Multiresolution analysis: theory and applications

Analisi multirisoluzione: teoria e applicazioni

Course overview

Course structure

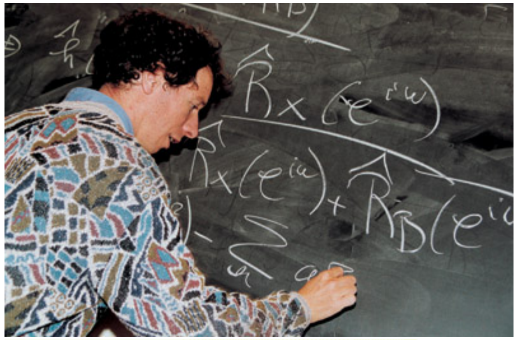
- The course is about wavelets and multiresolution
  - Theory: 4 hours per week (4 CFU)
    - Mon.11**.30-13.30**, room G
    - Tue. **8.30-10.30**, room I
  - Laboratory (2 CFU)
    - Wed**. 14.30-17.30** (Lab. Gamma) LM32
- Exam
  - Theory: Oral (in general)
  - Lab: Evaluation of lab. sessions and questions during the exam
  - Projects: only in case of diploma project

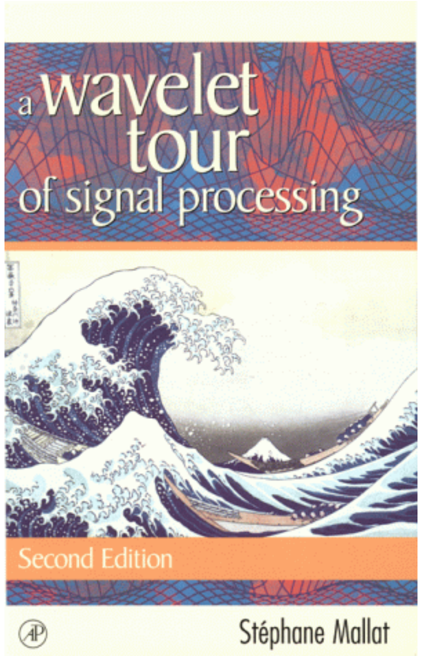
Contents

- Review of Fourier theory
- Wavelets and multiresolution
- Review of Information theoretic concepts
- Applications
  - Image coding (JPEG2000)
  - Feature extraction and signal/image analysis
- Wavelets and sparsity in neuroimaging

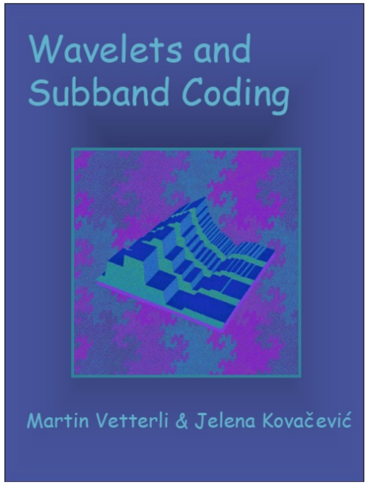
Books

Stephane Mallat
(Ecole Polytechnique)





Martin Vetterli (EPFL)



Multiresolution analysis

Good news

- It is fun!
- Get in touch with the state-of-the-art technology
- Convince yourself that the time spent on maths&stats was not wasted
- Learn how to map theories into applications
- Acquiring the tools for doing good research!

Bad news

- Some theoretical background is unavoidable
  - Mathematics
    - Fourier transform
    - Linear operators
    - Digital filters
    - Wavelet transform
  - (some) Information theory

Keywords & Concepts

Basis functions

Multiresolution

Dictionaries and pursuit

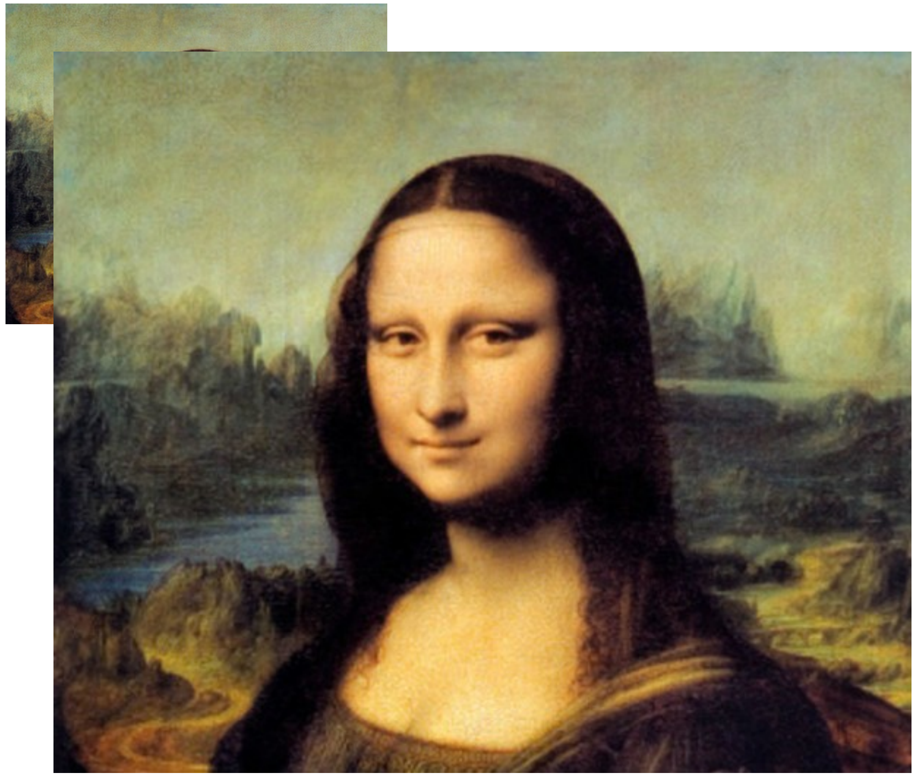
Scale

Sparsity

Vector spaces

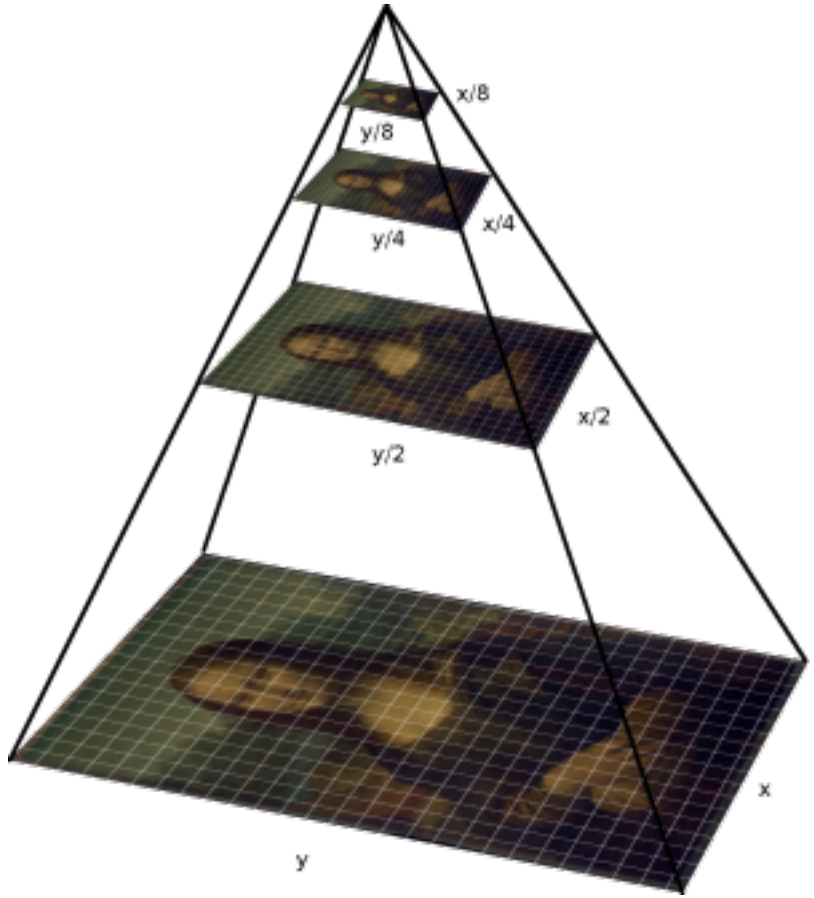
Approximation theory

Multiresolution

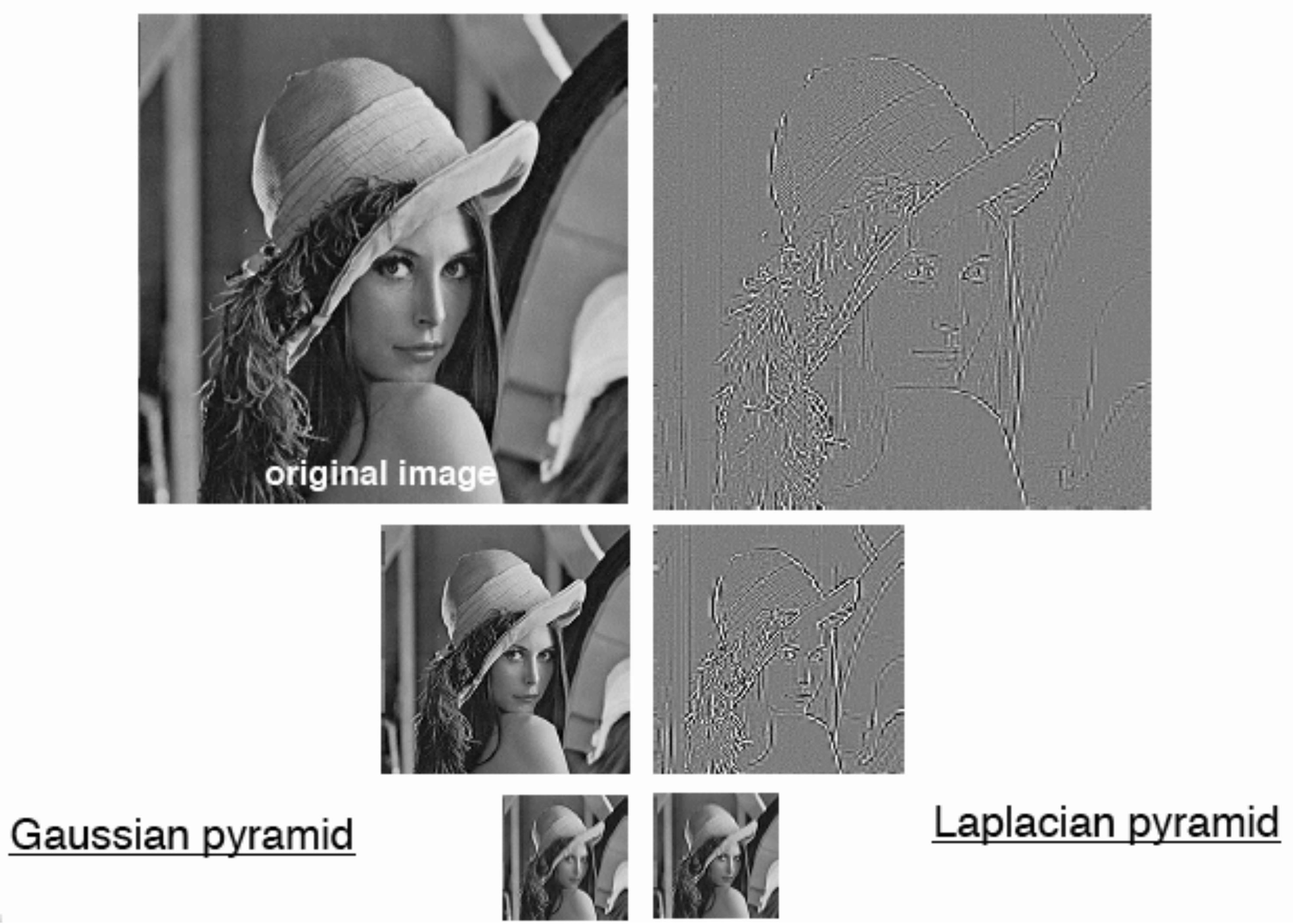


Multiresolution

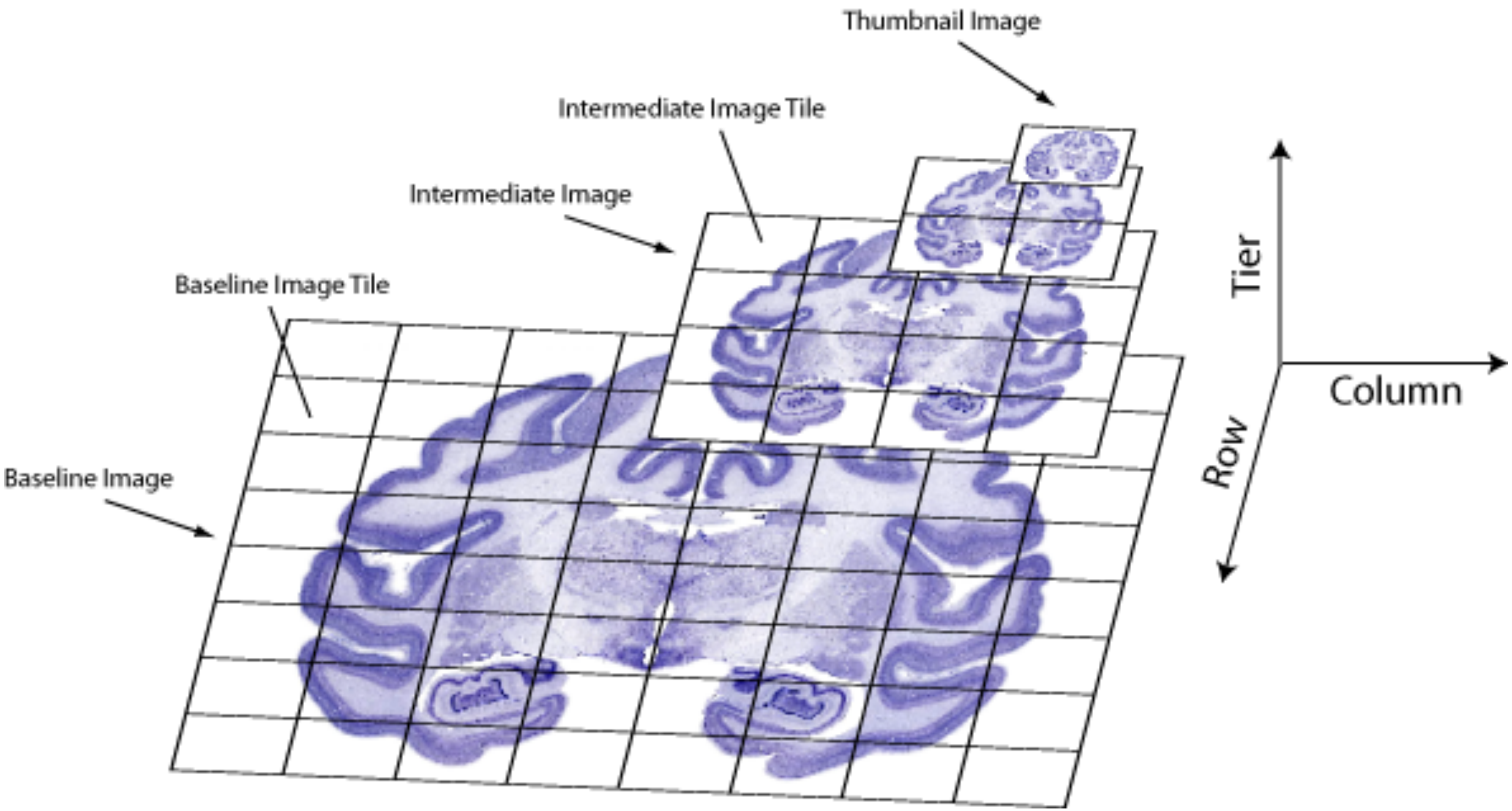
Embedded grids of approximations



Approximation vs Details

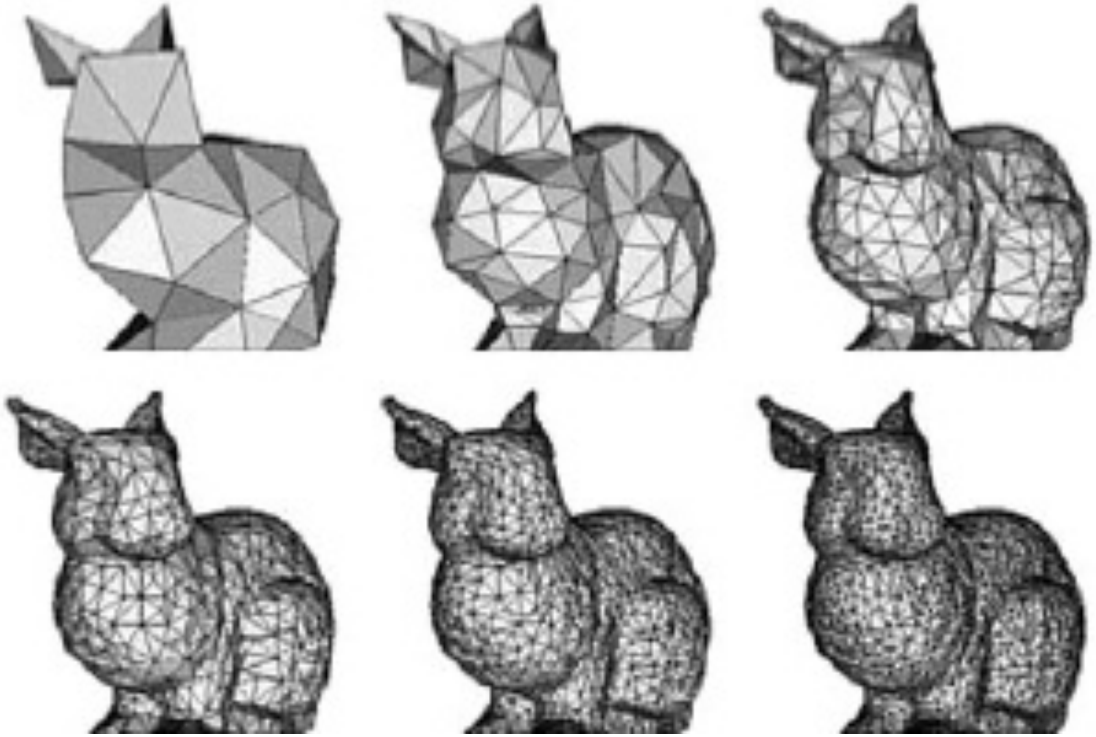


Multiresolution



Multiresolution

Multiresolution

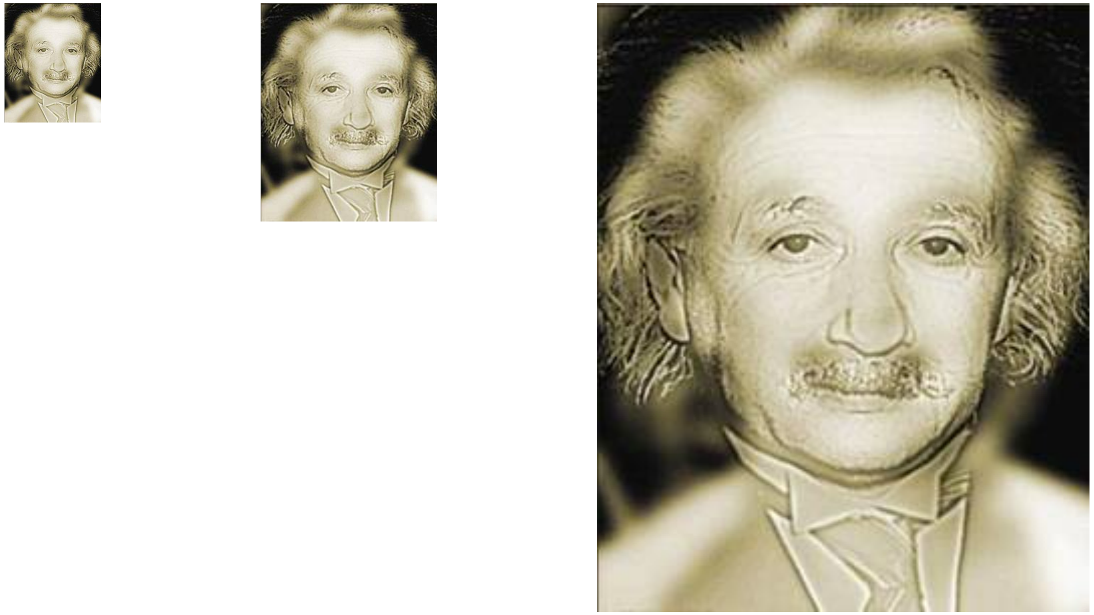


“Scale”

“Scale”

“Scale”

Which scale should we trust?

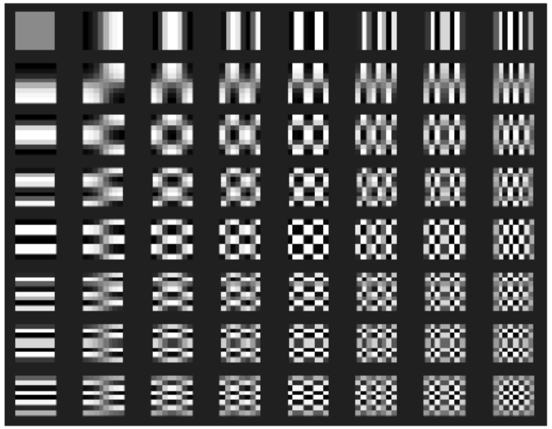


Sparsity

More for less!

Best bases: **1** coefficient, **NO** generalization!

Wavelet bases: **few** coefficients, **good** generalization!

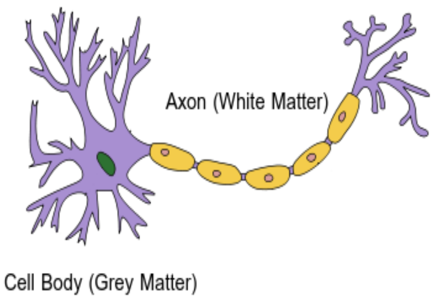


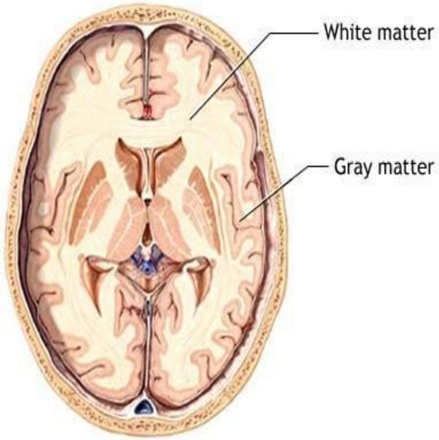
Fourier (DCT) bases: **many** coefficients, **good** generalization!

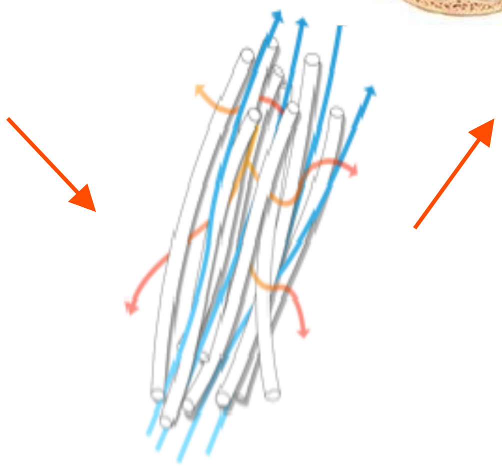
Some applications

Brain tissue microstructure

- The brain is principally composed of a type of cells called ***neurons*.**
- A neuron is composed of a cellular body called ***soma*** and a tail called ***axon*** that is physical link between the neurons.
- The axons are usually group in bundles called ***fibers*.**
- In the brain the *soma* are positioned in the cortex and are generally called ***gray matter*** (GM), while the *fibers* are positioned in the central regions and are called ***white matter*** (WM).







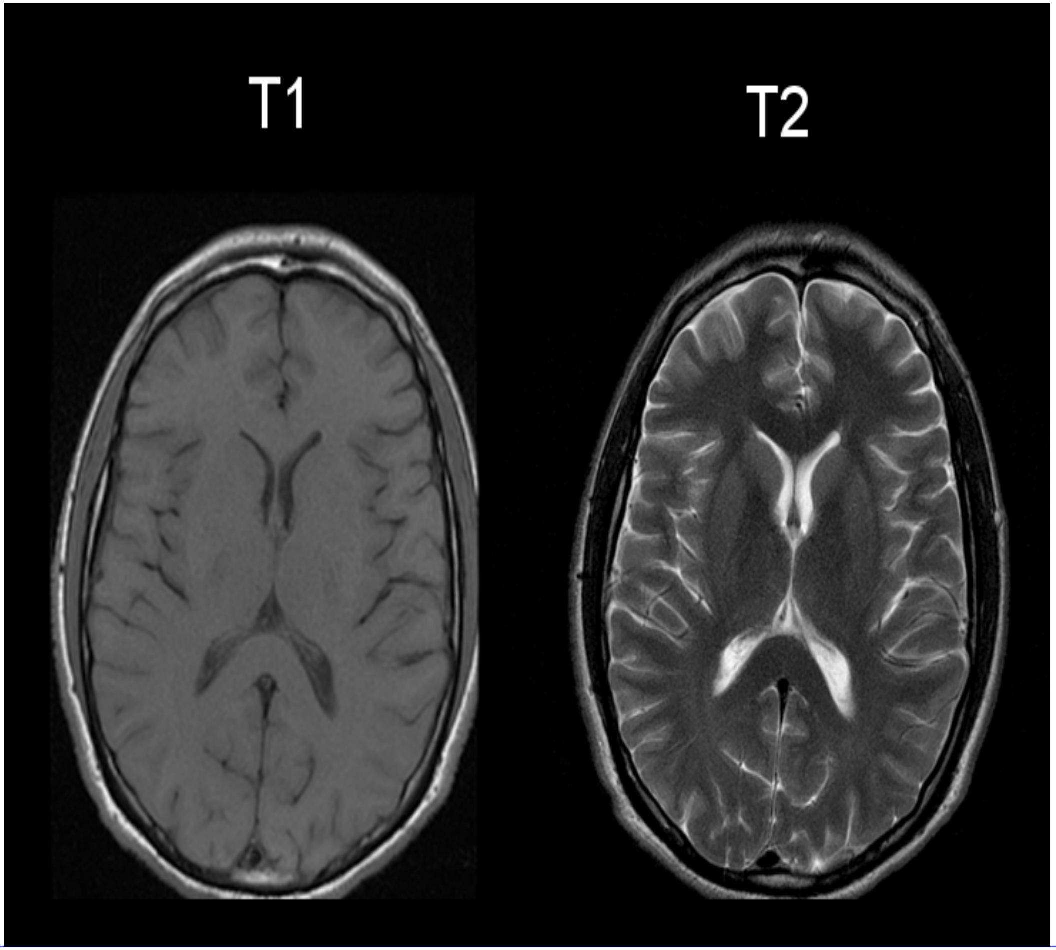
Axons bundles

Magnetic Resonance Imaging

- **Standard MRI** is the principal non-invasive imaging technique used for clinical purposes.

- Using standard MRI techniques is possible to distinguish between GM, WM and CSF but not the **complex structure** of the White Matter fibers bundles.

- To overcome this limitation, using an additional pulse is possible to obtain a different type of images called **Diffusion Weighted MRI**.

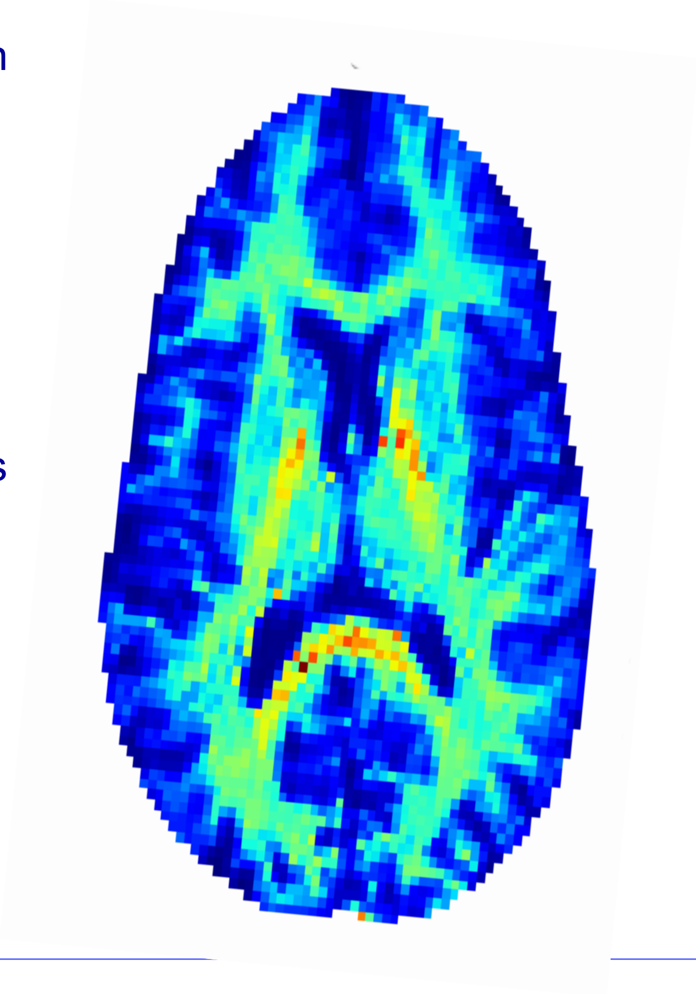


Diffusion Weighted MRI

- Diffusion MRI was born to observe the **diffusion of water molecules** in soft tissues.
- The diffusion signal can be modelled using some mathematical algorithms called **reconstruction techniques.**
- From the reconstructed signal is possible to calculate numerous measures to characterize the tissue and to calculate the **orientation of the fibers** tract in the voxel.
- From the single voxel orientation profile is possible to reconstruct the brain fibers tracts topology, this operation is called **tractography**.

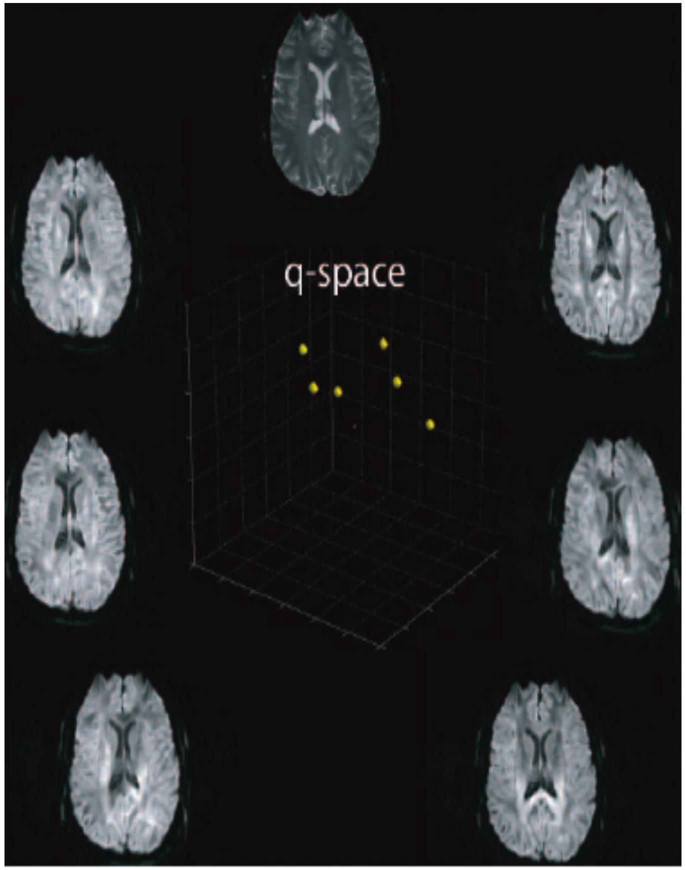
Objectives

- Find the **optimal** reconstruction technique for Diffusion MRI data
- Definition of a standard criterion for validation
  - Synthetic data
- Identification of **new scalar indices** as numerical biomarkers of the **structural properties** of brain tissues
  - Anatomically and biophysically plausible besides being objectively measurable
  - Supporting and improving cortical connectivity modeling
- Uses of this indices features
  - Tissues characterization by pattern recognition
  - Patient vs Control classification



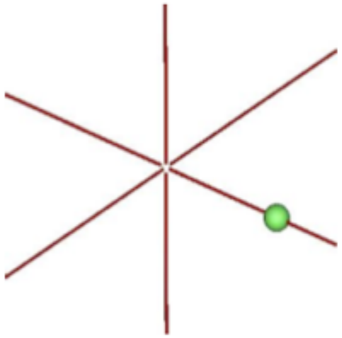
Diffusion signal

- Invented by Stejskal and Tanner (**1965**)
- It exploits an additional sequence of pulses: Pulse Gradient Spin Echo (PSGE) to measure the **attenuation** of the signal due to the diffusion of water in the soft tissues
- Changing the gradient direction (***u***) and strength (*b-value*) it is possible to obtain different volumes called **DWI**, each one representing the attenuation of the diffusion in the chosen direction
- The *b-value* depends on the duration of the pulse τ and the **pulse frequency** q:

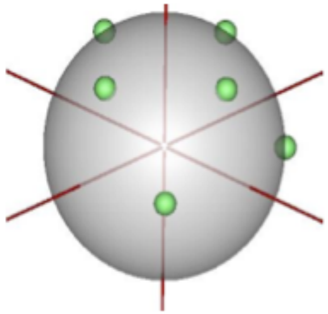


$$b = 4\pi^2 q^2 \tau \ (s/mm^2)$$

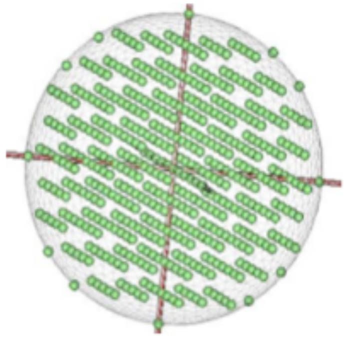
Sampling topologies



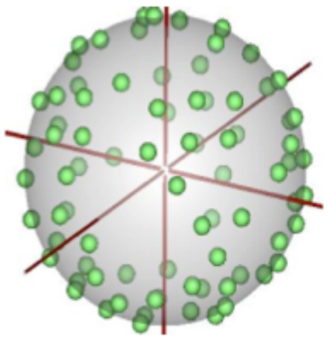
Pulsed Gradient Spin Echo
Stejskal & Tanner, 1965



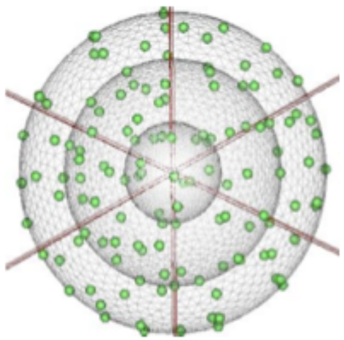
Diffusion tensor imaging
Basser, 1994



Diffusion spectrum imaging
Van Wedeen, 2000



Single-Shell High Angular
Resolution Diffusion Imaging
2000-2008



Multiple-Shell, sparse
Hybrid Diffusion Imaging
2008-now

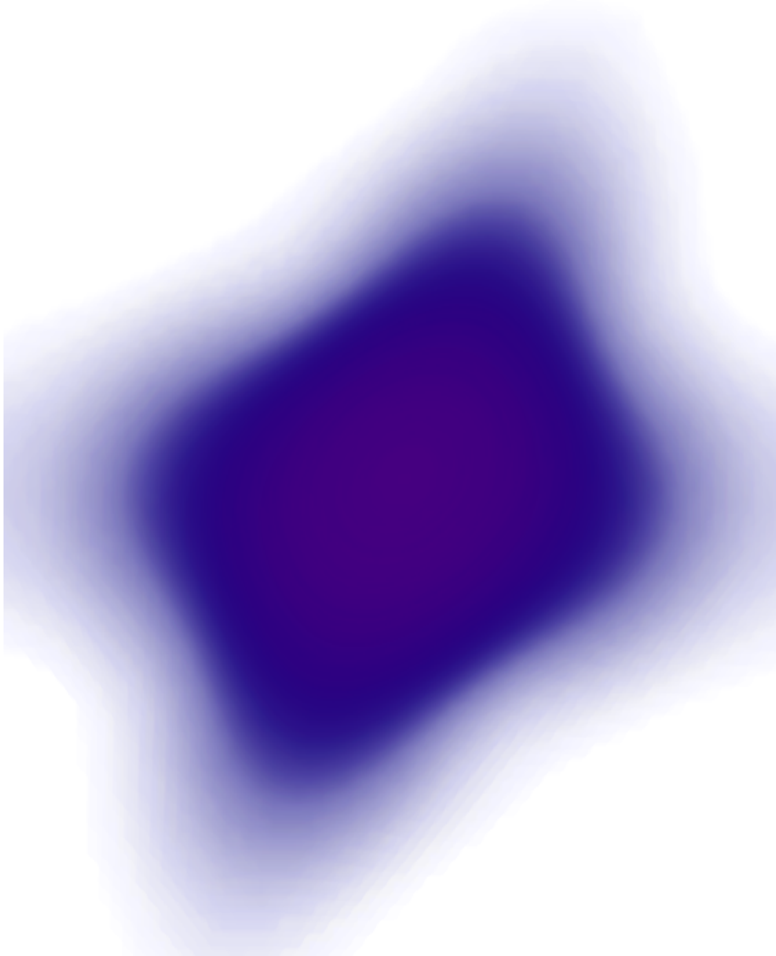
From diffusion signal to water molecules pdf

- The signal attenuation *E(**q**)* is related to Ensemble Average Propagator (EAP) by a Fourier relationship:

$$P(\mathbf{r}) = \int_{\mathbf{q}\in\Re^3} E(\mathbf{q}) exp(+2\pi i \mathbf{q}\cdot\mathbf{r}) d\mathbf{q}$$

r: nit time
q: reciprocal vector

- The EAP represents the probability of a net displacement **r** in the unit time



Continuous Analytical Basis for Diffusion Imaging

- Continuous analytical basis besides SH have been proposed to find an accurate **mathematical description** of the diffusion signal and its derivations

- Analytical models aim at approximating the signal ***E(q)*** by a truncated linear combination of **basis functions** $\Phi(\mathbf{q})$ up to the order N:

$$\mathbf{E}(\mathbf{q}) = \sum_{j=0}^{N} c_j \Phi_j(\mathbf{q})$$

c_j are the **transform coefficients** characterizing the signal. Usually these coefficients are obtained by linear fitting, e.g. using regularized mean squares

Continuous Analytical Basis for Diffusion Imaging

The principal advantages of Continuous Basis are:

- Continuous analytical signal representation in *q-space* **independently** from the acquisition sampling scheme

- Possibility to calculate the EAP and the ODF **analytically**, obtaining an exact solution for all the computations

Principal open issues:

- Identification of the sampling topology
- Identification of the optimal *basis* for signal approximation

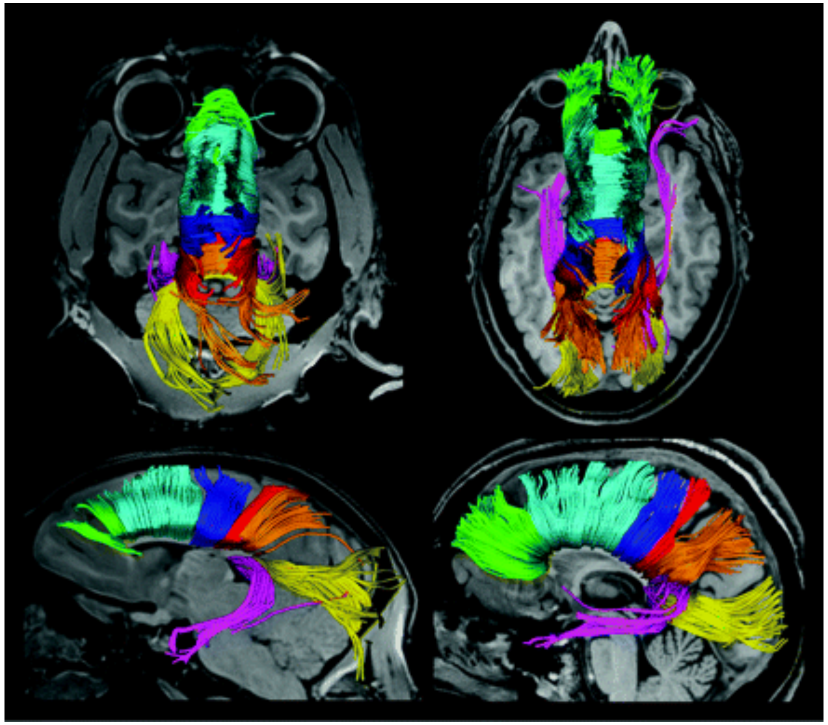
Simple Harmonic Oscillator based Reconstruction and Estimation

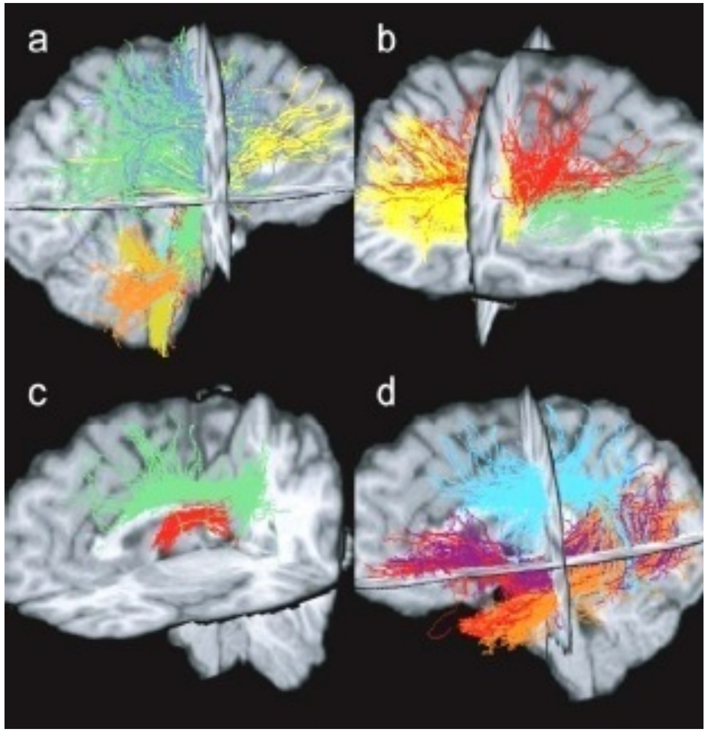
- **SHORE** is a continuous analytical basis introduced by Ozarslan in 2009
- The signal is approximated using a combination of orthonormal functions which are the solutions of the 3D *quantum mechanical harmonic oscillator*
- Separable solution (Merlet 2013): *Laguerre Polynomials* for the **radial part** and *Spherical Harmonics* for the **angular part**

$$\mathbf{E}(\mathbf{q}) = \sum_{n=0}^{N_{max}} \sum_{l=0}^{n} \sum_{m=-l}^{l} c_{nlm} \Phi_{nlm}(\mathbf{q})$$

$$\Phi_{nlm}(q\mathbf{u}) = \left[\frac{2(n-l)!}{\zeta^{3/2}\Gamma(n+3/2)} \right]^{1/2} \left(\frac{q^2}{\zeta} \right)^{l/2} exp\left(\frac{-q^2}{2\zeta} \right) L_{n-l}^{l+1/2}\left(\frac{q^2}{\zeta} \right) Y_l^m(\mathbf{u})$$

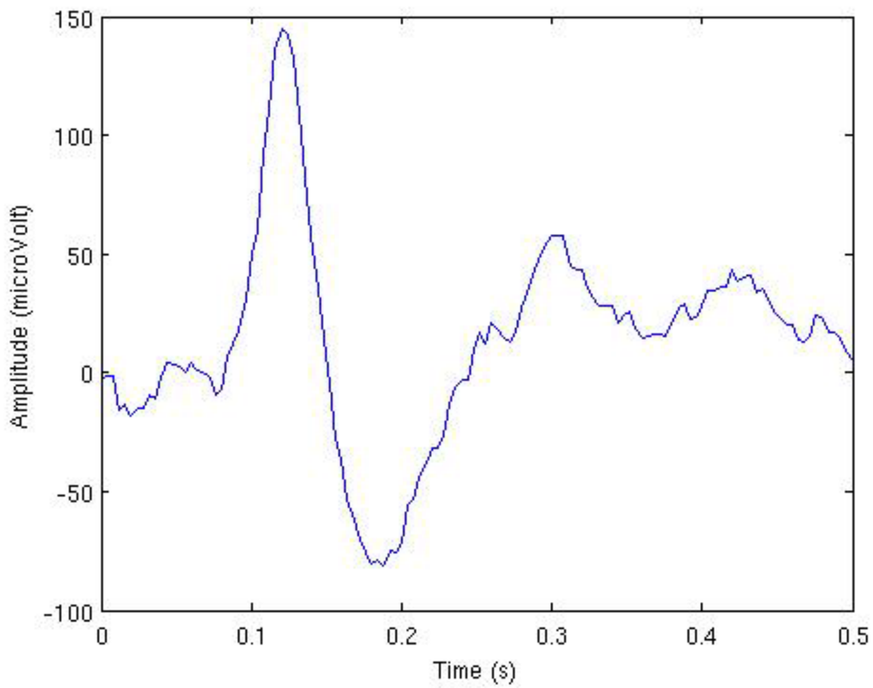
Wiring the brain





Modeling and recognition of waveforms by multiresolution methods with application to hdEEG

The purpose of this work was to focus on a particular pathology, namely temporal lobe epilepsy, in order to detect, analyze and model the so-called interictal spikes.



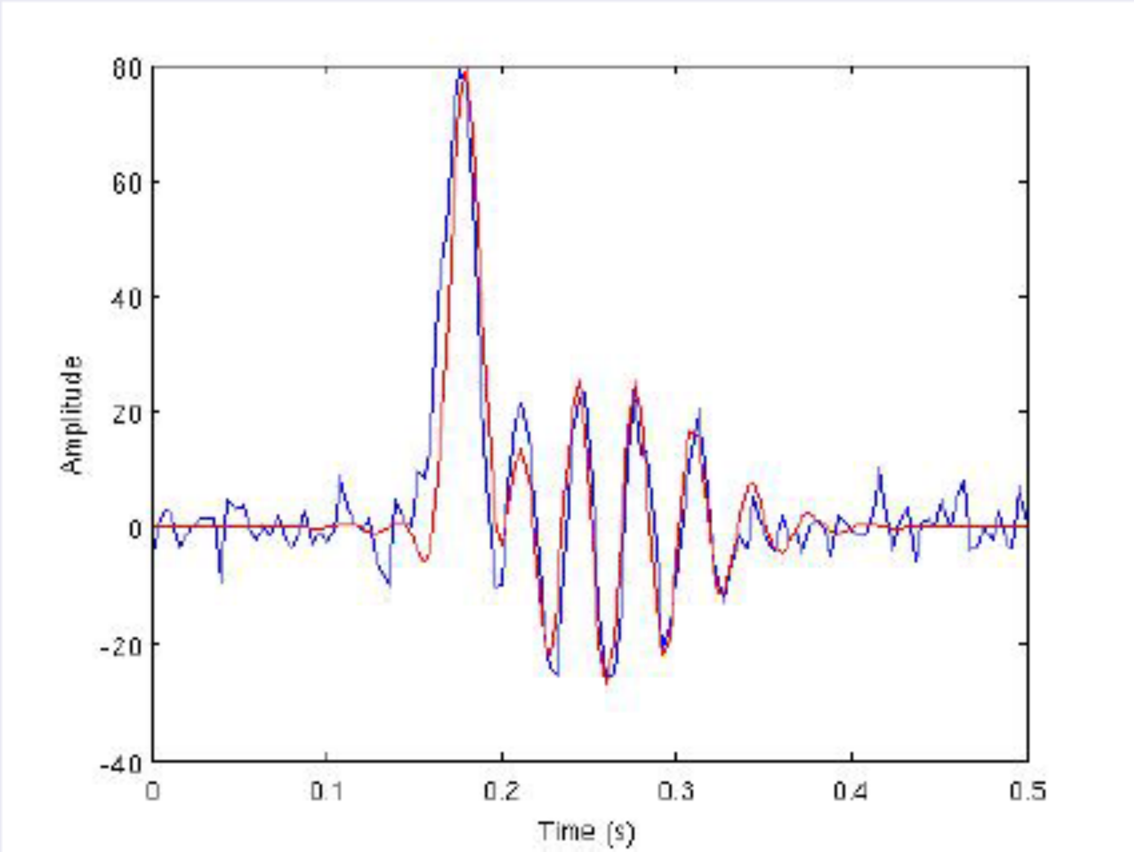

Fp1 – F7
F7 – T3
T3 – T5
T5 – 01
Fp2 – F8
F8 – T4
T4 – T6
T6 – 02
Fz – Cz

Matching Pursuit

MP

Given a dictionary of waveforms $D = \Psi(\vec{p})$ of size P which at least contains N linearly independent functions (with $P > N$), the corresponding sparse regression problem aims at finding signal expansions of the form:

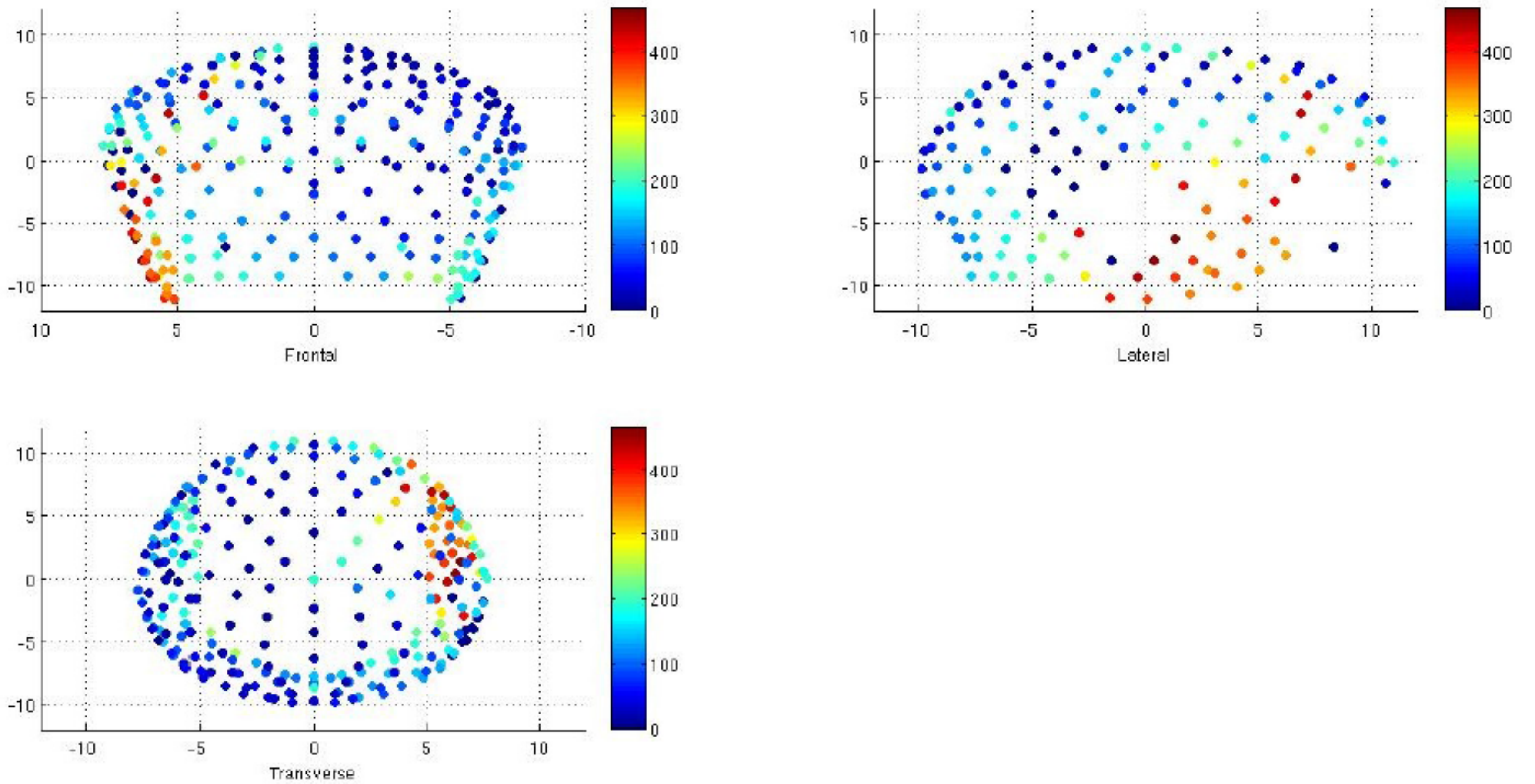
$$s(t) = \sum_{i=1}^{I} a_i \Psi_{\vec{p}_i}(t) + N(t) \quad (1)$$



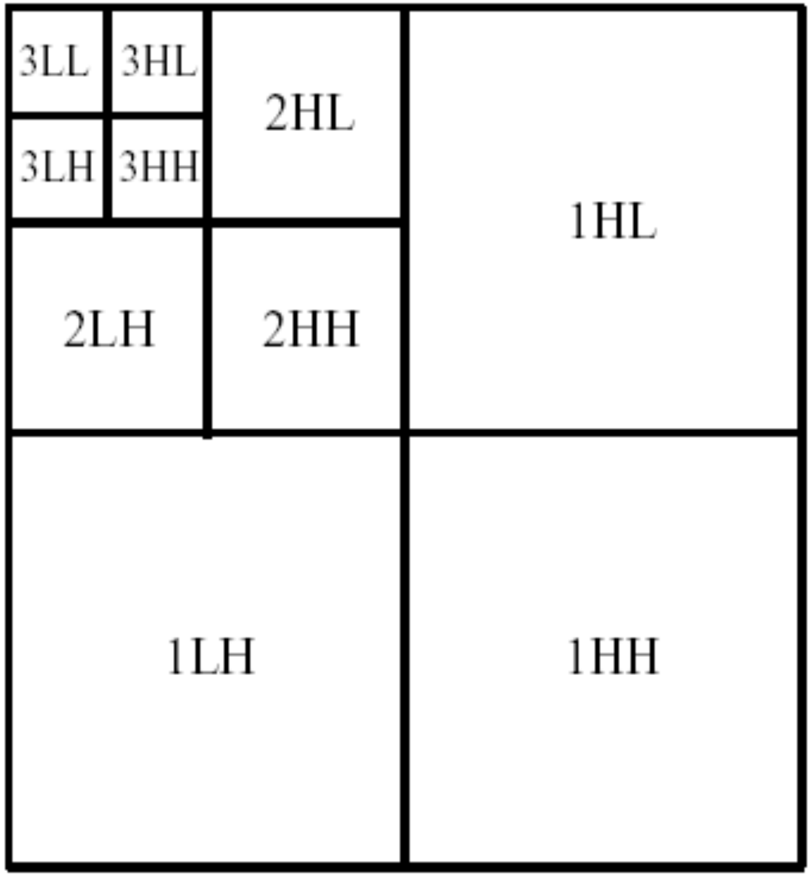
Reconstruction

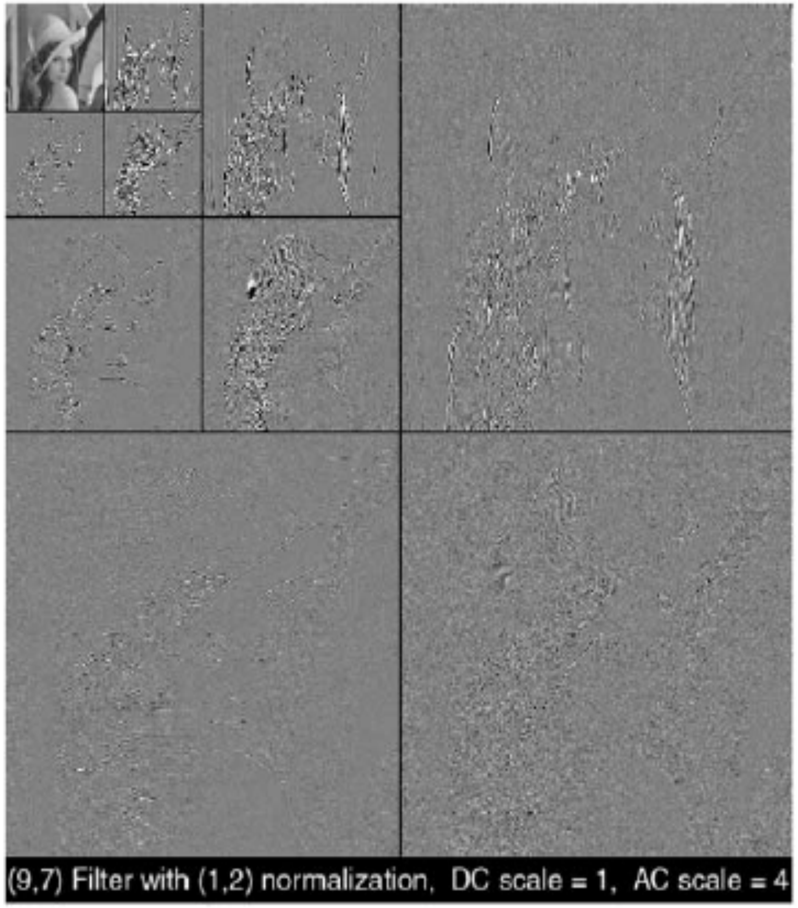
Real dataset

All channels classification: $M(\hat{p}_1)$



JPEG2000


3LL
3HL
3LH
3HH
2HL
2LH
2HH
1HL
1LH
1HH


(9,7) Filter with (1,2) normalization, DC scale = 1, AC scale = 4

Mathematical tools

Introduction

- Sparse representations: few coefficients reveal the information we are looking for.
  - Such representations can be constructed by decomposing signals over elementary waveforms chosen in a family called a *dictionary*.
  - An **orthogonal** basis is a dictionary of **minimum size** that can yield a sparse representation if designed to concentrate the signal energy over a set of few vectors. This set gives a *geometric* signal description.
    - Signal compression and noise reduction
  - Dictionaries of vectors that are **larger** than bases are needed to build sparse representations of complex signals. But choosing is difficult and requires more complex algorithms.
    - Sparse representations in redundant dictionaries can improve pattern recognition, compression and noise reduction
- Basic ingredients: Fourier and Wavelet transforms
  - They decompose signals over oscillatory waveforms that reveal many signal properties and provide a path to sparse representations

Signals as functions

- CT analogue signals (real valued functions of continuous independent variables)
  - 1D: $f=f(t)$
  - 2D: $f=f(x,y)$ x,y
  - Real world signals (audio, ECG, pictures taken with an analog camera)
- DT analogue signals (real valued functions of discrete variables)
  - 1D: $f=f[k]$
  - 2D: $f=f[i,j]$
  - *Sampled* signals
- Digital signals (discrete valued functions of DT variables)
  - 1D: $y=y[k]$
  - 2D: $y=y[i,j]$
  - *Sampled and discretized* signals

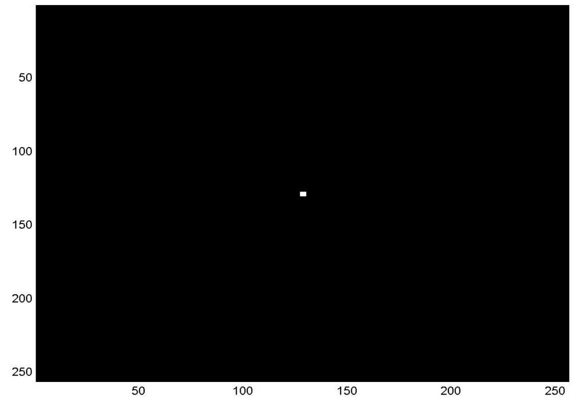
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of *(x,y)* values for which *f(x,y)* is defined : 2D lattice *[i,j]* defining the pixel locations
  - Set of values taken by the function : gray levels

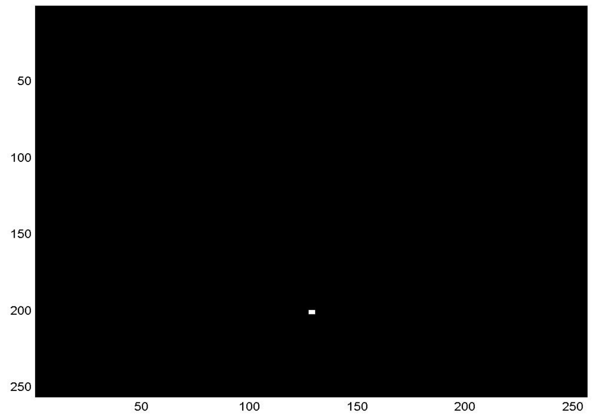
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - *I,J:* number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position *[i,j]*

Example 1: δ function

$$\delta[i,j]=\begin{cases}1 & i=j=0\\ 0 & i,j\neq 0; i\neq j\end{cases}$$



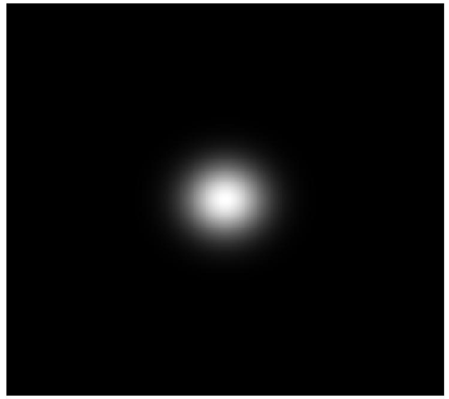
$$\delta[i,j-J]=\begin{cases}1 & i=0; j=J\\ 0 & otherwise\end{cases}$$



Example 2: Gaussian

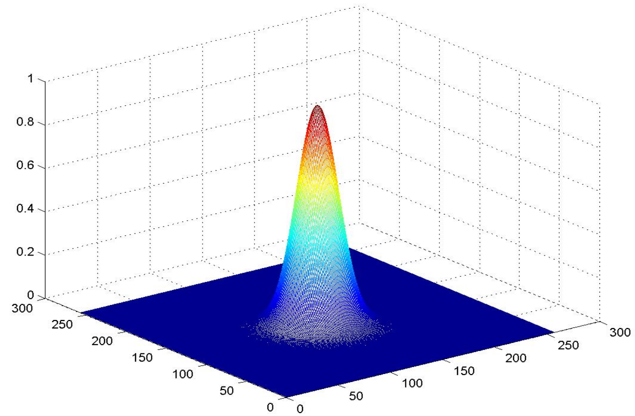
Continuous function

$$f(x,y)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{x^2+y^2}{2\sigma^2}}$$

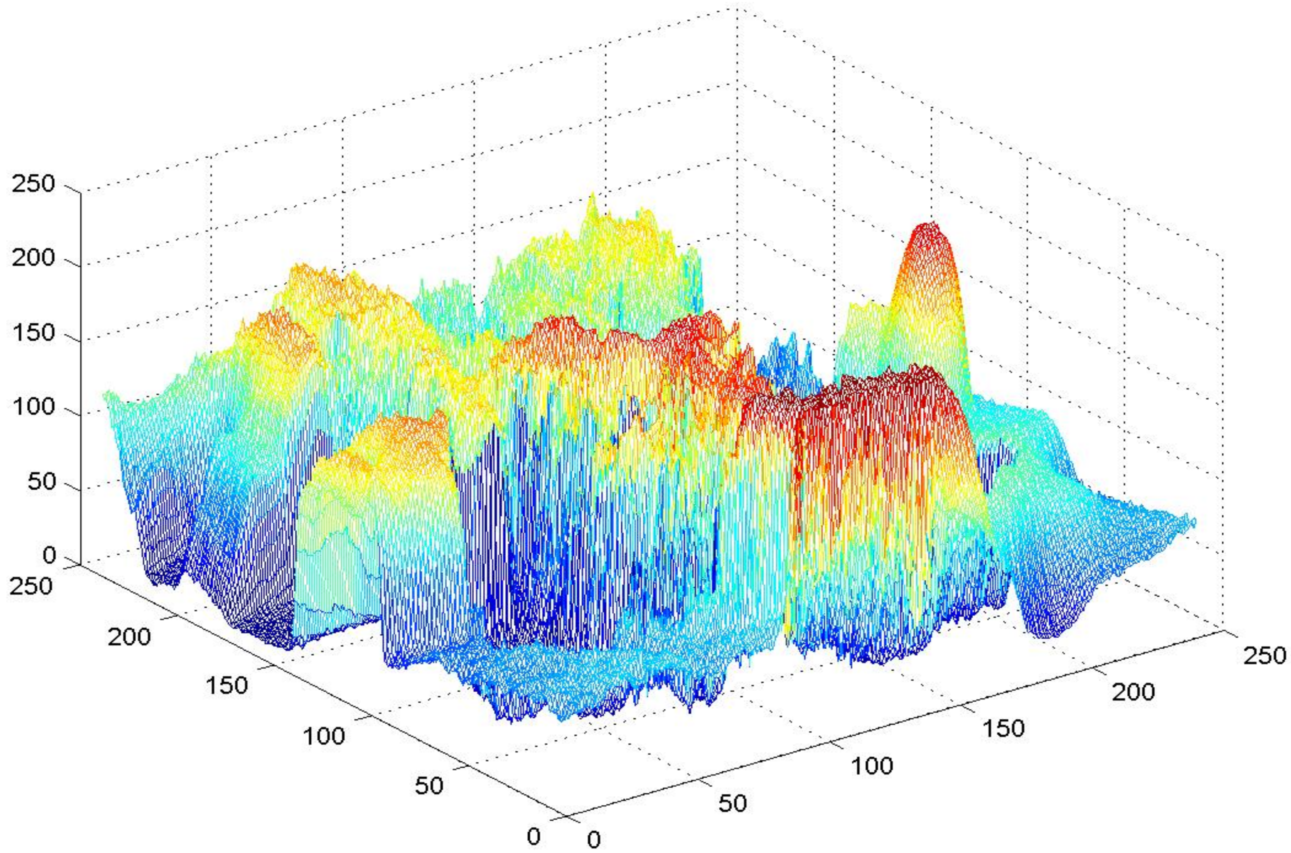


Discrete version

$$f[i,j]=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{i^2+j^2}{2\sigma^2}}$$



Example 3: Natural image



Example 3: Natural image

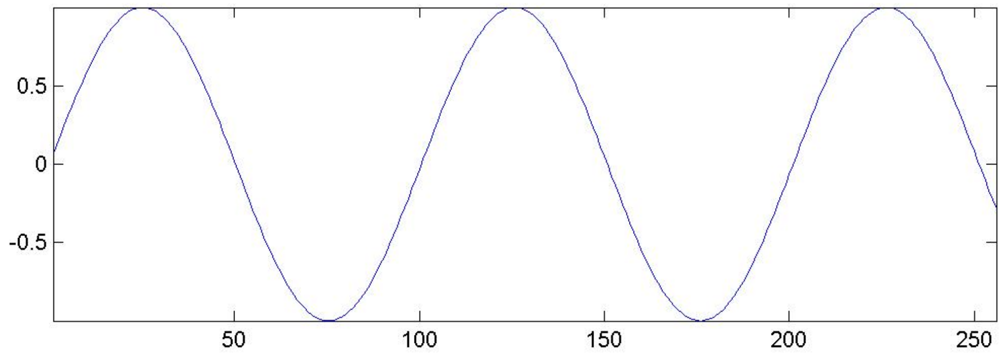
The Fourier kingdom qui

- Frequency domain characterization of signals

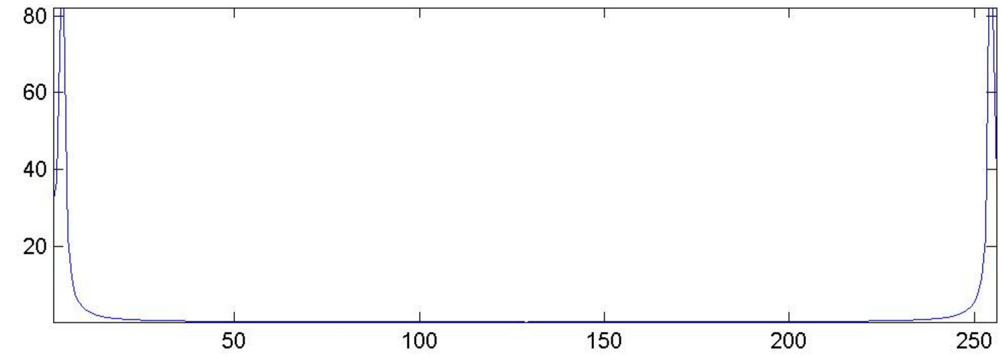
$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}\,dt$$

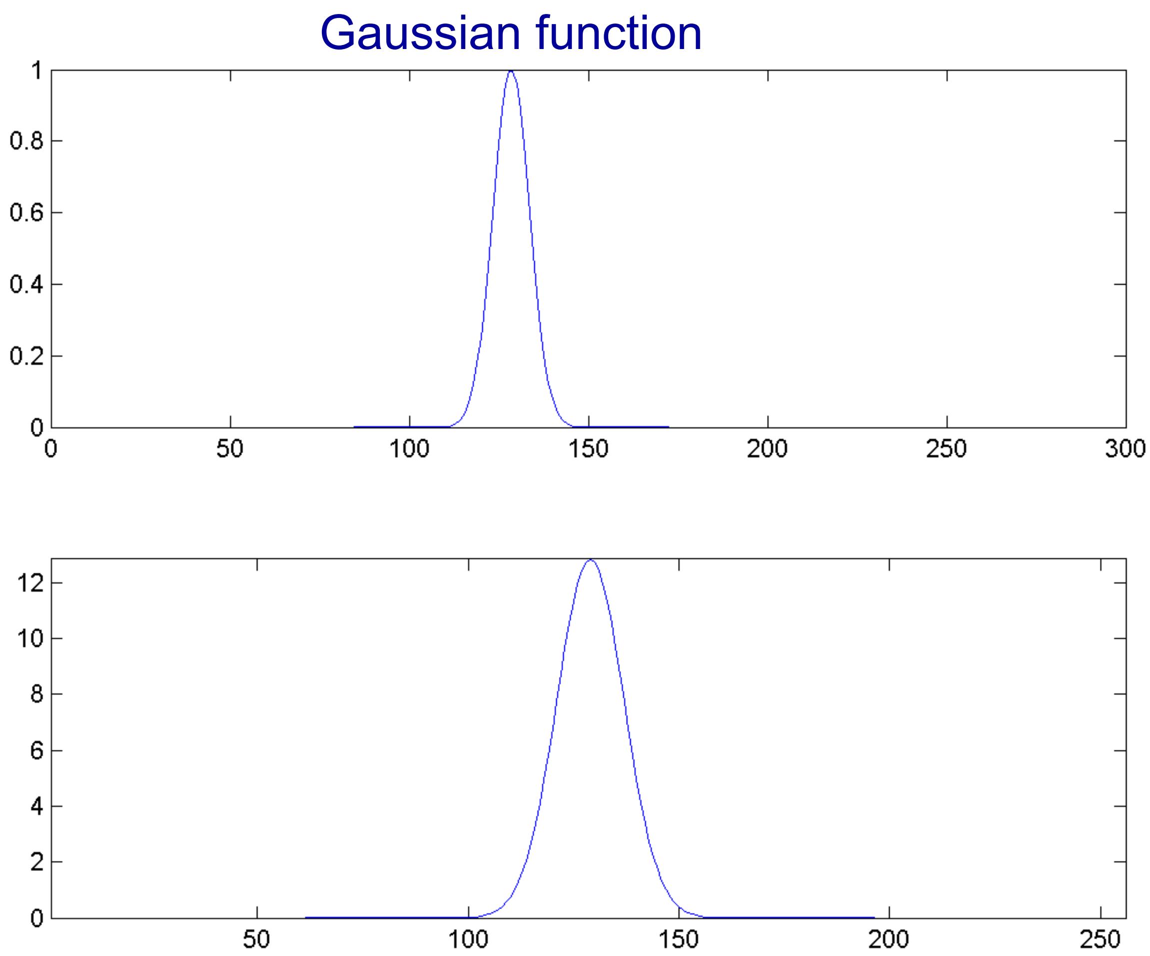
Signal domain



Frequency domain

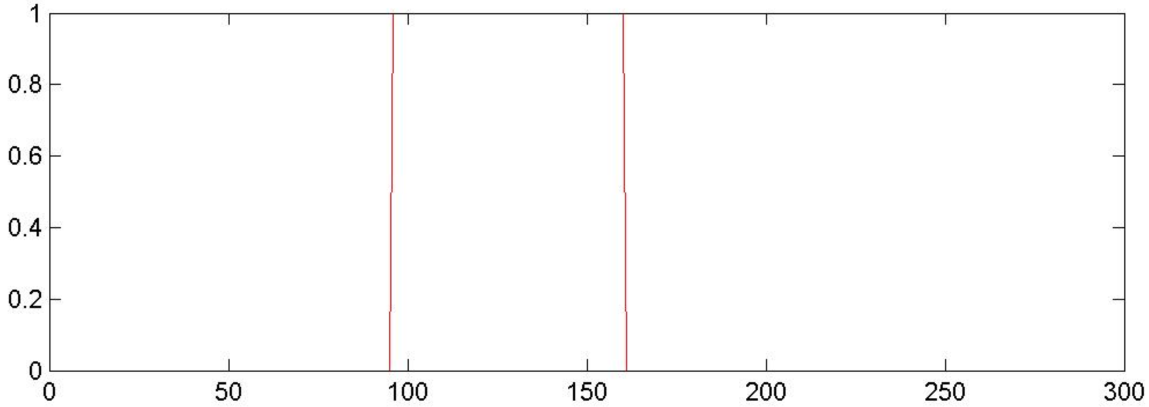


The Fourier kingdom

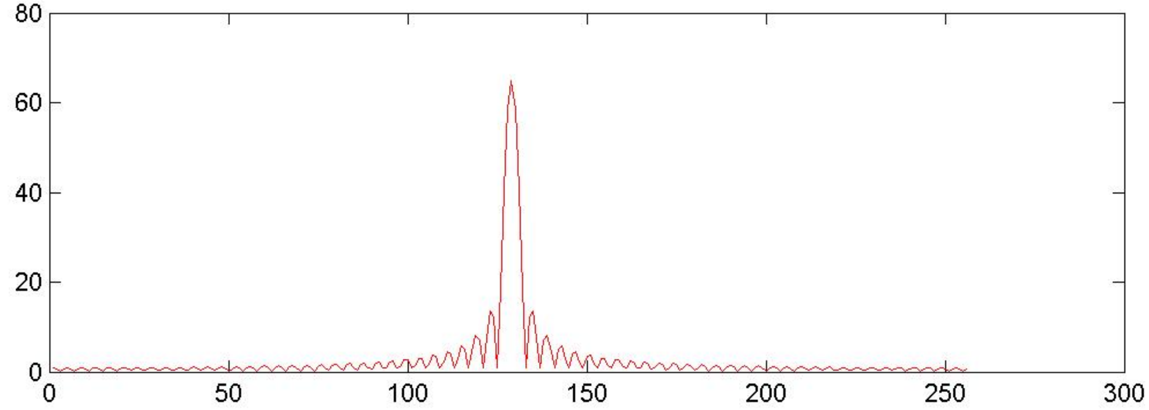


The Fourier kingdom

rect function



sinc function



2D Fourier transform

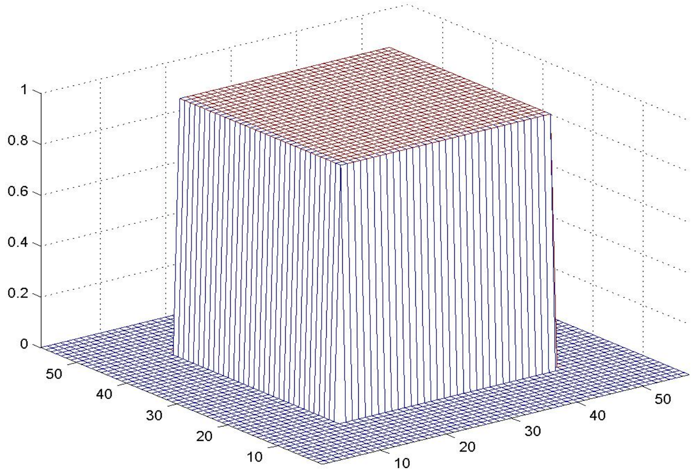
$$\hat{f}(\omega_x,\omega_y) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-j(\omega_x x+\omega_y y)}\, dxdy$$

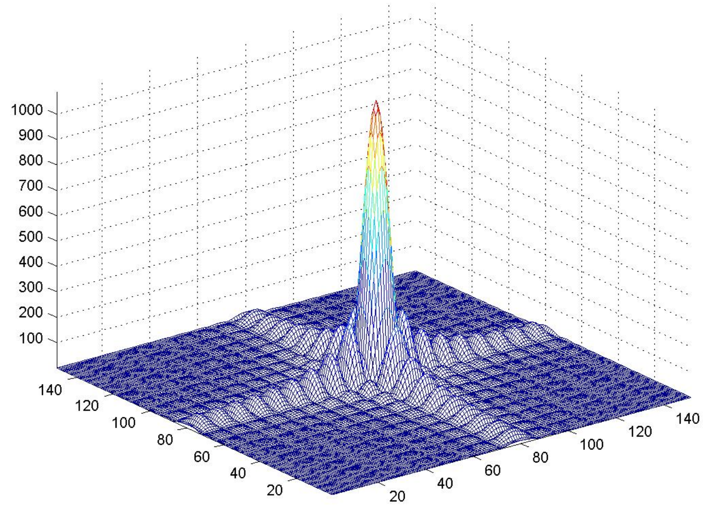
$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \hat{f}(\omega_x,\omega_y)\, e^{j(\omega_x x+\omega_y y)}\, d\omega_x d\omega_y$$

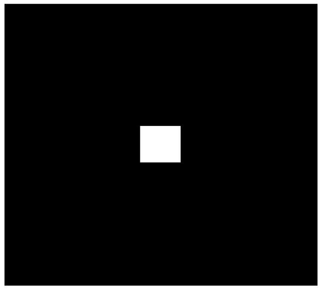
$$\iint f(x,y)\, g^*(x,y)\, dxdy = \frac{1}{4\pi^2}\iint \hat{f}(\omega_x,\omega_y)\hat{g}^*(\omega_x,\omega_y)\, d\omega_x d\omega_y$$ Parseval formula

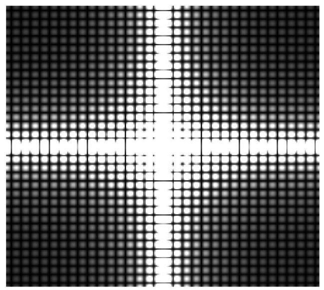
$$f = g \rightarrow \iint |f(x,y)|^2\, dxdy = \frac{1}{4\pi^2}\iint \left|\hat{f}(\omega_x,\omega_y)\right|^2 d\omega_x d\omega_y$$ Plancherel equality

The Fourier kingdom

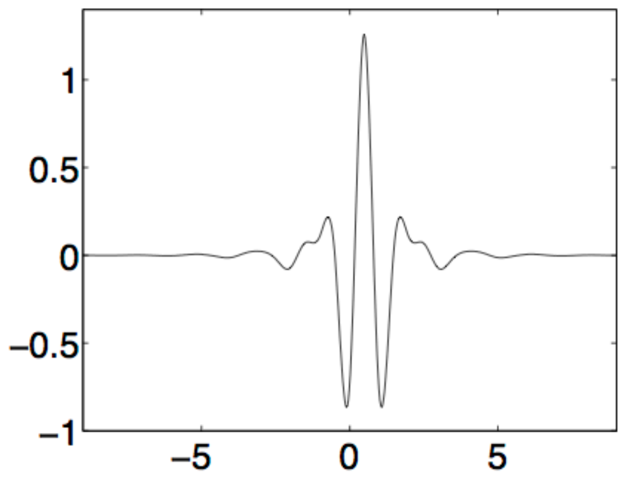




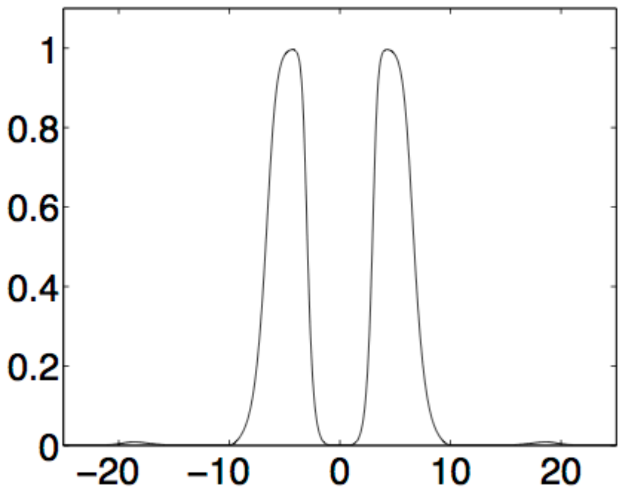




Wavelets

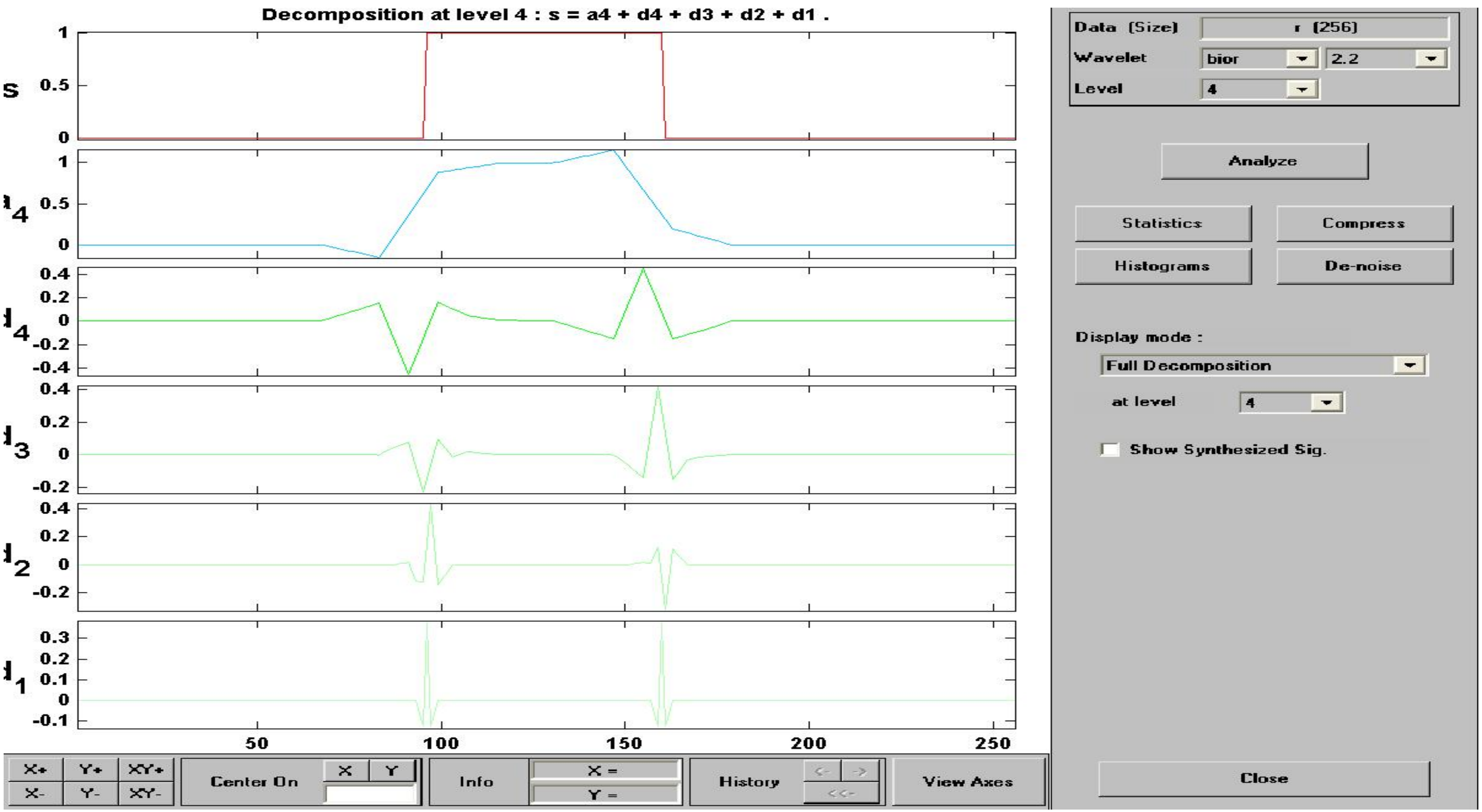


Wavalet in signal (time or space) domain

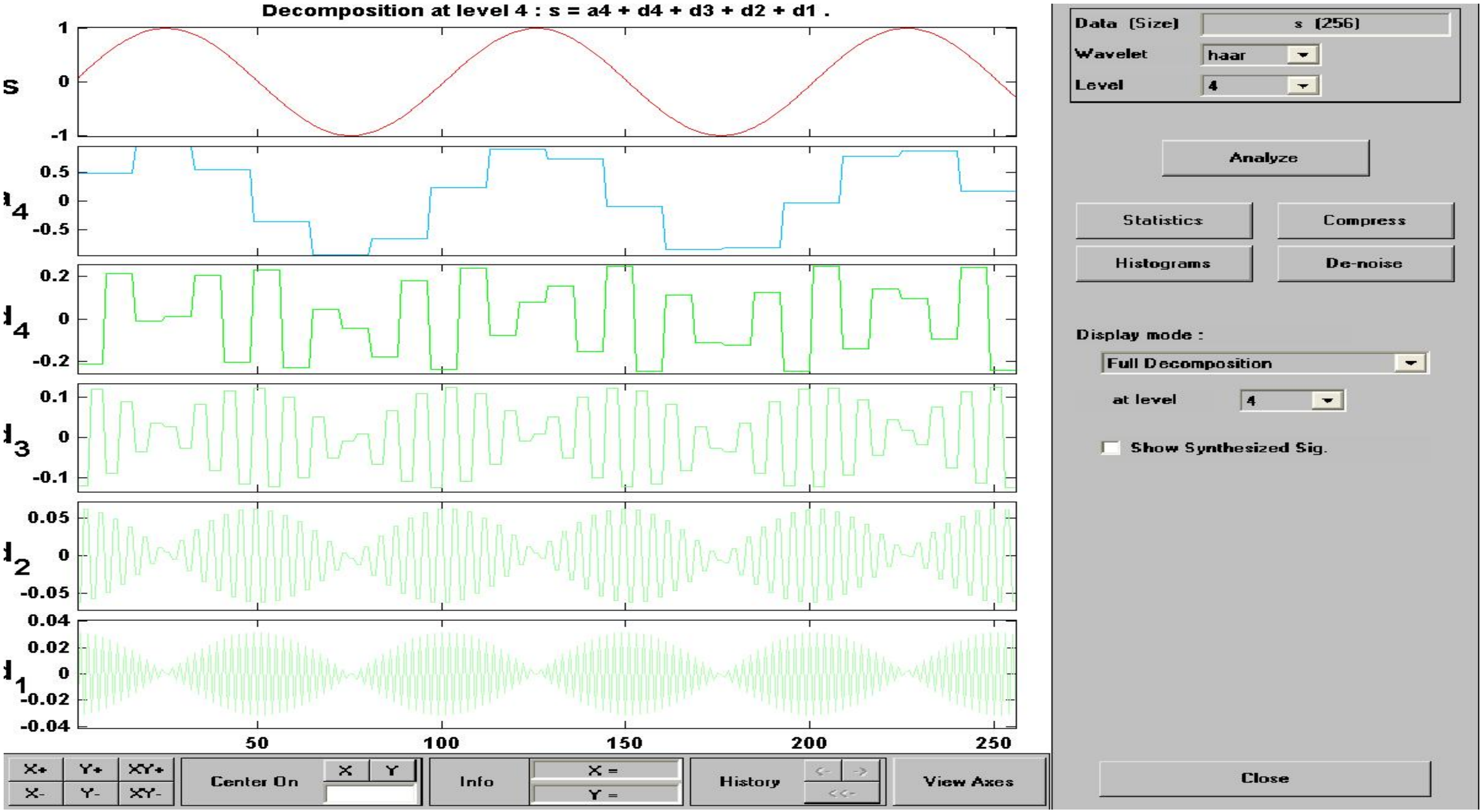


Wavalet in frequency (Fourier) domain

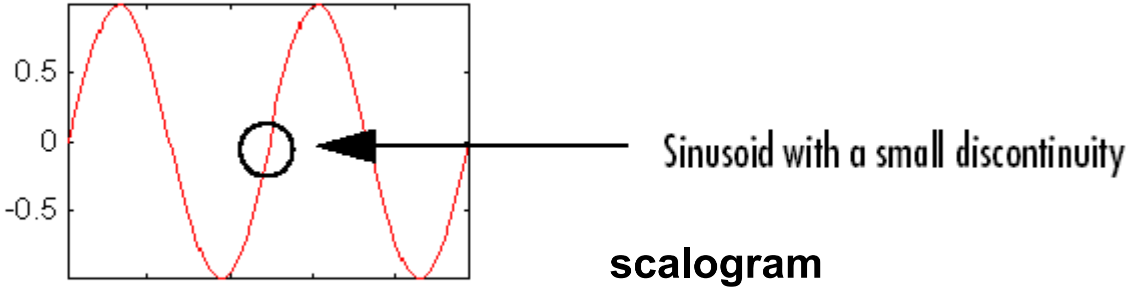
Wavelet representation

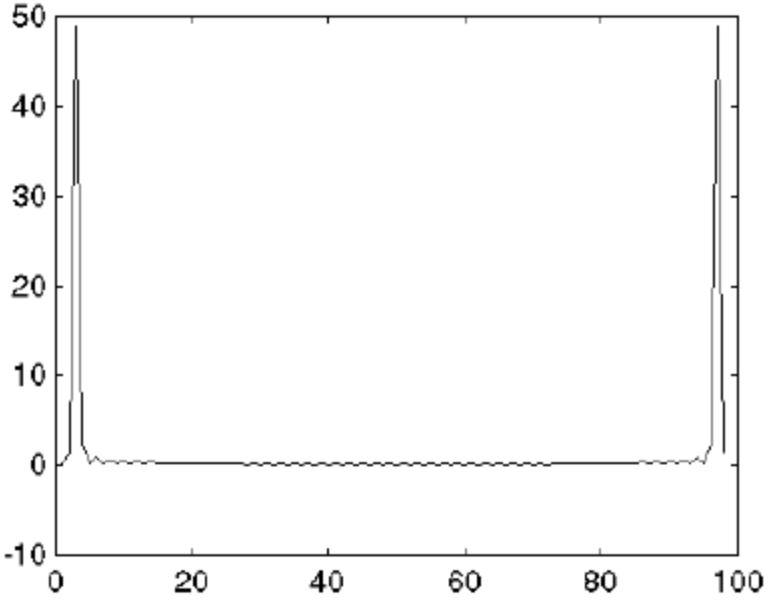


Wavelet representation



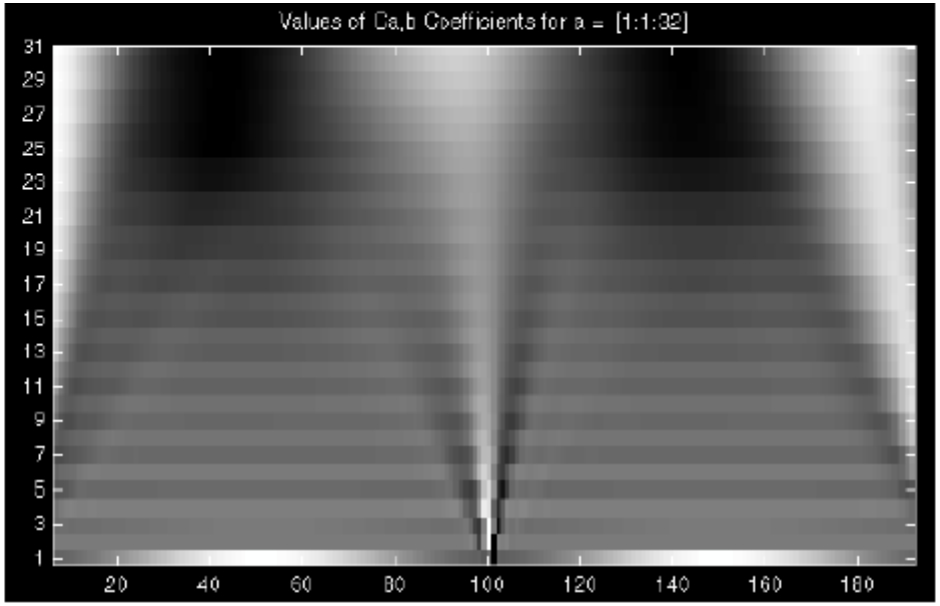
Wavelets are good for transients





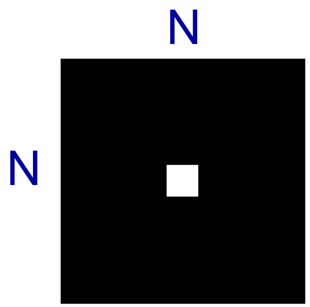
Fourier Coefficients

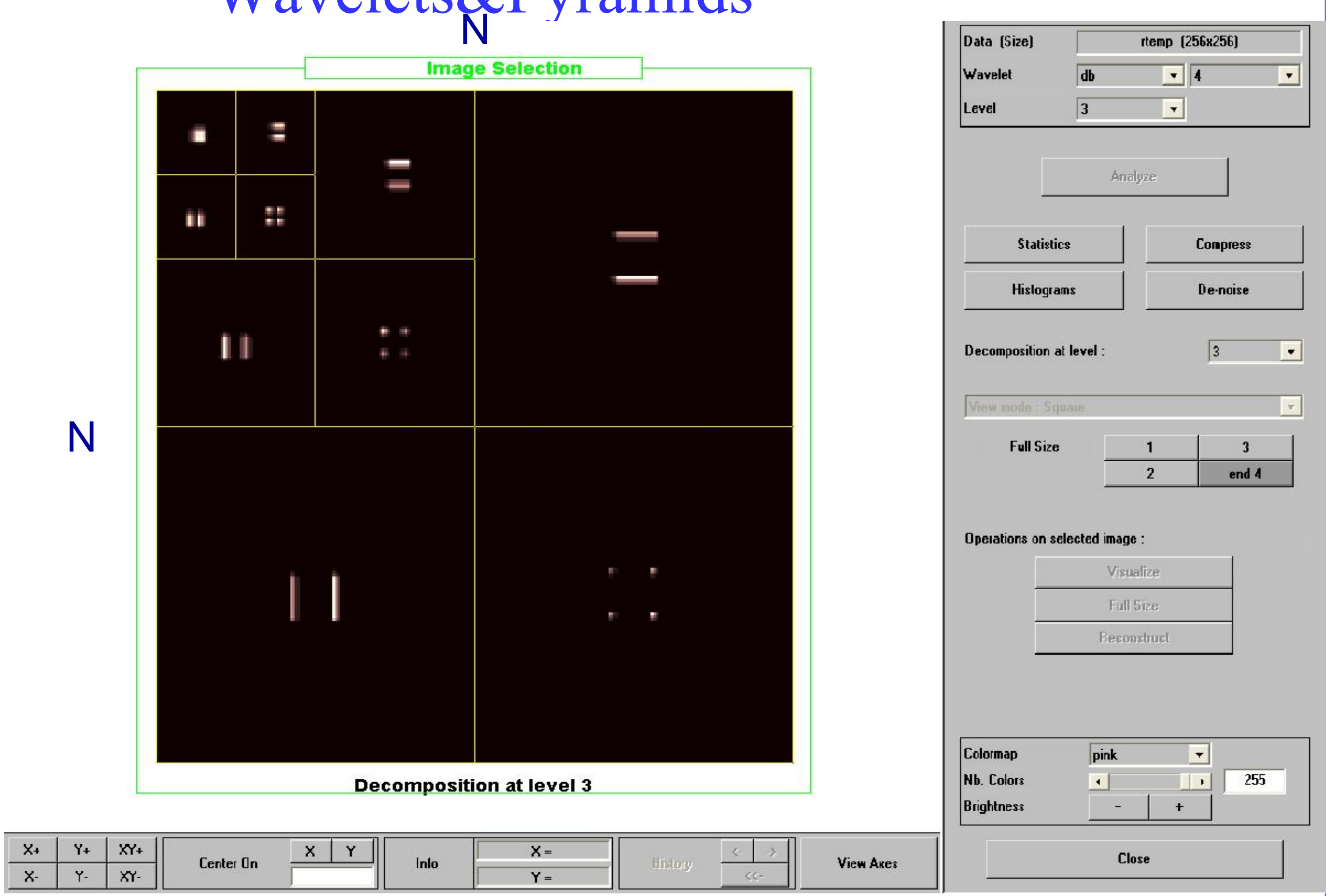
scalogram



Wavelet Coefficients

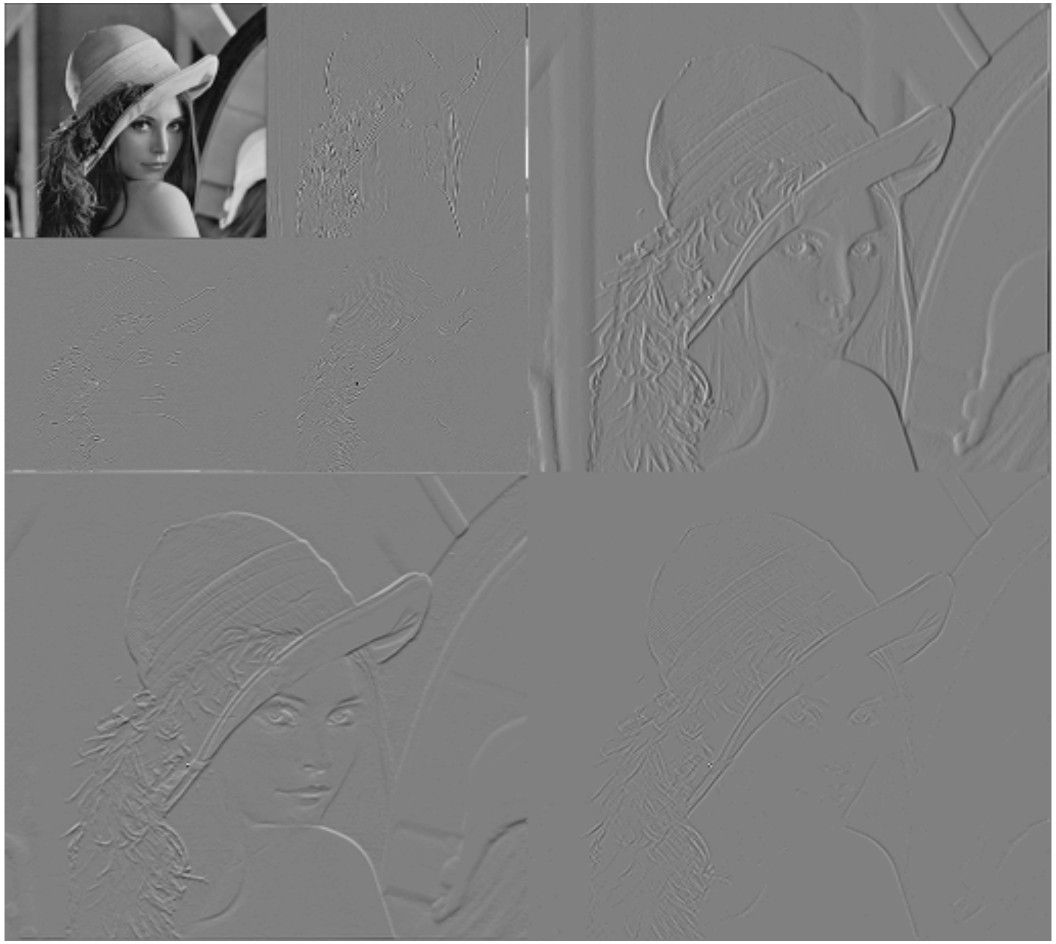
Wavelets&Pyramids



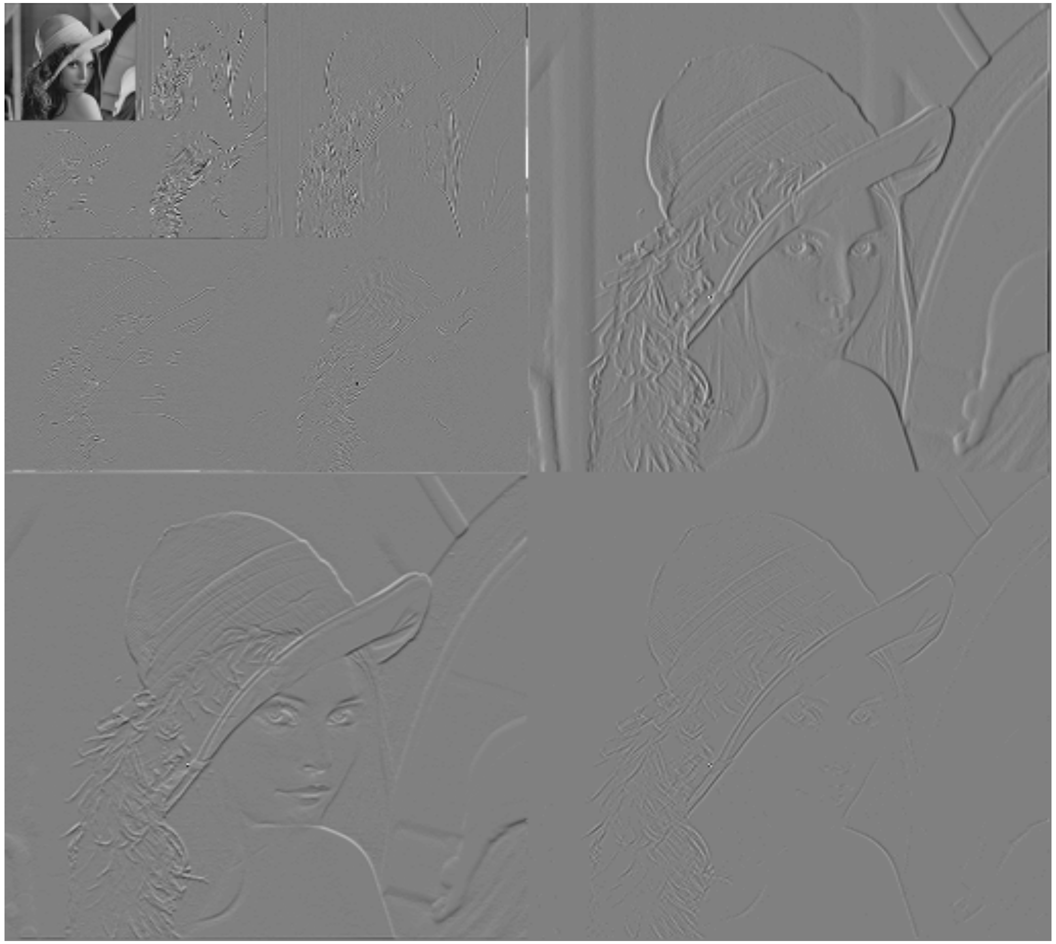


Waveletts&Pyramids

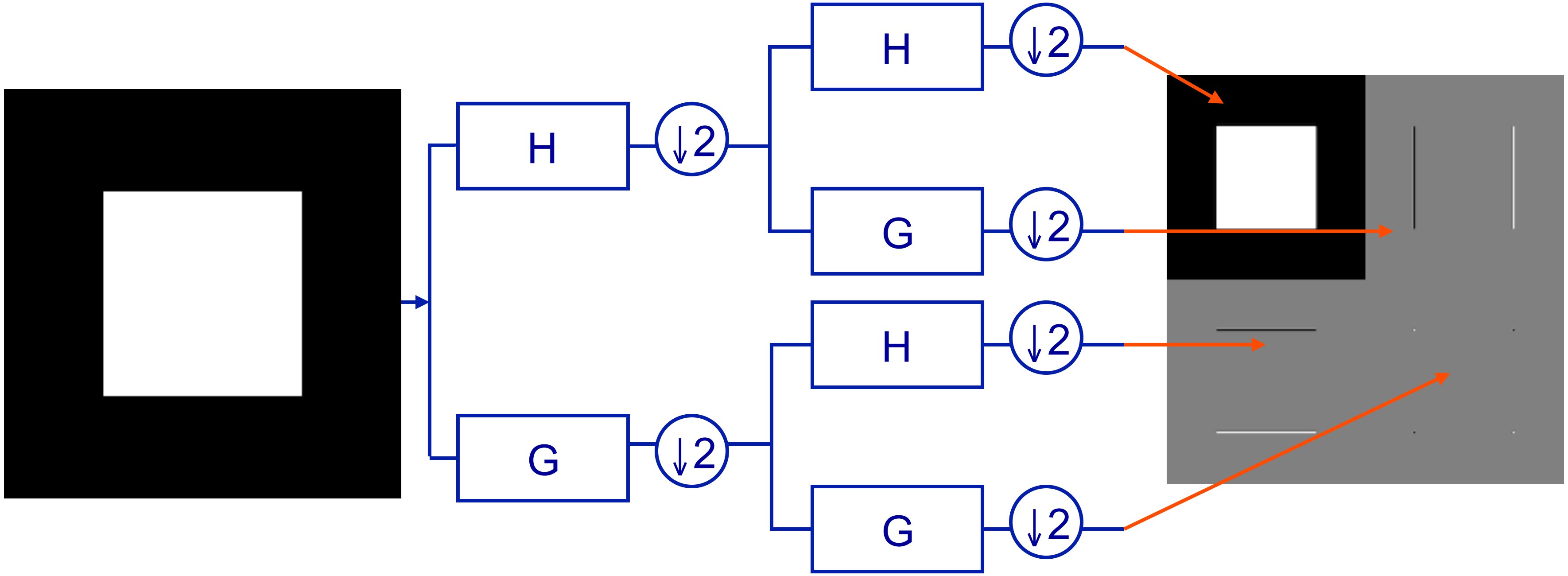
Wavelets&Pyramids



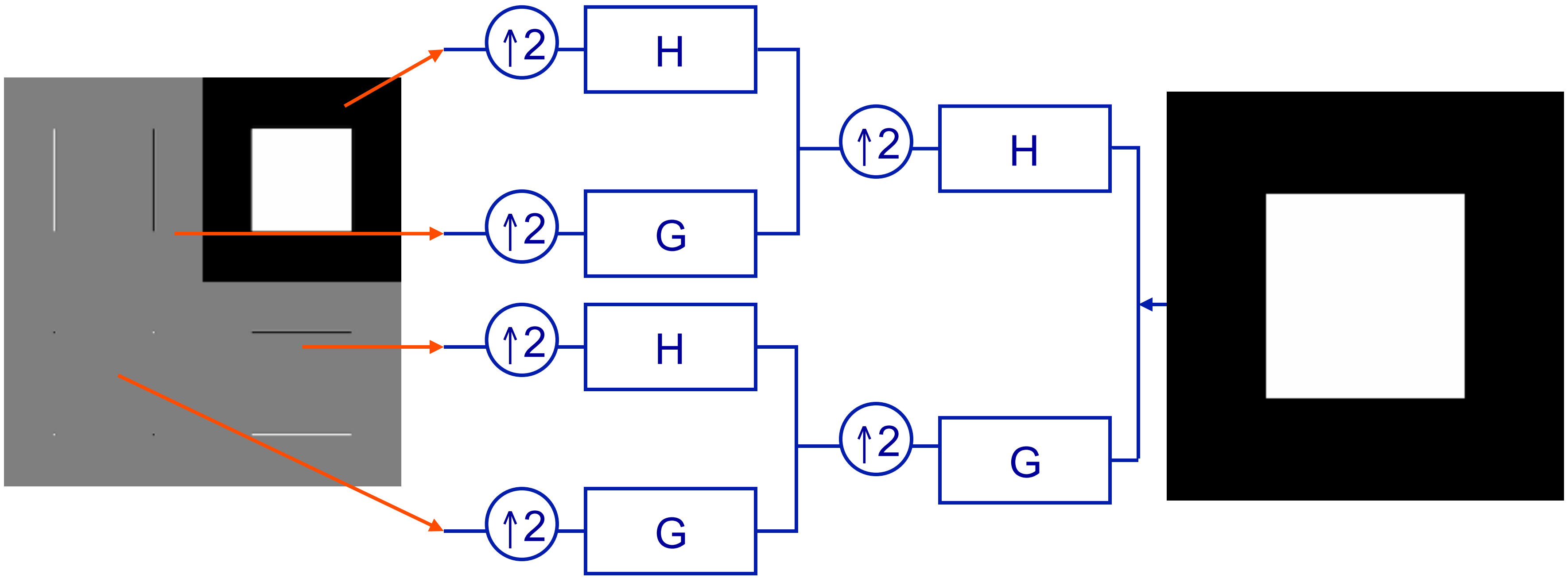
Waveletts&Pyramids



Wavelets&Filterbanks



Waveletss&Filterbanks



Very efficient implementation by recursive filtering

Fourier versus Wavelets

Fourier

- Basis functions are sinusoids
  - More in general, complex exponentials
- Switching from signal domain *t* to frequency domain *f*
  - Either spatial or temporal
- Good localization either in time or in frequency
  - Transformed domain: Information on the sharpness of the transient but not on its position
- Good for stationary signals but unsuitable for transient phenomena

Wavelets

- Different families of basis functions are possible
  - Haar, Daubechies', biorthogonal
- Switching from the signal domain to a *multiresolution* representation
- *Good localization in time and frequency*
  - Information on *both* the *sharpness* of the transient and the *point* where it happens
- Good for any type of signal

Applications

- Compression and coding
  - Critically sampled representations (discrete wavelet transforms, DWT)
- Feature extraction for signal analysis
  - Overcomplete bases (continuous wavelet transform, wavelet frames)
- Image modeling
  - Modeling the human visual system: Objective metrics for visual quality assessment
  - Texture synthesis
- Image enhancement
  - Denoising by wavelet thresholding, deblurring, hole filling
- Image processing on manyfolds
  - Wavelet transform on the sphere: applications in diffusion MRI