$.

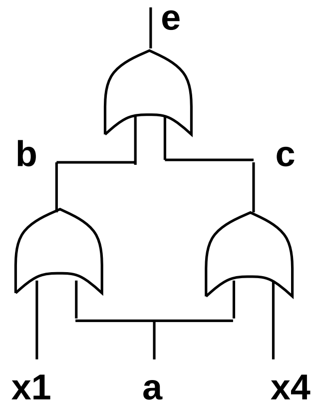
- $ODC_b = (\partial f_e / \partial b)' = (b + c)|_{b=1} \overline{\oplus} (b + c)|_{b=0} = c$.

- $ODC_c = (\partial f_e / \partial c)' = b$.

- $ODC_{x_1} = ODC_b + (\partial f_b / \partial x_1)' = c + a_1$.

- ...

General networks

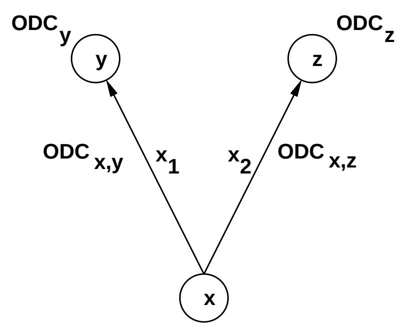


- Fanout reconvergence.

- For each vertex with two (or more) fanout stems:
  - The contribution of the ODC along the stems cannot be added *tout court*.
  - Interplay of different paths.

- More elaborate analysis.

Two-way fanout stem



- Compute ODC sets associated with edges.

- Combine ODCs at vertex.

- Formula derivation:

  – Assume two equal perturbations on the edges.

  – $\mathbf{ODC}_x = \mathbf{f}^{x_1,x_2}(1,1) \ \overline{\oplus} \ \mathbf{f}^{x_1,x_2}(0,0)$

ODC formula derivation

$$\begin{aligned}
\mathbf{ODC}_x &= \mathbf{f}^{x_1,x_2}(1,1) \;\overline{\oplus}\; \mathbf{f}^{x_1,x_2}(0,0) \\
&= \mathbf{f}^{x_1,x_2}(1,1) \;\overline{\oplus}\; \mathbf{f}^{x_1,x_2}(0,0) \\
&\quad \overline{\oplus}\; (\mathbf{f}^{x_1,x_2}(0,1) \;\overline{\oplus}\; \mathbf{f}^{x_1,x_2}(0,1)) \\
&= (\mathbf{f}^{x_1,x_2}(1,1) \;\overline{\oplus}\; \mathbf{f}^{x_1,x_2}(0,1)) \\
&\quad \overline{\oplus}\; (\mathbf{f}^{x_1,x_2}(0,1) \;\overline{\oplus}\; \mathbf{f}^{x_1,x_2}(0,0)) \\
&= \mathbf{ODC}_{x,y}|_{\delta_2=1} \;\overline{\oplus}\; \mathbf{ODC}_{x,z}|_{\delta_1=0} \\
&= \mathbf{ODC}_{x,y}|_{x_2=x'} \;\overline{\oplus}\; \mathbf{ODC}_{x,z}|_{x_1=x} \\
&= \mathbf{ODC}_{x,y}|_{x=x'} \;\overline{\oplus}\; \mathbf{ODC}_{x,z}
\end{aligned}$$

- Because $x = x_1 = x_2$.

Multi-way stems
Theorem

- Let $v_x \in V$ be any internal or input vertex.

- Let $\{x_i, i = 1, 2, \ldots, p\}$ be the edge vars corresponding to $\{(x, y_i)\ ;\ i = 1, 2, \ldots, p\}$.

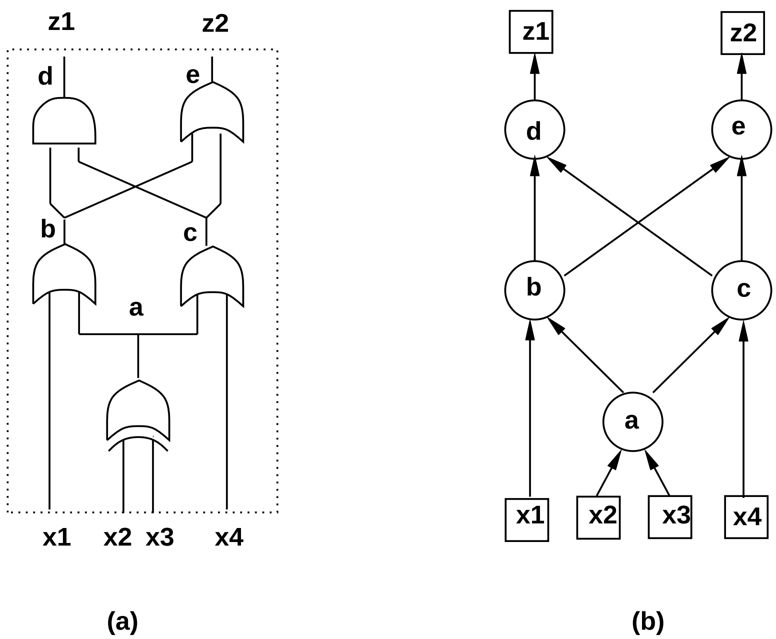
- Let $\mathbf{ODC}_{x,y_i}$, $i = 1, 2, \ldots, p$ the edge ODCs.

- $\mathbf{ODC}_x = \overline{\bigoplus}_{i=1}^{p} \mathbf{ODC}_{x,y_i}|_{x_{i+1}=\cdots=x_p\ =x'}$

Observability *don't care* **algorithm**

OBSERVABILITY($G_n(V, E)$, $\mathbf{ODC}_{out}$) {
  foreach vertex $v_x \in V$ in reverse topological order {
    for ($i = 1$ to p)
      $\mathbf{ODC}_{x,y_i} = (\partial f_{y_i}/\partial x)'\mathbf{1} + \mathbf{ODC}_{y_i}$;
    $\mathbf{ODC}_x = \overline{\bigoplus}_{i=1}^{p} \mathbf{ODC}_{x,y_i}|_{x_{i+1}=\cdots=x_p\ =x'}$;
  }
}

Example



$$\mathbf{ODC}_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;;\mathbf{ODC}_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\mathbf{ODC}_c = \begin{pmatrix} b' \\ b \end{pmatrix} \;;\mathbf{ODC}_b = \begin{pmatrix} c' \\ c \end{pmatrix}$$

$$\mathbf{ODC}_{a,b} = \begin{pmatrix} c' + x_1 \\ c + x_1 \end{pmatrix} = \begin{pmatrix} a'x_4' + x_1 \\ a + x_4 + x_1 \end{pmatrix}$$

$$\mathbf{ODC}_{a,c} = \begin{pmatrix} b' + x_4 \\ b + x_4 \end{pmatrix} = \begin{pmatrix} a'x_1' + x_4 \\ a + x_1 + x_4 \end{pmatrix}$$

$$\mathbf{ODC}_a = \mathbf{ODC}_{a,b}|_{a=a'} \overline{\oplus} \mathbf{ODC}_{a,c} = \begin{pmatrix} ax_4' + x_1 \\ a' + x_4 + x_1 \end{pmatrix} \overline{\oplus} \begin{pmatrix} a'x_1' + x_4 \\ a + x_1 + x_4 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1x_4 \\ x_1 + x_4 \end{pmatrix}$$

Transformations with *don't cares*

- Boolean simplification:
  - Use standard minimizer (Espresso).
  - Minimize the number of literals.

- Boolean substitution:
  - Simplify a function by adding an extra input.
  - Equivalent to simplification
    with global *don't care* conditions.

Example
Boolean substitution

- Substitute $q = a + cd$ into $f_h = a + bcd + e$ to get $f_h = a + bq + e$.

- SDC set: $q \oplus (a + cd) = q'a + q'cd + qa'(cd)'$.

- Simplify $f_h = a+bcd+e$ with $q'a+q'cd+qa'(cd)'$ as *don't care* .

- Simplification yields $f_h = a + bq + e$.

- One literal less by changing the support of f_h.

Single-vertex optimization

$SIMPLIFY\_SV(\ G_n(V,E)\ )\{$
  repeat $\{$
    $v_x =$ selected vertex ;
    Compute the local *don't care* set DC_x;
    Optimize the function f_x ;
  $\}$**until** (no more reduction is possible)
$\}$

Optimization and perturbations

- Replace f_x by g_x.

- Perturbation $\delta_x = f_x \oplus g_x$.

- Condition for feasible replacement:

  – Perturbation bounded by local *don't care* set

  – $\delta_x \subseteq \mathbf{DC}_{ext} + \mathbf{ODC}_x$

  – If x not a primary input
  consider also CDC set.

Example

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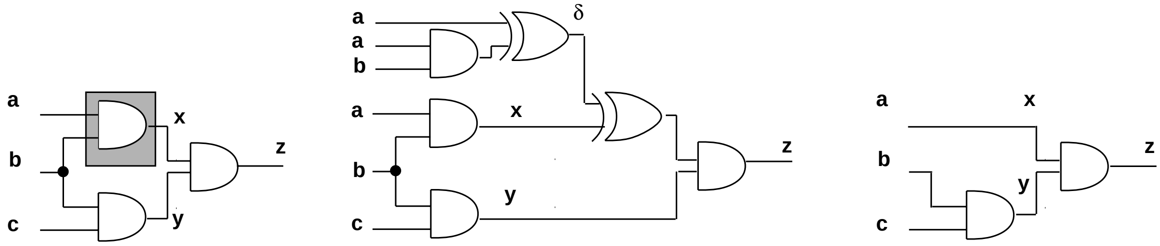


(a) (b) (c)

- No external *don't care* set.

- Replace AND by wire: $g_x = a$

- Analysis:

  - $\delta = f_x \oplus g_x \; = \; ab \oplus a \; = \; ab'$.
  - $ODC_x = y' = b' + c'$.
  - $\delta = ab' \subseteq DC_x = b' + c' \Rightarrow$ feasible!

Degrees of freedom

- Fully represented by *don't care* conditions:
  - External *don't cares* .
  - Internal observability and controllability.
- *Don't cares* represent an *upper bound* on the perturbation.
- Approximations:
  - Use smaller *don't care* sets to speed-up the computation.

Multiple-vertex optimization

- Simplify more than one local function at a time.

- Potentially better (more general) approach.

- Analysis:

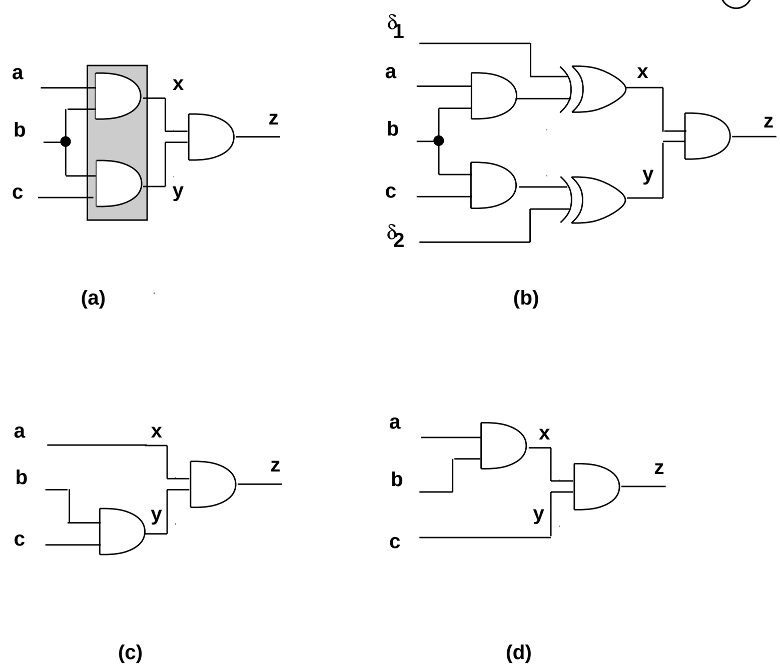
  - Multiple perturbations.

- Condition for feasible replacement:

  - *Upper and lower bounds* on the perturbation.

  - Boolean relation model.

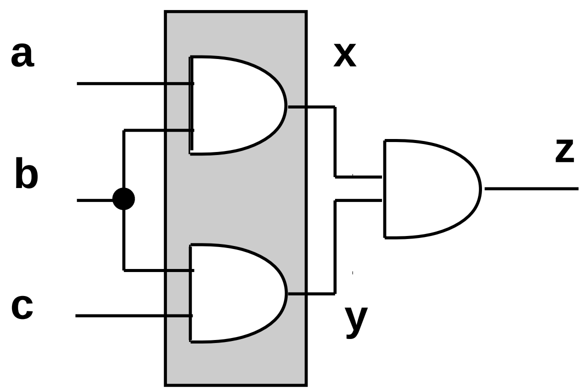
Example



- The two perturbations are related.

- Cannot change simultaneously:
  - $ab \to a$.
  - $cb \to c$.

Multiple-vertex optimization
Boolean relation model



a	b	c	x, y
0	0	0	{ 00, 01, 10 }
0	0	1	{ 00, 01, 10 }
0	1	0	{ 00, 01, 10 }
0	1	1	{ 00, 01, 10 }
1	0	0	{ 00, 01, 10 }
1	0	1	{ 00, 01, 10 }
1	1	0	{ 00, 01, 10 }
1	1	1	{ 11 }

Multiple-vertex optimization
Boolean relation model

- Compute Boolean relation:
  - Flatten the network. Analyze patterns.
  - Derive equivalence relation from ODCs.
- Use relation minimizer.
- Example of minimum function:

a	b	c	x, y
1	$*$	$*$	10
$*$	1	1	01

Multiple-vertex optimization
Boolean relation model

$SIMPLIFY\_MVR(\ G_n(V,E)\ )\{$
repeat $\{$
$U =$ selected vertex subset;
foreach vertex $v_x \in U$
Compute OCD_x;
Determine the equiv. classes of the Boolean relation of the subnetwork induced by U;
Find an optimal function compatible with the relation using a relation minimizer;
$\}$**until** (no more reduction is possible);
$\}$

Multiple-vertex optimization compatible *don't cares*

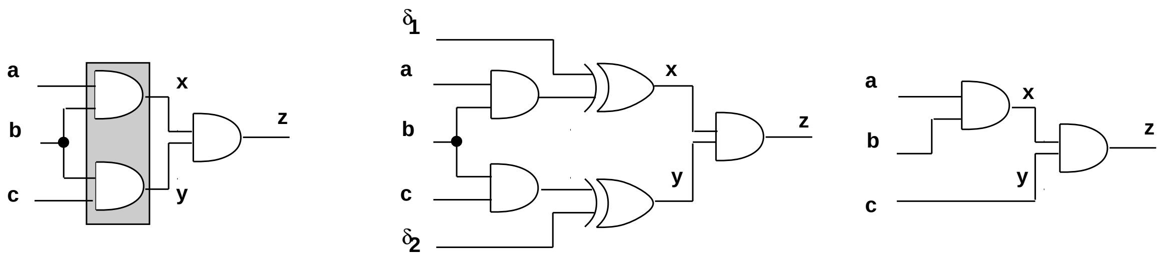
- Determine *compatible don't cares* :
  - CODCs: subset of ODCs.
  - Decouple dependencies.
  - Reduced degrees of freedom.

- Using compatible ODCs, only *upper bounds* on the perturbation need to be satisfied.

Example
two perturbations

- First vertex:
  - CODC equal to its ODC set.
  - $CODC_{x_1} = ODC_{x_1}$.
- The second vertex:
  - CODC smaller than its ODC to be safe enough to allow transformations permitted by the first ODC.
  - $CODC_{x_2} = \mathcal{C}_{x_1}(ODC_{x_2}) + ODC_{x_2} ODC'_{x_1}$
- Order dependence.

Example
first vertex v_y



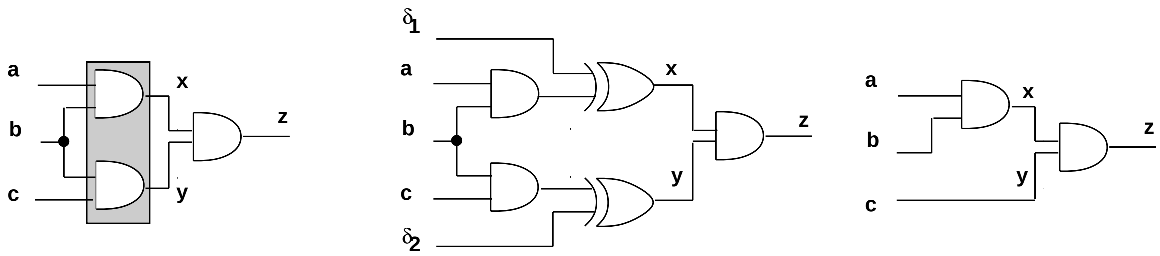
(a) (b)

- $CODC_y = ODC_y = x' = b' + a'$

- $ODC_x = y' = b' + c'$

- $CODC_x = \mathcal{C}_y(ODC_x) + ODC_x(ODC_y)' = \mathcal{C}_y(y') + y'x = y'x = (b' + c')ab = abc'$.

Example (2)



(a) (b)

- Allowed perturbation:
  - $f_y = bc \to g_y = c$.
  - $\delta_y = bc \oplus c = b'c \subseteq CODC_y = b' + a'$.
- Disallowed perturbation:
  - $f_x = ab \to g_x = a$.
  - $\delta_x = ab \oplus a = ab' \not\subseteq CODC_x = abc'$.
- The converse holds if v_x is the first vertex.

Multiple-vertex optimization compatible *don't cares*

$SIMPLIFY\_MV(\ G_n(V,E)\ )\{$
repeat $\{$
$U =$ selected vertex subset;
foreach vertex $v_x \in U$
Compute $COCD_x$ and the corresponding local *don't care* subset $\widetilde{DC}_x$;
Optimize simultaneously the functions at U;
$\}$**until** (no more reduction is possible);
$\}$

Summary
Boolean methods

- Boolean methods exploit *don't care* sets and simplification of logic representations.

- *Don't care* set computation:
  - Controllability and observability.

- Single and multiple transformations.

Synthesis and testability

- Testability:
  - Ease of testing a circuit.
- Assumptions:
  - Combinational circuit.
  - Single or multiple *stuck-at* faults.
- Full testability:
  - Possible to generate test set for all faults.
  - Restrictive interpretation.

Test for *stuck-at*s

- Net y stuck-at 0.
  - Input pattern that sets y to true.
  - Observe output.
  - Output of faulty circuit differs.
- Net y stuck-at 1.
  - Same, but set y to false.
- Need *controllability* and *observability*.

Test sets
don't care interpretation

- Stuck-at 0 on net y.
  - $\{\mathbf{t} | y(\mathbf{t}) \cdot ODC'_y(\mathbf{t}) = 1\}$.
- Stuck-at 1 on net y.
  - $\{\mathbf{t} | y'(\mathbf{t}) \cdot ODC'_y(\mathbf{t}) = 1\}$.

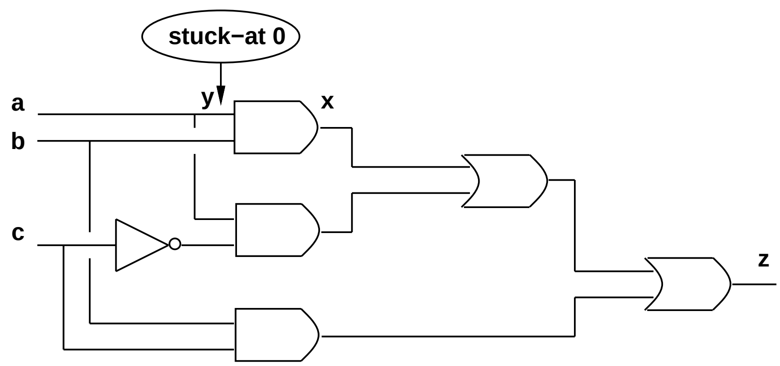
Using testing methods for synthesis

- Redundancy removal.
  - Use TPG to search for untestable faults.

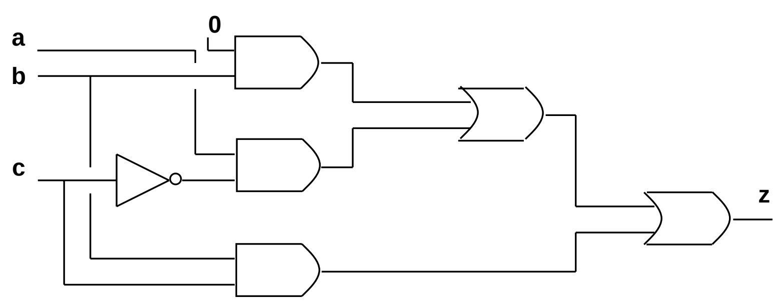
- If stuck-at 0 on net y is untestable:
  - Set $y = 0$.
  - Propagate constant.

- If stuck-at 1 on y is untestable:
  - Set $y = 1$.
  - Propagate constant.

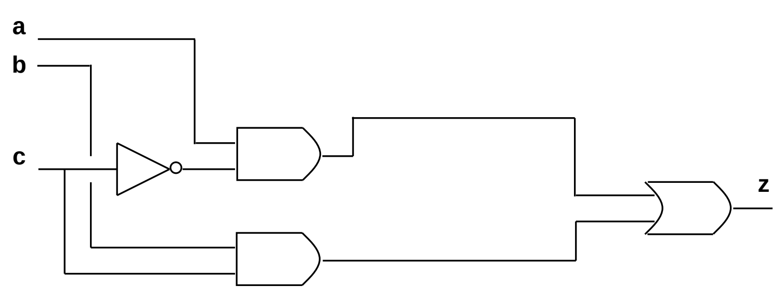
Example



(a)

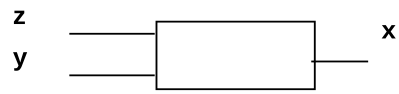


(b)



(c)

Redundancy removal and perturbation analysis



- Stuck-at 0 on y.
  - y set to 0. Namely $g_x = f_x|_{y=0}$.
  - Perturbation:
    - $\delta = f_x \oplus f_x|_{y=0} \; = \; y \cdot \partial f_x / \partial y$.

- Perturbation is feasible $\Leftrightarrow$ fault is untestable.
  - No input vector $\mathbf{t}$ can make $y(\mathbf{t}) \cdot ODC'_y(\mathbf{t})$ true.
  - No input vector can make $y(\mathbf{t}) \cdot ODC'_x(\mathbf{t}) \cdot \partial f_x / \partial y$ true.
    - because $ODC_y = ODC_x + (\partial f_x / \partial y)'$.

Redundancy removal and perturbation analysis

- Assume untestable stuck-at 0 fault.
- $y \cdot ODC'_x \cdot \partial f_x / \partial y \subseteq SDC$.
- Local *don't care* set:
  - $DC_x \supseteq ODC_x + y \cdot ODC'_x \cdot \partial f_x / \partial y$.
  - $DC_x \supseteq ODC_x + y \cdot \partial f_x / \partial y$.
- Perturbation $\delta \;=\; y \cdot \partial f_x / \partial y$.
  - Included in the local *don't care* set.

Synthesis for testability

- Synthesize networks that are fully testable.
  - Single stuck-at faults.
  - Multiple stuck-at faults.

- Two-level forms.

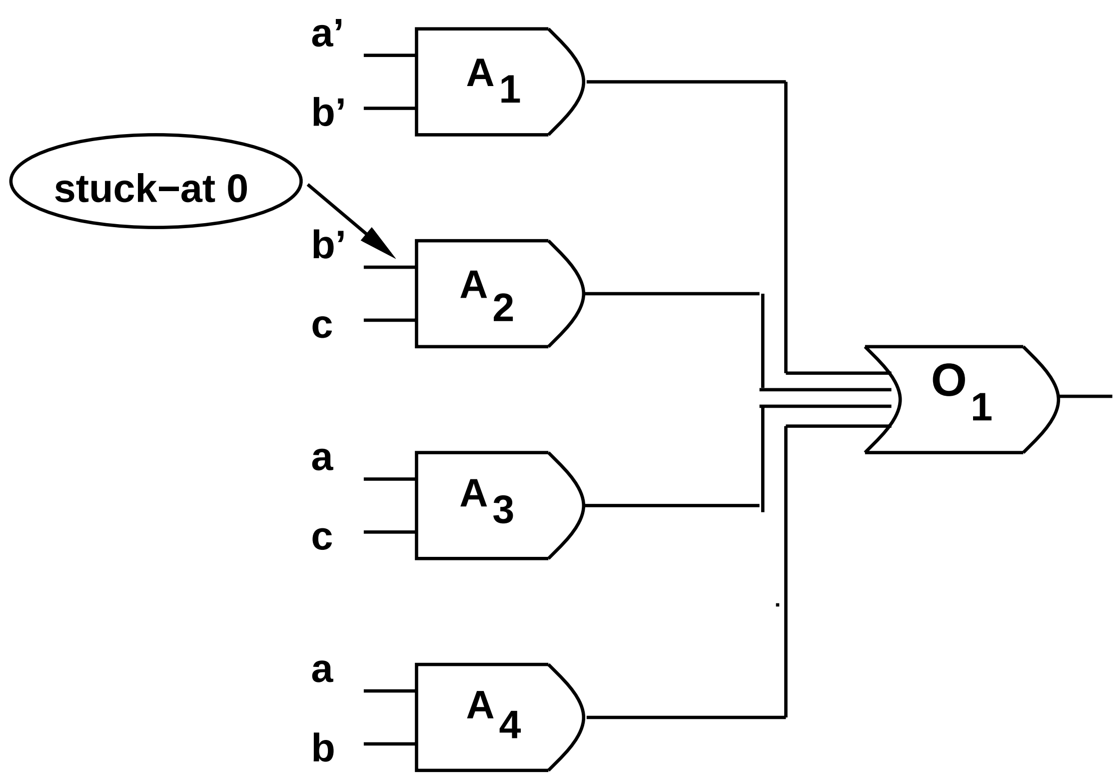
- Multiple-level networks.

Two-level forms

- Full testability for single stuck-at faults:
  - Prime and irredundant cover.
- Full testability for multiple stuck-at faults:
  - Prime and irredundant cover when:
    - Single-output function.
    - No product term sharing.
    - Each component is PI.

Example

$$f = a'b' + b'c + ac + ab$$



Multiple-level networks
Definitions

- A logic network $G_n(V, E)$,
  with local functions in *sum of product* form.

- Prime and irredundant (PI):
  - No literal nor implicant of any local function can be dropped.

- Simultaneously prime and irredundant (SPI):
  - No subset of literals and/or implicants can be dropped.

Multiple-level networks
Theorems

- A logic network is PI and only if:
  - its AND-OR implementation is fully testable for single stuck-at faults.

- A logic network is SPI if and only if:
  - its AND-OR implementation is fully testable for multiple stuck-at faults.

- Compute full local *don't care* sets.
  - Make all local functions PI w.r. to *don't care* sets.
- Pitfall:
  - *Don't cares* change as functions change.
- Solution:
  - Iteration (Espresso-MLD).
- If iteration converges, network is fully testable.

- Flatten to two-level form.
  - When possible – no size explosion.

- Make SPI by disjoint logic minimization.

- Reconstruct multiple-level network:
  - *Algebraic transformations preserve multifault testability.*

Summary

- Synergy between synthesis and testing.
- Testable networks correlate to small-area networks.
- *Don't care* conditions play a major role.