Wavelets and filterbanks

Mallat 2009, Chapter 7

Outline

- Wavelets and Filterbanks
- Biorthogonal bases
- The dual perspective: from FB to wavelet bases
  - Biorthogonal FB
  - Perfect reconstruction conditions
- Separable bases (2D)
- Overcomplete bases
  - Wavelet frames (algorithme à trous, DDWF)
  - Curvelets

Wavelets and Filterbanks

Wavelet side

- Scaling function
  - Design (from multiresolution priors)
  - Signal approximation
  - Corresponding filtering operation
    - Condition on the filter h[n] → Conjugate Mirror Filter (CMF)
- Corresponding wavelet families

Filterbank side

- Perfect reconstruction conditions (PR)
  - Reversibility of the transform
- Equivalence with the conditions on the wavelet filters
  - Special case: CMFs → Orhogonal wavelets
  - General case → Biorthogonal wavelets

Wavelets and filterbanks

- The decomposition coefficients in a wavelet orthogonal basis are computed with a fast algorithm that cascades discrete convolutions with h and g, and subsample the output
- Fast orthogonal WT

$$f(t) = \sum_n a_0[n]\varphi(t-n) \in V_0$$

Since $\{\varphi(\text{t-n})\}_{n\in Z}$ is an orthonormal basis

$$a_0[n] = \langle f(t), \varphi(t-n)\rangle = \int_{-\infty}^{+\infty} f(t)\varphi^*(t-n)dt = \int_{-\infty}^{+\infty} f(t)\bar{\varphi}^*(n-t)dt = f * \bar{\varphi}(n)$$

$$\bar{\varphi}(t) = \varphi(-t)$$

Linking the domains

$$z = e^{j\omega}$$

$$\hat{f}(\omega) = \hat{f}(e^{j\omega}) \leftrightarrow f(z)$$

$$\hat{f}(\omega+\pi) = \hat{f}(e^{j(\omega+\pi)}) = \hat{f}(-e^{j\omega}) \leftrightarrow f(-z)$$

$$\hat{f}(-\omega) = \hat{f}(e^{-j\omega}) \leftrightarrow f(z^{-1})$$

$$\hat{f}^*(\omega) = \hat{f}(-\omega) \leftrightarrow f(z^{-1})$$

Switching between the Fourier and the z-domain

$$f[n] \leftrightarrow f(z) = \sum_{k=-\infty}^{+\infty} f[k] z^{-k}$$

$f[n-1] \leftrightarrow z^{-1} f(z)$ unit delay

$f[-n] \leftrightarrow f\left(z^{-1}\right)$ reverse the order of the coefficients

$(-1)^n f[n] \leftrightarrow f(-z)$ negate odd terms

Switching between the time and the z-domain

Fast orthogonal wavelet transform

- Fast FB algorithm that computes the orthogonal wavelet coefficients of a discrete signal $a_0[n]$. Let us define

$$f(t) = \sum_n a_0[n]\varphi(t-n) \in V_0$$

Since $\{\varphi(t-n)\}_{n\in Z}$ is orthonormal, then

$$a_0[n] = \langle f(t), \varphi(t-n)\rangle = f * \bar{\varphi}(n)$$

$$\Rightarrow \quad a_j[n] = \langle f, \varphi_{j,n}\rangle \quad \text{since } \varphi_{j,n} \text{ is an orthonormal basis for } V_j$$

$$d_j[n] = \langle f, \psi_{j,n}\rangle$$

- *A fast wavelet transform decomposes successively each approximation $PV_j f$ in the coarser approximation $PV_{j+1} f$ plus the wavelet coefficients carried by $PW_{j+1} f$.*
- *In the reconstruction, $PV_j f$ is recovered from $PV_{j+1} f$ and $PW_{j+1} f$ for decreasing values of j starting from J (decomposition depth)*

Fast wavelet transform

- Theorem 7.7
  - At the decomposition

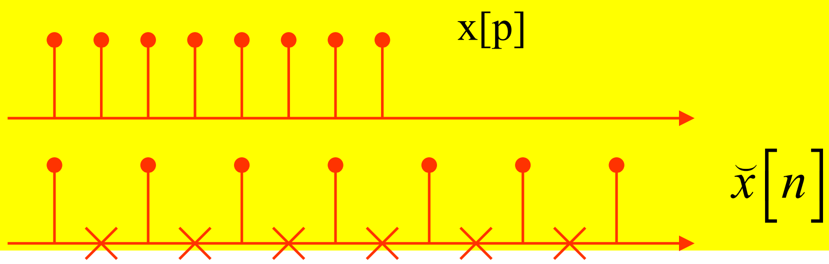
$$a_{j+1}[p] = \sum_{n=-\infty}^{+\infty} h[n-2p]a_j[n] = a_j * \bar{h}[2p] \qquad (1)$$

$$d_{j+1}[p] = \sum_{n=-\infty}^{+\infty} g[n-2p]a_j[n] = a_j * \bar{g}[2p] \qquad (2)$$

  - At the reconstruction

$$a_j[p] = \sum_{n=-\infty}^{+\infty} h[p-2n]a_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n]d_{j+1}[n] = \breve{a}_{j+1} * h[n] + \breve{d}_{j+1} * g[n] \qquad (4)$$

$$\breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$



Proof: decomposition (1)

$$\varphi_{j+1}[p] \in V_{j+1} \subset V_j \rightarrow \varphi_{j+1}[p] = \sum_n \langle \varphi_{j+1}[p], \varphi_j[n] \rangle \varphi_j[n] \text{ (b)}$$

but

$$\langle \varphi_{j+1}[p], \varphi_j[n] \rangle = \int \frac{1}{\sqrt{2^{j+1}}} \varphi\left(\frac{t-2^{j+1}p}{2^{j+1}}\right) \frac{1}{\sqrt{2^j}} \varphi^*\left(\frac{t-2^j n}{2^j}\right) dt \text{ (a)}$$

let

$$t' = 2^{-j}t - 2p \rightarrow t = 2^j t' + 2^{j+1}p \rightarrow t - 2^{j+1}p = 2^j t' \rightarrow \frac{t-2^{j+1}p}{2^{j+1}} = \frac{t'}{2}$$

then

$$\varphi\left(\frac{t-2^{j+1}p}{2^{j+1}}\right) = \varphi\left(\frac{t'}{2}\right)$$

$$\varphi^*\left(\frac{t-2^j n}{2^j}\right) = \varphi^*(t'+2p-n) \qquad \frac{t'}{2} = \frac{t}{2^{j+1}} - p \rightarrow \frac{t}{2^{j+1}} = \frac{t'}{2} + p \rightarrow \frac{t}{2^j} = t'+2p$$

replacing into (a)

$$\langle \varphi_{j+1}[p], \varphi_j[n] \rangle = \int \frac{1}{\sqrt{2}} \varphi\left(\frac{t'}{2}\right) \varphi^*(t'+2p-n)\, dt' = \left\langle \frac{1}{\sqrt{2}} \varphi\left(\frac{t}{2}\right), \varphi(t+2p-n) \right\rangle = h[n-2p] \qquad (3)$$

thus (b) becomes

$$\boxed{\varphi_{j+1}[p] = \sum_n h[n-2p] \varphi_j[n]}$$

Proof: decomposition (2)

- Coming back to the projection coefficients

$$a_{j+1}[p] = \langle f, \varphi_{j+1,p} \rangle = \left\langle f, \sum_n h[n-2p]\varphi_{j,n} \right\rangle = \int_{-\infty}^{+\infty} f \sum_n h[n-2p]\varphi^*_{j,n} dt =$$

$$= \sum_n h[n-2p] \int_{-\infty}^{+\infty} f(t)\varphi^*_{j,n}(t)dt = \sum_n h[n-2p] \langle f, \varphi_{j,n} \rangle = \sum_n h[n-2p] a_j[n] \rightarrow$$

$$\boxed{a_{j+1}[p] = a_j * \bar{h}[2p]}$$

- Similarly, one can prove the relations for both the details and the reconstruction formula

Proof: decomposition (3)

- Details

$$\psi_{j+1,p} \in W_{j+1} \subset V_j \rightarrow \psi_{j+1,p} = \sum_n \langle \psi_{j+1,n}, \varphi_{j,n} \rangle \varphi_{j,n}$$

$$t' = 2^{-j} t - 2p \rightarrow$$

(3bis) $$\langle \psi_{j+1,n}, \varphi_{j,n} \rangle = \left\langle \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right), \varphi(t - n + 2p) \right\rangle = g[n-2p] \rightarrow$$

$$\psi_{j+1,p} = \sum_n g[n-2p] \varphi_{j,n} \rightarrow$$

$$\langle f, \psi_{j+1,n} \rangle = \sum_n g[n-2p] \langle f, \varphi_{j,n} \rangle \rightarrow$$

$$d_{j+1}[p] = \sum_n g[n-2p] a_j[n]$$

Proof: Reconstruction

Since W_{j+1} is the orthonormal complement of V_{j+1} in V_j, the union of the two respective basis is a basis for V_j. Hence

$$V_j = V_{j+1} \oplus W_{j+1} \rightarrow \varphi_{j,p} = \sum_n \langle \varphi_{j,p}, \varphi_{j+1,n} \rangle \varphi_{j+1,n} + \sum_n \langle \varphi_{j,p}, \psi_{j+1,n} \rangle \psi_{j+1,n}$$

but $\langle \varphi_{j,p}, \varphi_{j+1,n} \rangle = h[p-2n]$ (see (3) and (3bis), the analogous one for g)

$$\langle \varphi_{j,p}, \psi_{j+1,n} \rangle = g[p-2n]$$

thus

$$\varphi_{j,p} = \sum_n h[p-2n]\varphi_{j+1,n} + \sum_n g[p-2n]\psi_{j+1,n}$$

Taking the scalar product with f at both sides:

$$a_j[p] = \sum_{n=-\infty}^{+\infty} h[p-2n]a_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n]d_{j+1}[n] = \breve{a}_{j+1} * h[n] + \breve{d}_{j+1} * g[n]$$ CVD

$$\breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$

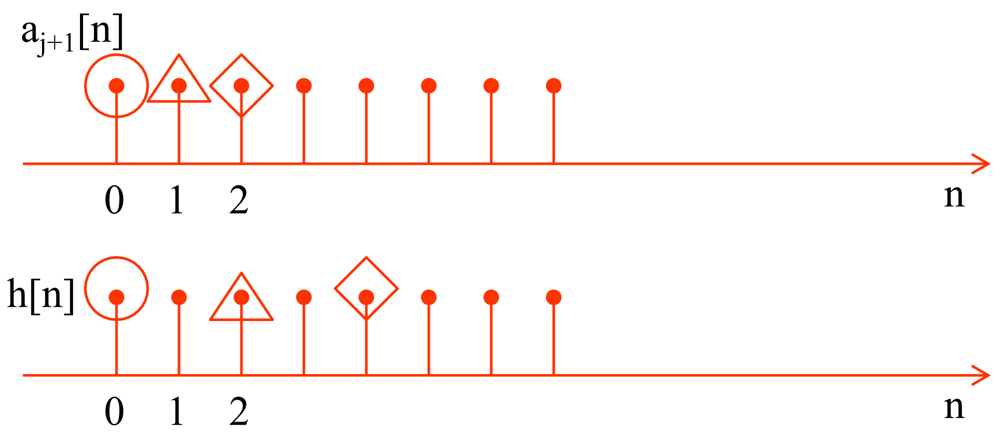
Graphically

$$a_j[p] = \sum_n h[p-2n]a_{j+1}[n] = \sum_n a_{j+1}[n]h[p-2n]$$

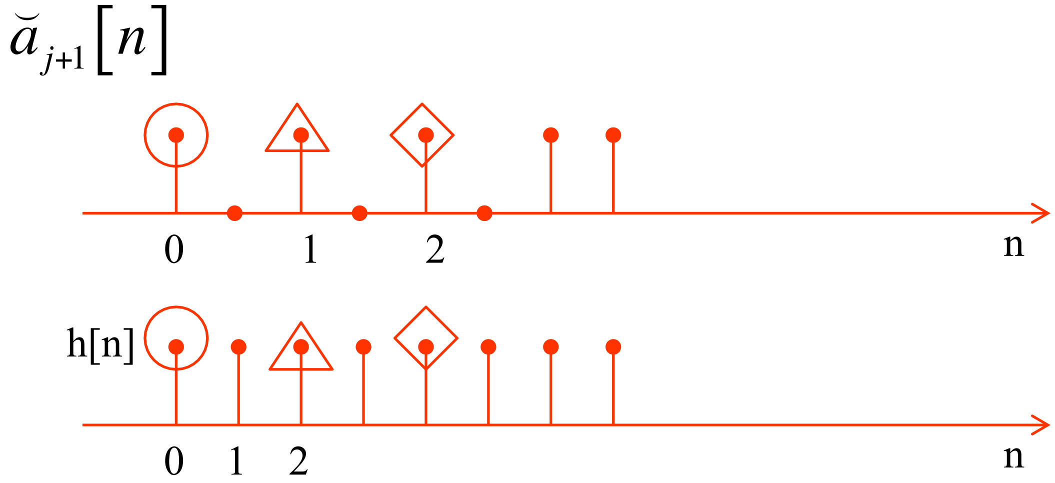
$$a_j[0] = \sum_n h[-2n]a_{j+1}[n] = \sum_n a_{j+1}[n]h[-2n]$$

Let's assume that h is symmetric

$$a_j[0] = \sum_n a_{j+1}[n]h[2n]$$



Graphically

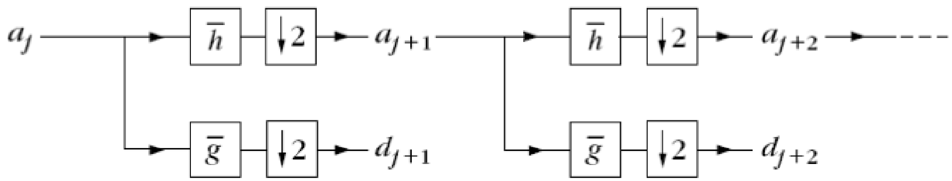


$$a_j[0] = \sum_n a_{j+1}[n]h[2n] = \sum_n \breve{a}_{j+1}[n]h[n]$$

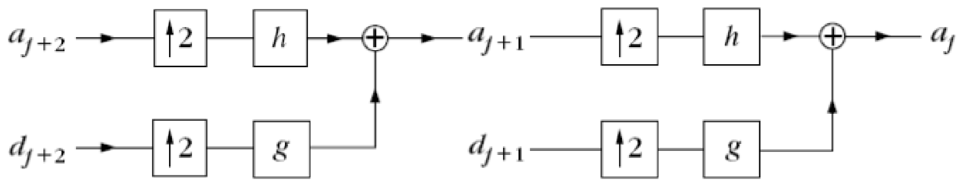
Summary

$$\boxed{a_{j+1}[p] = a_j * \bar{h}[2p]}$$

- The coefficients a_{j+1} and d_{j+1} are computed by taking every other sample of the convolution of a_j with $\bar{h}$ and $\bar{g}$ respectively.
- The filter $\bar{h}$ *removes* the higher frequencies of the inner product sequence a_j , whereas $\bar{g}$ is a high-pass filter that *collects* the remaining highest frequencies.
- The reconstruction is an interpolation that inserts zeroes to expand a_{j+1} and d_{j+1} and filters these signals, as shown in Figure.

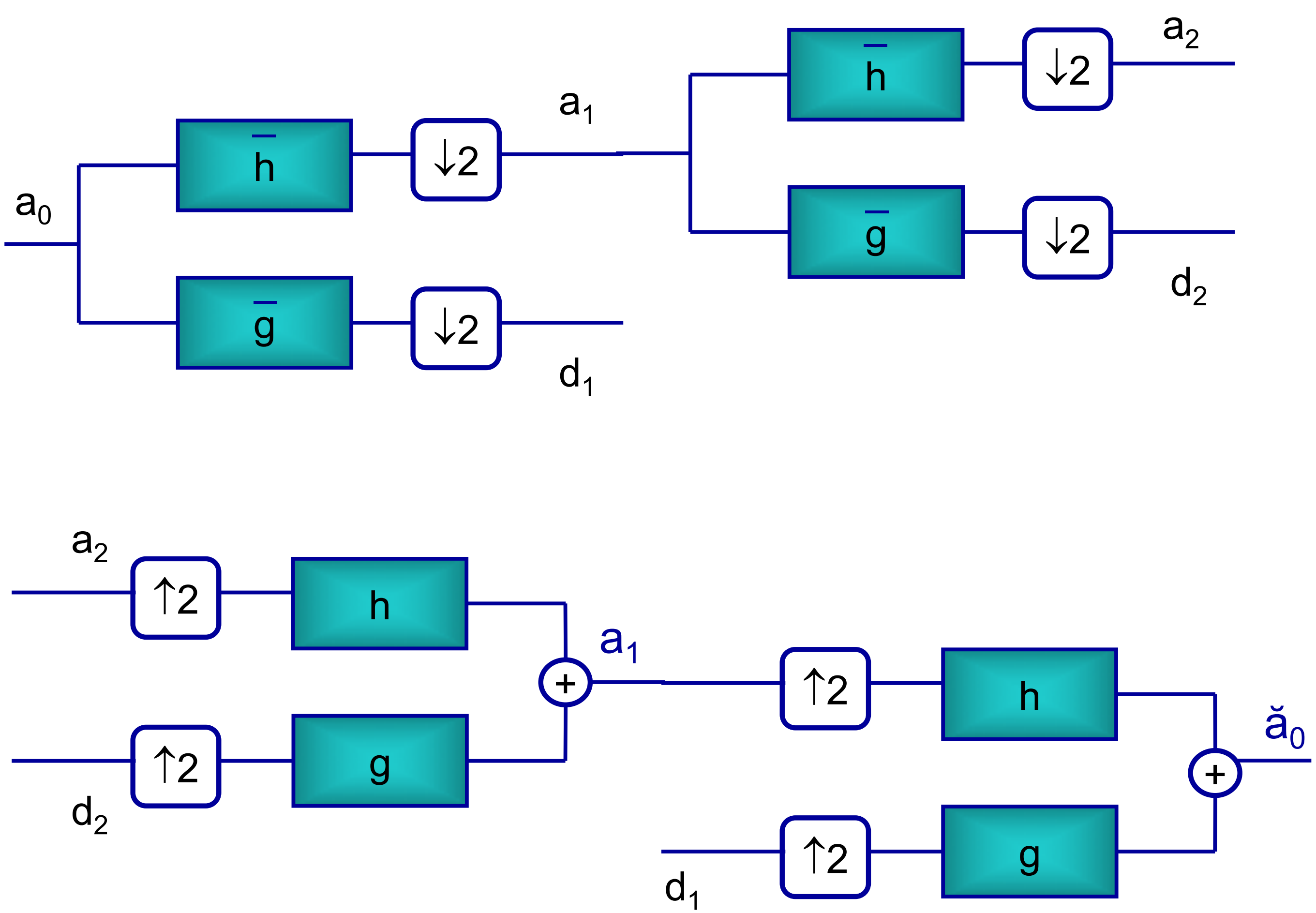


(a)



(b)

Filterbank implementation



Fast DWT

- Theorem 7.10 proves that a_{j+1} and d_{j+1} are computed by taking every other sample of the convolution on a_j with $\bar{h}$ and $\bar{g}$ respectively
- The filter h removes the higher frequencies of the inner product and the filter g is a band-pass filter that collects such residual frequencies
- An orthonormal **wavelet representation** is composed of wavelet coefficients at scales

$$1 \leq 2^j \leq 2^J$$

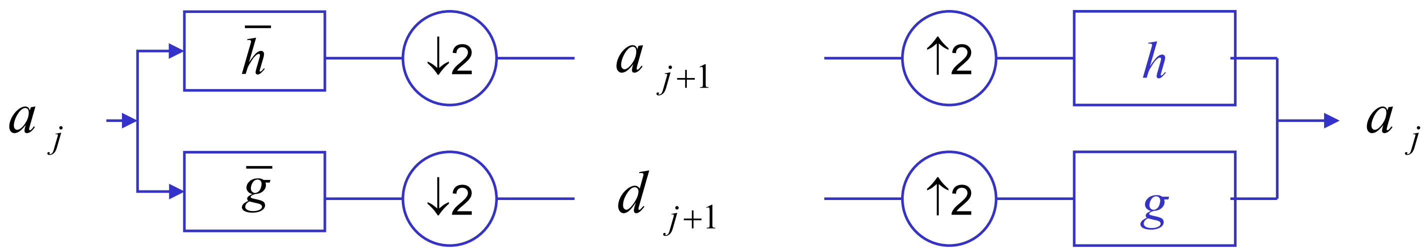
plus the remaining approximation at scale 2^J

$$\left[\left\{ d_j \right\}_{1 \leq j \leq J}, a_J \right]$$

Summary

Analysis or decomposition

Synthesis or reconstruction



Teorem 7.2 (Mallat&Meyer) and **Theorem 7.3** [Mallat&Meyer]

$$\forall \omega \in \mathbb{R}, \qquad \left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2$$

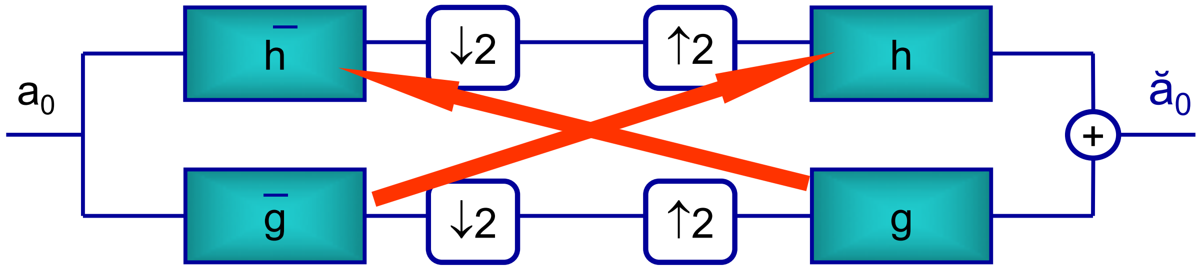
and

$$\hat{h}(0) = \sqrt{2}$$

$$\hat{g}(\omega) = e^{-j\omega}\hat{h}^*(\omega+\pi) \leftrightarrow g[n] = (-1)^{1-n} h[1-n]$$

The fast orthogonal WT is implemented by a filterbank that is completely specified by the filter *h*, which is a CMF
The filters are the same for every *j*

Filter bank perspective



Taking h[n] as reference (which amounts to choosing **the synthesis low-pass filter**) the following relations hold for an orthogonal filter bank:

$$\bar{h}[n] = h[-n]$$

$$g[n] = (-1)^{1-n} h[1-n] = (-1)^{1-n} \bar{h}[n-1]$$

$$\bar{g}[n] = g[-n] = (-1)^{-(1-n)} h[-(1-n)]$$

neglecting the unitary shift, as usually done in applications

$$g[n] = (-1)^{-n} h[-n] = (-1)^{-n} \bar{h}[n]$$

$$\bar{g}[n] = g[-n] = (-1)^{n} h[n]$$

Finite signals

- Issue: signal extension at borders
- Possible solutions:
  - Periodic extension
    - Works with any kind of wavelet
    - Generates large coefficients at the borders
  - Symmetryc/antisymmetric extension, depending on the wavelet symmetry
    - More difficult implementation
    - Haar filter is the only symmetric filter with compact support
  - Use different wavelets at boundary (boundary wavelets)
  - Implementation by *lifting steps*

Wavelet graphs

The graphs of ϕ and ψ are computed numerically with the inverse wavelet transform. If $f=\phi$, then $a_0[n]=\delta[n]$ and $d_j[n]=0$ for all $L<j\leqslant 0$. The inverse wavelet transform computes a_L and (7.111) shows that

$$N^{1/2}\,a_L[n]\approx\phi(N^{-1}n).$$

If ϕ is regular and N is large enough, we recover a precise approximation of the graph of ϕ from a_L.

Similarly, if $f=\psi$, then $a_0[n]=0$, $d_0[n]=\delta[n]$, and $d_j[n]=0$ for $L<j<0$. Then $a_L[n]$ is calculated with the inverse wavelet transform and $N^{1/2}\,a_L[n]\approx\psi(N^{-1}n)$. The Daubechies wavelets and scaling functions in Figure 7.10 are calculated with this procedure.

Initialization

Initialization Most often the discrete input signal $b[n]$ is obtained by a finite resolution device that averages and samples an analog input signal. For example, a CCD camera filters the light intensity by the optics and each photo-receptor averages the input light over its support. A pixel value thus measures average light intensity. If the sampling distance is N^{-1}, to define and compute the wavelet coefficients, we need to associate to $b[n]$ a function $f(t) \in \mathbf{V}_L$ approximated at the scale $2^L = N^{-1}$, and compute $a_L[n] = \langle f, \phi_{L,n} \rangle$. Problem 7.6 explains how to compute $a_L[n] = \langle f, \phi_{L,n} \rangle$ so that $b[n] = f(N^{-1}n)$.

- Initialization
  - Let *b[n]* be the discrete time input signal and let N^{-1} be the sampling period, such that the corresponding scale is $2^L=N^{-1}$
  - Then:

N^{-1}: discrete sample distance

$2^L= N^{-1}$ scale

$$f(t) = \sum_{n=-\infty}^{+\infty} b[n]\, \phi\left(\frac{t-2^L n}{2^L}\right) \in \mathbf{V}_L .$$

original continuous time signal

discrete time signal

interpolation function

Initialization

$$f(t)=\sum_{n=-\infty}^{+\infty} b[n]\,\phi\left(\frac{t-2^L n}{2^L}\right)\in \mathbf{V}_L .$$

following the definition:

N^{-1}: discrete sample distance
$2^L = N^{-1}$ scale

$$\varphi_{L,n}=\frac{1}{\sqrt{2^L}}\varphi\left(\frac{t-2^L n}{2^L}\right)$$ Basis for V_L

$$2^L=\frac{1}{N}\rightarrow\frac{1}{\sqrt{2^L}}=N^{1/2}=\sqrt{N}\rightarrow\varphi_{L,n}=\sqrt{N}\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)\rightarrow\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)=\frac{1}{\sqrt{N}}\varphi_{L,n}$$

but

$$f(t)=\sum_{n=-\infty}^{+\infty} b[n]\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)=\frac{1}{\sqrt{N}}\sum_{n=-\infty}^{+\infty} b[n]\varphi_{L,n}(t)$$

$$b[n]=\left\langle f,\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)\right\rangle=\left\langle f,\frac{1}{\sqrt{N}}\varphi_{L,n}\right\rangle=\frac{1}{\sqrt{N}}a_L[n]$$

$$a_L[n]=\langle f,\varphi_{L,n}\rangle$$

since

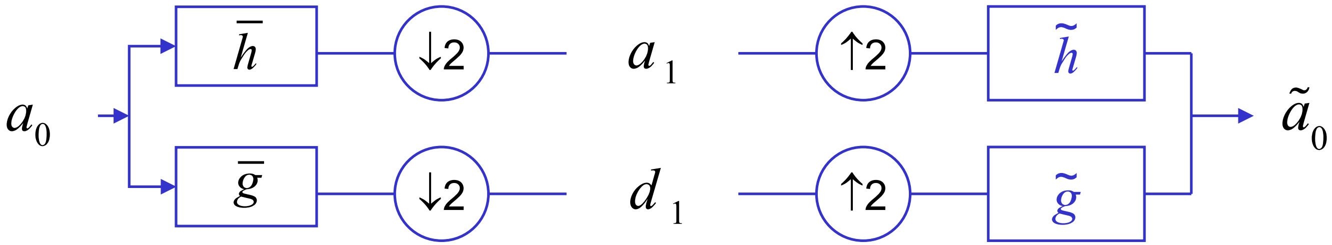
$$a_L[n]=\int_{-\infty}^{+\infty} f(t)\sqrt{N}\varphi\left(\frac{t-N^{-1}n}{N^{-1}}\right)dt$$ by definition, then

$$a_L[n]\approx\sqrt{N}f\left(N^{-1}n\right)$$ if f is regular, the sampled values can be considered as a local average in the neighborhood of $f(N^{-1}n)$

The filter bank perspective

Perfect reconstruction FB

- **Dual perspective**: given a filterbank consisting of 4 filters, we derive the *perfect reconstruction conditions*



- Goal: determine the conditions on the filters ensuring that

$$\boxed{\tilde{a}_0 \equiv a_0}$$

PR Filter banks

- The decomposition of a discrete signal in a multirate filter bank is interpreted as an expansion in $l^2(Z)$

since

$$a_1[l] = a_0 * \bar{h}[2l] = \sum_n a_0[n]\bar{h}[2l-n] = \sum_n a_0[n]h[n-2l]$$

then

$$a_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n]\,h[n-2l] = \langle a_0[n], h[n-2l]\rangle,$$

$$d_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n]\,g[n-2l] = \langle a_0[n], g[n-2l]\rangle.$$

and the signal is recovered by the reconstruction filter

$$a_0[n] = \sum_{l=-\infty}^{+\infty} a_1[l]\,\tilde{h}[n-2l] + \sum_{l=-\infty}^{+\infty} d_1[l]\,\tilde{g}[n-2l].$$

dual family of vectors

thus

$$a_0[n] = \sum_{l=-\infty}^{+\infty} \langle f[k], h[k-2l]\rangle \boxed{\tilde{h}[n-2l]} + \sum_{l=-\infty}^{+\infty} \langle f[k], g[k-2l]\rangle \boxed{\tilde{g}[n-2l].}$$

points to biorthogonal wavelets

The two families are biorthogonal

Theorem 7.13. If h, g, $\tilde{h}$, and $\tilde{g}$ are perfect reconstruction filters, and their Fourier transforms are bounded, then $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l\in\mathbb{Z}}$ and $\{h[n-2l], g[n-2l]\}_{l\in\mathbb{Z}}$ are biorthogonal Riesz bases of $\ell^2(\mathbb{Z})$.

Thus, a PR FB projects a discrete time signals over a biorthogonal basis of $l^2(Z)$.
If the dual basis is the same as the original basis than the projection is orthonormal.

Discrete Wavelet basis

- Question: why bother with the construction of wavelet basis if a PR FB can do the same easily?
- Answer: because conjugate mirror filters are most often used in filter banks that cascade several levels of filterings and subsamplings. Thus, it is necessary to understand the behavior of such a cascade

N^{-1}: discrete sample distance

$2^L = N^{-1}$ scale

$$a_L[n] = \langle f, \varphi_{L,n} \rangle$$ discrete signal at scale 2^L

$$\varphi\left(\frac{t - N^{-1}n}{N^{-1}}\right) = \frac{1}{\sqrt{N}}\varphi_{L,n}$$

for depth j>L

$$a_j[l] = a_L \star \bar{\phi}_j[2^{j-L}l] = \langle a_L[n], \phi_j[n - 2^{j-L}l] \rangle$$

$$d_j[l] = a_L \star \bar{\psi}_j[2^{j-L}l] = \langle a_L[n], \psi_j[n - 2^{j-L}l] \rangle.$$

$$\hat{\phi}_j(\omega) = \prod_{p=0}^{j-L-1} \hat{h}(2^p\omega)$$

$$\hat{\psi}_j(\omega) = \hat{g}(2^{j-L-1}\omega) \prod_{p=0}^{j-L-2} \hat{h}(2^p\omega).$$

Discrete wavelet basis

For conjugate mirror filters, one can verify that this family is an orthonormal basis of $\ell^2(\mathbb{Z})$. These discrete vectors are close to a uniform sampling of the continuous time-scaling functions $\phi_j(t) = 2^{-j/2}\phi(2^{-j}t)$ and wavelets $\psi_j(t) = 2^{-j/2}\phi(2^{-j}t)$. When the number $L-j$ of successive convolutions increases, one can verify that $\phi_j[n]$ and $\psi_j[n]$ converge, respectively, to $N^{-1/2}\,\phi_j(N^{-1}n)$ and $N^{-1/2}\,\psi_j(N^{-1}n)$.

The factor $N^{-1/2}$ normalizes the $\ell^2(\mathbb{Z})$ norm of these sampled functions. If $L-j=4$, then $\phi_j[n]$ and $\psi_j[n]$ are already very close to these limit values. Thus, the impulse responses $\phi_j[n]$ and $\psi_j[n]$ of the filter bank are much closer to continuous time-scaling functions and wavelets than they are to the original conjugate mirror filters h and g. This explains why wavelets provide appropriate models for understanding the applications of these filter banks. Chapter 8 relates more general filter banks to wavelet packet bases.

Perfect reconstruction FB

- **Theorem 7.7 (Vetterli)** The FB performs an exact reconstruction for any input signal iif

$$\boxed{\begin{aligned}&\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega)\hat{\tilde{g}}(\omega)=2\\&\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega+\pi)\hat{\tilde{g}}(\omega)=0\end{aligned}}$$

(alias free)

Matrix notations

$$\begin{pmatrix}\hat{\tilde{h}}^*(\omega)\\ \hat{\tilde{g}}^*(\omega)\end{pmatrix}=\frac{2}{\Delta(\omega)}\begin{pmatrix}\hat{g}(\omega+\pi)\\ -\hat{h}(\omega+\pi)\end{pmatrix}$$

$$\Delta(\omega)=\hat{h}(\omega)\hat{g}(\omega+\pi)-\hat{h}(\omega+\pi)\hat{g}(\omega)$$

When all the filters are FIR, the determinant can be evaluated, which yields simpler relations between the decomposition and the reconstruction filters.

Changing the sampling rate

- Downsampling

$$\hat{y}(2\omega) = \frac{1}{2}\left(\hat{x}(\omega) + \hat{x}(\omega + \pi)\right) = \sum_{n=-\infty}^{+\infty} x[2n] e^{-jn\omega}$$

$$\hat{x}(\omega) \longrightarrow \boxed{\downarrow 2} \longrightarrow \hat{x}_{down}(\omega) = \hat{y}(\omega)$$

$$\hat{y}(\omega) = \frac{1}{2}\left(\hat{x}\left(\frac{\omega}{2}\right) + \hat{x}\left(\frac{\omega}{2} + \pi\right)\right)$$

- Upsampling

$$\hat{y}(\omega) = \hat{x}(2\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2n\omega}$$

$$\hat{x}(\omega) \longrightarrow \boxed{\uparrow 2} \longrightarrow \hat{x}_{up}(\omega) = \hat{y}(\omega)$$

Subsampling: proof

$$\hat{y}(\omega) = \ldots y[0] + y[1]e^{-j\omega} + y[2]e^{-j2\omega} + \ldots =$$

$$= \ldots x[0] + x[2]e^{-j\omega} + x[4]e^{-j2\omega} + \ldots \rightarrow$$

thus

$$\hat{y}(2\omega) = \ldots x[0] + x[2]e^{-j2\omega} + x[4]e^{-j4\omega} + \ldots$$

but

$$x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)} = 0 \rightarrow \frac{1}{2}\left(x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)}\right) = 0$$

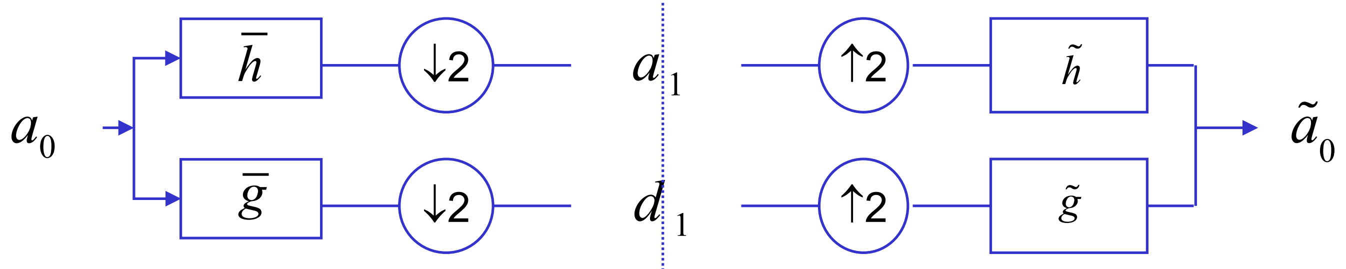
$$x[2]e^{-j2\omega} = \frac{1}{2}\left(x[2]e^{-j2\omega} + x[2]e^{-j2(\omega+\pi)}\right)$$

thus

$$\hat{y}(2\omega) = \ldots x[0] + \frac{1}{2}\left(x[1]e^{-j\omega} + x[1]e^{-j(\omega+\pi)}\right) + \frac{1}{2}\left(x[2]e^{-j2\omega} + x[2]e^{-j2(\omega+\pi)}\right) + \ldots =$$

$$\hat{y}(2\omega) = \frac{1}{2}\left(\hat{x}(\omega) + \hat{x}(\omega+\pi)\right)$$

Perfect Reconstruction conditions



$$a_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{\bar{h}}(\omega) + a_0(\omega+\pi)\hat{\bar{h}}(\omega+\pi)\right)$$

since h and g are real

$$h[n] \rightarrow h(\omega)$$

$$h[-n] = \bar{h}[n] \rightarrow \hat{\bar{h}}(\omega) = \hat{h}(-\omega) = h^*(\omega)$$

thus, replacing in the first equation

$$a_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{h}^*(\omega) + a_0(\omega+\pi)\hat{h}^*(\omega+\pi)\right)$$

Similarly, for the high-pass branch

$$d_1(2\omega) = \frac{1}{2}\left(a_0(\omega)\hat{g}^*(\omega) + a_0(\omega+\pi)\hat{g}^*(\omega+\pi)\right)$$

$$\hat{\tilde{a}}_0(\omega) = \hat{a}_1(2\omega)\hat{\tilde{h}}(\omega) + \hat{d}_1(2\omega)\hat{\tilde{g}}(\omega)$$

Perfect Reconstruction conditions

- Putting all together

$$\hat{\tilde{a}}_0(\omega) = \hat{a}_1(2\omega)\hat{\tilde{h}}(\omega) + \hat{d}_1(2\omega)\hat{\tilde{g}}(\omega) =$$

$$= \frac{1}{2}\left(a_0\left(\omega\right)\hat{h}^*\left(\omega\right) + a_0\left(\omega+\pi\right)\hat{h}^*\left(\omega+\pi\right)\right)\hat{\tilde{h}}(\omega)$$

$$+ \frac{1}{2}\left(a_0\left(\omega\right)\hat{g}^*\left(\omega\right) + a_0\left(\omega+\pi\right)\hat{g}^*\left(\omega+\pi\right)\right)\hat{\tilde{g}}(\omega)$$

$$\hat{\tilde{a}}_0(\omega) = \frac{1}{2}\underbrace{\left(\hat{h}^*\left(\omega\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega\right)\hat{\tilde{g}}(\omega)\right)}_{=1} a_0\left(\omega\right) + \frac{1}{2}\underbrace{\left(\hat{h}^*\left(\omega+\pi\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega+\pi\right)\hat{\tilde{g}}(\omega)\right)}_{=0 \quad \text{(alias-free)}} a_0\left(\omega+\pi\right)$$

$$\boxed{\begin{aligned} &\hat{h}^*\left(\omega\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega\right)\hat{\tilde{g}}(\omega) = 2 \\ &\hat{h}^*\left(\omega+\pi\right)\hat{\tilde{h}}(\omega) + \hat{g}^*\left(\omega+\pi\right)\hat{\tilde{g}}(\omega) = 0 \end{aligned}}$$

(alias free)

Matrix notations

$$\begin{pmatrix} \hat{\tilde{h}}^*(\omega) \\ \hat{\tilde{g}}^*(\omega) \end{pmatrix} = \frac{2}{\Delta(\omega)} \begin{pmatrix} \hat{g}(\omega+\pi) \\ -\hat{h}(\omega+\pi) \end{pmatrix}$$

$$\Delta(\omega) = \hat{h}(\omega)\hat{g}(\omega+\pi) - \hat{h}(\omega+\pi)\hat{g}(\omega)$$

Perfect reconstruction biorhogonal filters

- Theorem 7.8. Perfect reconstruction filters also satisfy

$$\boxed{\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2}$$

Furthermore, if the filters have a finite impulse response there exists *a in R* and *l in Z* such that

$$\hat{g}(\omega)=ae^{-i(2l+1)\omega}\hat{\tilde{h}}^*(\omega+\pi)$$
$$\hat{\tilde{g}}(\omega)=\frac{1}{a}e^{-i(2l+1)\omega}\hat{h}^*(\omega+\pi)$$

$\Rightarrow$ *a=1, l=0* $\Rightarrow$

$$\hat{g}(\omega)=e^{-j\omega}\hat{\tilde{h}}^*(\omega+\pi)$$
$$\hat{\tilde{g}}(\omega)=e^{-j\omega}h^*(\omega+\pi)$$

Correspondingly

$$g[n]=(-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n]=(-1)^{1-n}h[1-n]$$

- Conjugate Mirror Filters: $\tilde{h}=h\rightarrow\boxed{\left|\hat{h}(\omega)\right|^2+\left|\hat{h}(\omega+\pi)\right|^2=2}$

Perfect reconstruction biorthogonal filters

Given h and $\tilde{h}$ and setting *a=1* and *l=0* in (2) the remaining filters are given by the following relations

(3)
$$\hat{g}(\omega) = e^{-i\omega}\hat{\tilde{h}}^*(\omega+\pi)$$
$$\hat{\tilde{g}}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi)$$

- The filters h and $\tilde{h}$ are related to the scaling functions φ and ~φ via the corresponding two-scale relations, as was the case for the orthogonal filters (see eq. 1).

Switching to the z-domain

$$g(z) = z^{-1}\tilde{h}(-z^{-1})$$
$$\tilde{g}(z) = z^{-1}h(-z^{-1})$$

Signal domain

$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$

Biorthogonal filter banks

- A 2-channel **multirate** filter bank convolves a signal a_0 with

  a low pass filter $\bar{h}[n] = h[-n]$

  and a high pass filter $\bar{g}[n] = g[-n]$

  and sub-samples the output by 2

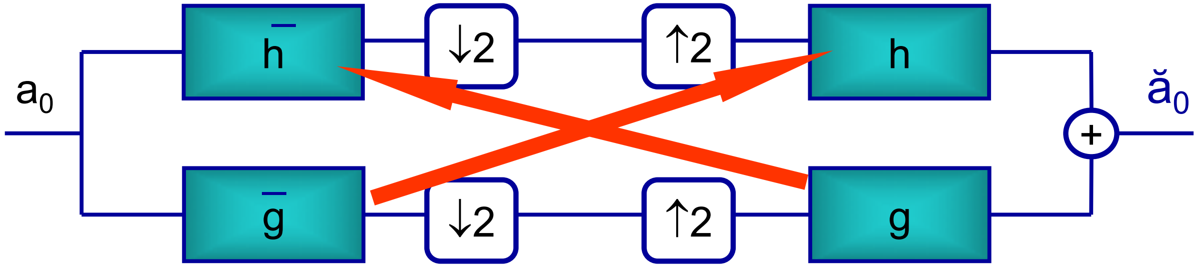
$$a_1[n] = a_0 * \bar{h}[2n]$$
$$d_1[n] = a_0 * \bar{g}[2n]$$

A reconstructed signal $\tilde{a}_0$ is obtained by filtering the zero-expanded signals with a *dual low-pass* $\tilde{h}[n]$ *and high pass filter* $\tilde{g}[n]$

$$\tilde{a}_0[n] = \breve{a}_1 * \tilde{h}[n] + \breve{d}_1 * \tilde{g}[n]$$
$$y[n] = \breve{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & n = 2p+1 \end{cases}$$

Imposing the PR condition (output signal=input signal) one gets the relations that the different filters must satisfy (Theorem 7.7)

Revisiting the orthogonal case (CMF)

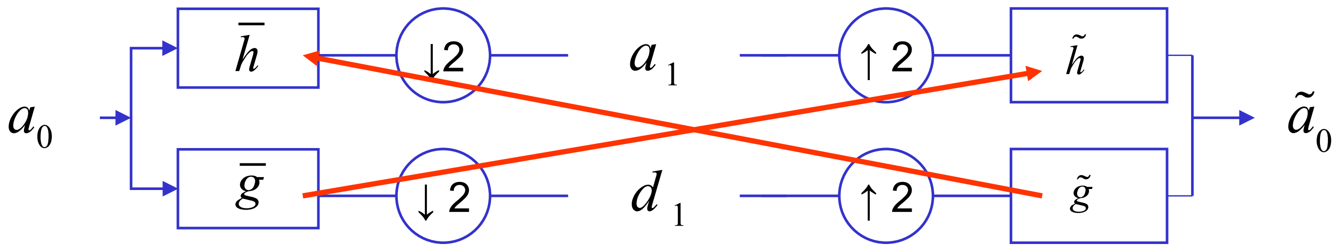


Taking $\bar{h}[n] = h[-n]$ as reference (which amounts to choosing the analysis low-pass filter) the following relations hold for an orthogonal filter bank:

$\bar{h}[n] = h[-n] \leftrightarrow h[n] = \bar{h}[-n]$ synthesis low-pass (interpolation) filter: reverse the order of the coefficients

$g[n] = (-1)^{1-n} h[1-n]$ negate every other sample

Orthogonal vs biorthogonal PRFB



$\tilde{h} \neq h$ *Biorthogonal* PRFB

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega) + \hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi) = 2$$

$$\hat{g}(\omega) = e^{-j\omega}\hat{\tilde{h}}^*(\omega+\pi)$$

$$\hat{\tilde{g}}(\omega) = e^{-j\omega}h^*(\omega+\pi)$$

In the signal domain

$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$

$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$

$\tilde{h} = h$ *Orthogonal* PRFB

$$\left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2$$

$$\tilde{g} = g$$

Fast BWT

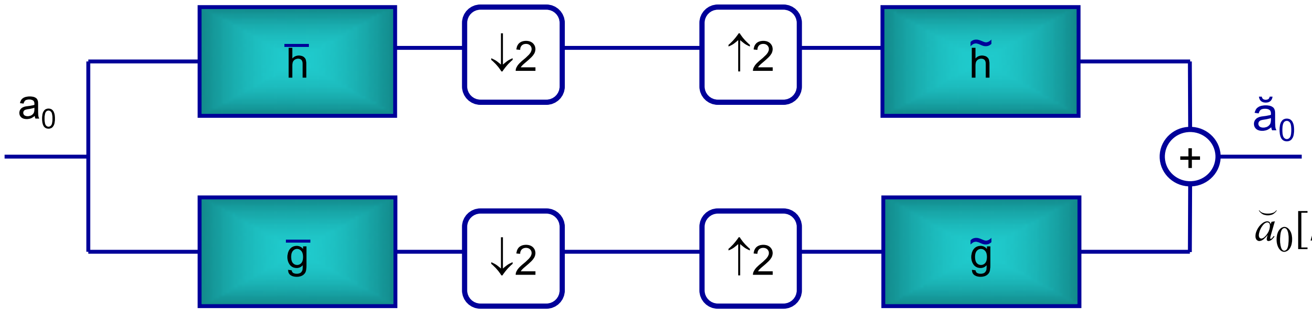
- Two different sets of basis functions are used for analysis and synthesis

$$a_{j+1}[n] = a_j * \bar{h}[2n]$$
$$d_{j+1}[n] = a_j * \bar{g}[2n]$$

$$a_j[n] = \breve{a}_{j+1} * \tilde{h}[n] + \breve{d}_{j+1} * \tilde{g}[n]$$

- PR filterbank

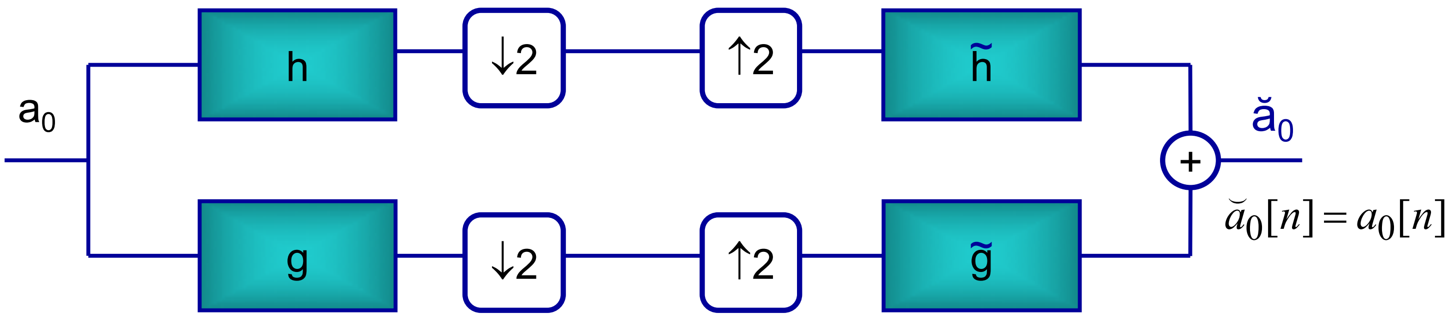
$$g[n] = (-1)^{1-n}\tilde{h}[1-n]$$
$$\tilde{g}[n] = (-1)^{1-n}h[1-n]$$



$$\breve{a}_0[n] = a_0[n]$$

Be careful with notations!

- In the simplified notation where
  - *h[n]* is the analysis low pass filter and *g[n]* is the analysis band pass filter, as it is the case in most of the literature;
  - the delay factor is not made explicit;
- The relations among the filters modify as follows



$$g[n] = (-1)^{-n} \tilde{h}[n]$$

$$\tilde{g}[n] = (-1)^{-n} h[n]$$

Slightly different formulation: the high pass filters are obtained by the low pass filters by negating the odd terms

Biorthogonal bases

Orthonormal basis

$\{e_n\}_{n \in N}$: basis of Hilbert space

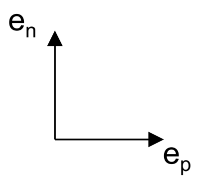
Ortogonality condition $\langle e_n, e_p \rangle = 0 \quad \forall n \neq p$

$\forall y \in H$,

There exists a sequence

$$\lambda[n] = \langle y, e_n \rangle :$$
$$y = \sum_n \lambda[n] e_n$$

$|e_n|^2 = 1$ ortho-*normal* basis



Bi-orthogonal basis

$\{e_n\}_{n \in N}$: linearly independent

$\forall y \in H, \quad \exists A > 0$ and $B > 0$:

$$\lambda[n] = \langle y, e_n \rangle :$$
$$y = \sum_n \lambda[n] \tilde{e}_n$$
$$\frac{|y|^2}{B} \leq \sum_n |\lambda[n]|^2 \leq \frac{|y|^2}{A}$$

Biorthogonality condition:

$$\langle e_n, \tilde{e}_p \rangle = \delta[n-p]$$

$$y = \sum_n \langle f, \tilde{e}_n \rangle e_n = \sum_n \langle f, e_n \rangle \tilde{e}_n$$

$A = B = 1 \Rightarrow$ orthogonal basis

Biorthogonal bases

If h and $\tilde{h}$ are FIR

$$\hat{\tilde{\Phi}}(\omega)=\prod_{p=1}^{+\infty}\frac{\hat{\tilde{h}}\left(2^{-p}\omega\right)}{\sqrt{2}}\hat{\tilde{\Phi}}(0), \qquad \hat{\Phi}(\omega)=\prod_{p=1}^{+\infty}\frac{\hat{h}\left(2^{-p}\omega\right)}{\sqrt{2}}\hat{\Phi}(0)$$

Though, some other conditions must be imposed to guarantee that φ^ and φ^tilde are FT of finite energy functions. The theorem from Cohen, Daubechies and Feaveau provides *sufficient* conditions (Theorem 7.10 in M1999 and Theorem 7.13 in M2009)

The functions $\hat{\phi}$ and $\hat{\tilde{\phi}}$ satisfy the biorthogonality relation

$$\boxed{\left\langle \varphi(t),\tilde{\varphi}(t-n)\right\rangle=\delta[n]}$$

The two wavelet families $\left\{\psi_{j,n}\right\}_{(j,n)\in Z^2}$ and $\left\{\tilde{\psi}_{j,n}\right\}_{(j,n)\in Z^2}$ are Riesz bases of $L^2(R)$

$$\boxed{\left\langle \psi_{j,n},\tilde{\psi}_{j',n'}\right\rangle=\delta[n-n']\delta[j-j']}$$

Any $f\in L^2\left(R\right)$ has two possible decompositions in these bases

$$\boxed{f=\sum_{n,j}\left\langle f,\psi_{j,n}\right\rangle\tilde{\psi}_{j,n}=\sum_{n,j}\left\langle f,\tilde{\psi}_{j,n}\right\rangle\psi_{j,n}}$$

Reminder

Theorem 7.13. If h, g, $\tilde{h}$, and $\tilde{g}$ are perfect reconstruction filters, and their Fourier transforms are bounded, then $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l\in\mathbb{Z}}$ and $\{h[n-2l], g[n-2l]\}_{l\in\mathbb{Z}}$ are biorthogonal Riesz bases of $\ell^2(\mathbb{Z})$.

Summary of Biorthogonality relations

- An infinite cascade of PR filter banks $(h,g),(\tilde{h},\tilde{g})$ yields two scaling functions and two wavelets whose Fourier transform satisfy

$$\hat{\Phi}(2\omega)=\frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\Phi}(\omega) \quad \leftrightarrow \quad \varphi\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} h[n]\varphi(t-n) \qquad (i)$$

$$\hat{\tilde{\Phi}}(2\omega)=\frac{1}{\sqrt{2}}\hat{\tilde{h}}(\omega)\hat{\tilde{\Phi}}(\omega) \quad \leftrightarrow \quad \tilde{\varphi}\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} \tilde{h}[n]\tilde{\varphi}(t-n) \qquad (ii)$$

$$\hat{\Psi}(2\omega)=\frac{1}{\sqrt{2}}\hat{g}(\omega)\hat{\Phi}(\omega) \quad \leftrightarrow \quad \psi\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} g[n]\varphi(t-n) \qquad (iii)$$

$$\hat{\tilde{\Psi}}(2\omega)=\frac{1}{\sqrt{2}}\hat{\tilde{g}}(\omega)\hat{\tilde{\Phi}}(\omega) \quad \leftrightarrow \quad \tilde{\psi}\left(\frac{t}{2}\right)=\sum_{n=-\infty}^{+\infty} \tilde{g}[n]\tilde{\varphi}(t-n) \qquad (iv)$$

Properties of biorthogonal filters

Imposing the zero average condition to ψ in equations *(iii)* and *(iv)*

$$\hat{\Psi}(0) = \hat{\tilde{\Psi}}(0) = 0 \rightarrow \boxed{\hat{g}(0) = \hat{\tilde{g}}(0) = 0}$$

replacing into the relations (3) (also shown below)

$$\hat{g}(\omega) = e^{-i\omega}\hat{\tilde{h}}^{*}(\omega+\pi) \qquad \hat{\tilde{g}}(\omega) = e^{-i\omega}\hat{h}^{*}(\omega+\pi) \rightarrow \boxed{\hat{h}^{*}(\pi) = \hat{\tilde{h}}(\pi) = 0}$$

Furthermore, replacing such values in the PR condition (1)

$$\hat{h}^{*}(\omega)\hat{\tilde{h}}(\omega) + \hat{g}^{*}(\omega)\hat{\tilde{g}}(\omega) = 2 \rightarrow \boxed{\hat{h}^{*}(0)\hat{\tilde{h}}(0) = 2}$$

It is common choice to set

$$\boxed{\hat{h}^{*}(0) = \hat{\tilde{h}}(0) = \sqrt{2}}$$

Biorthogonal bases

- If the decomposition and reconstruction filters are different, the resulting bases is non-orthogonal
- The cascade of J levels is equivalent to a signal decomposition over a non-orthogonal basis

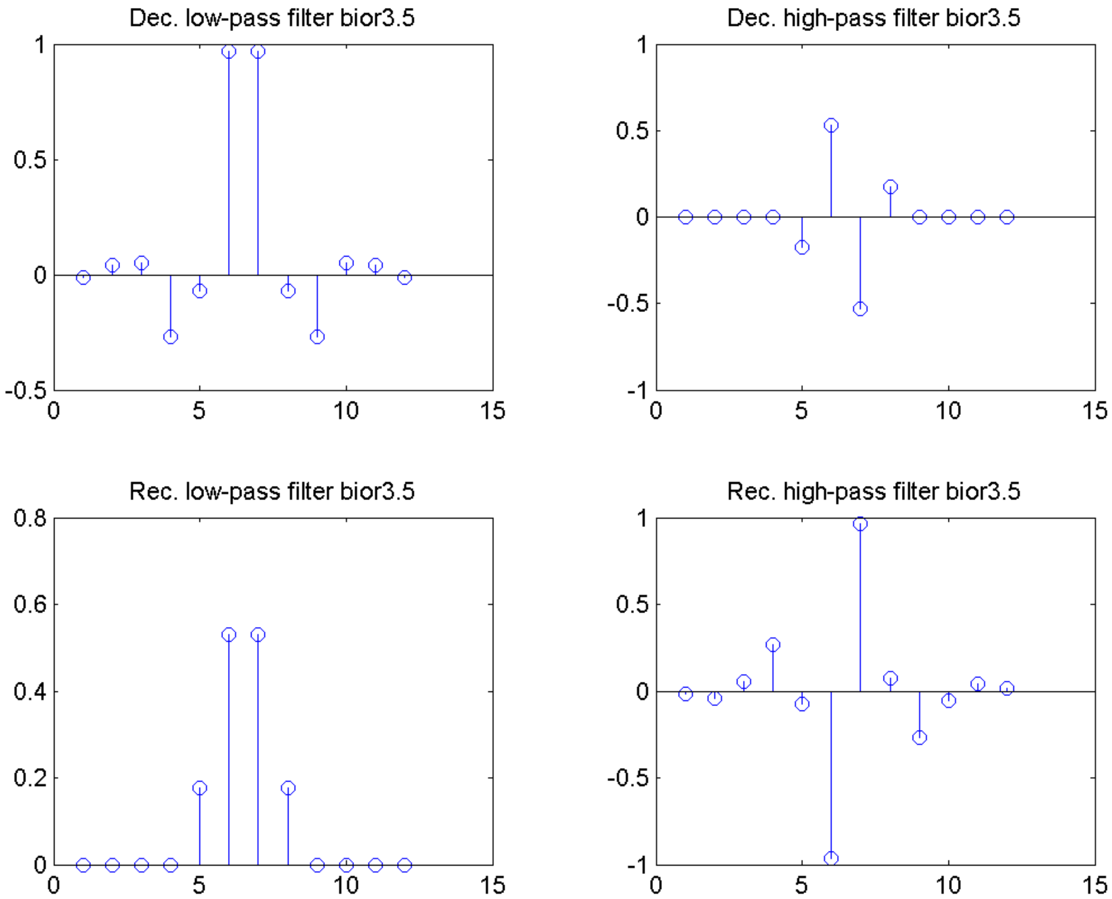
$$\left[\left\{\varphi_J\left[k-2^J n\right]\right\}_{n\in Z},\left\{\psi_j\left[k-2^j n\right]\right\}_{1\le j\le J,n\in Z}\right]$$

- The dual bases is needed for reconstruction

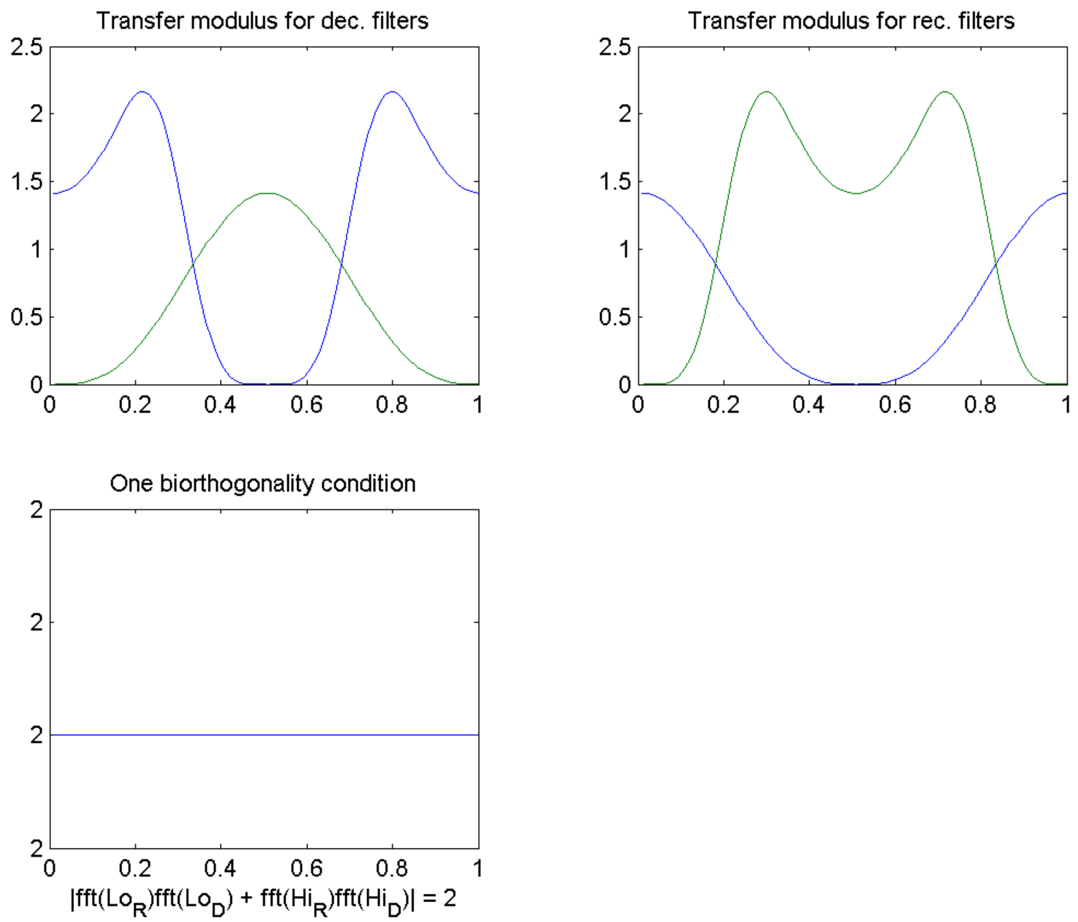
In Matlab

- Biorthogonal Wavelet Pairs: biorNr.Nd
- While the Haar wavelet is the only orthogonal wavelet with linear phase, you can design biorthogonal wavelets with linear phase.
- Biorthogonal wavelets feature a pair of scaling functions and associated scaling filters — one for analysis and one for synthesis.
- There is also a pair of wavelets and associated wavelet filters — one for analysis and one for synthesis.
- The analysis and synthesis wavelets can have different numbers of vanishing moments and regularity properties. You can use the wavelet with the greater number of vanishing moments for analysis resulting in a sparse representation, while you use the smoother wavelet for reconstruction.

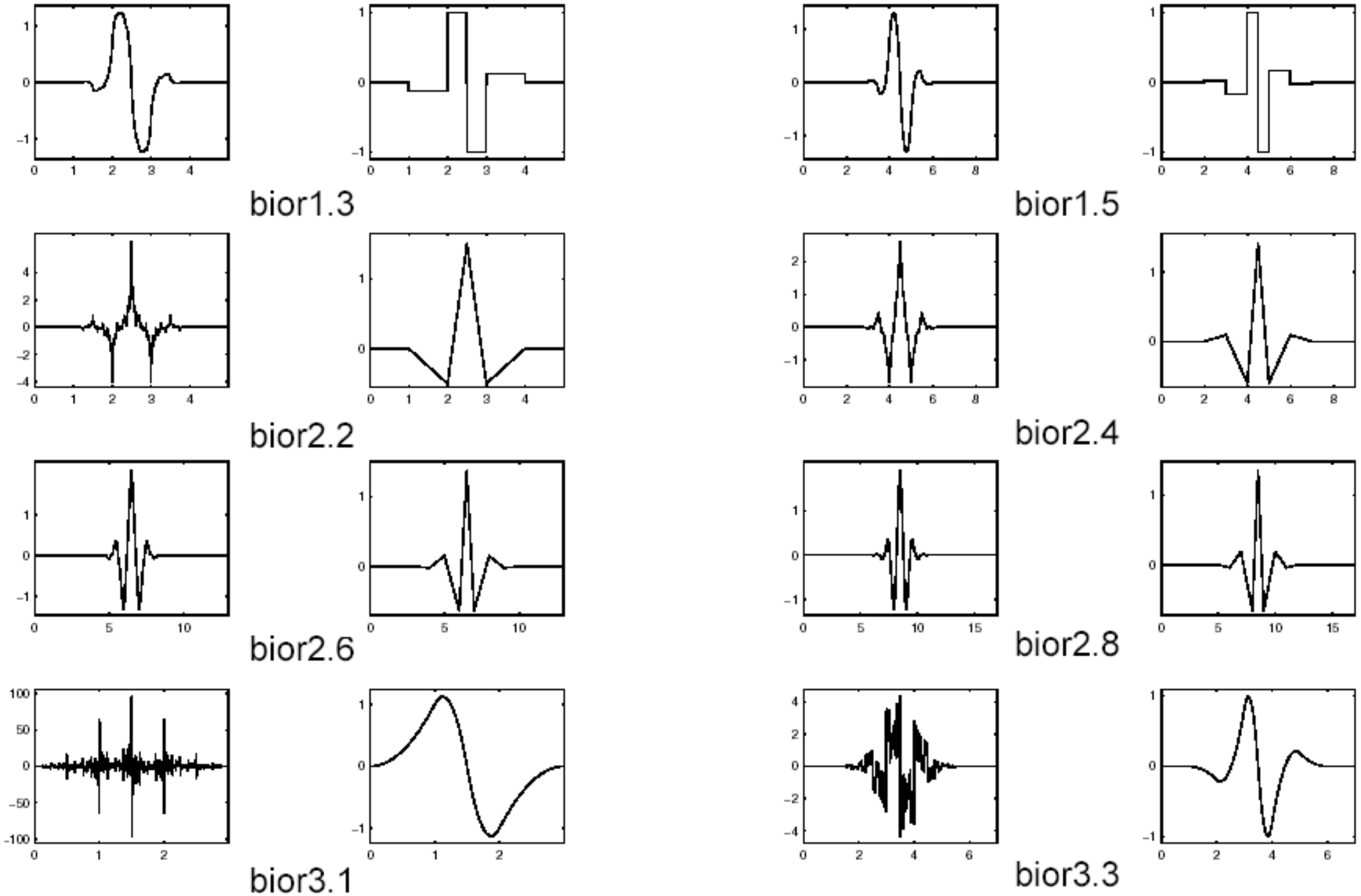
Example: bior3.5



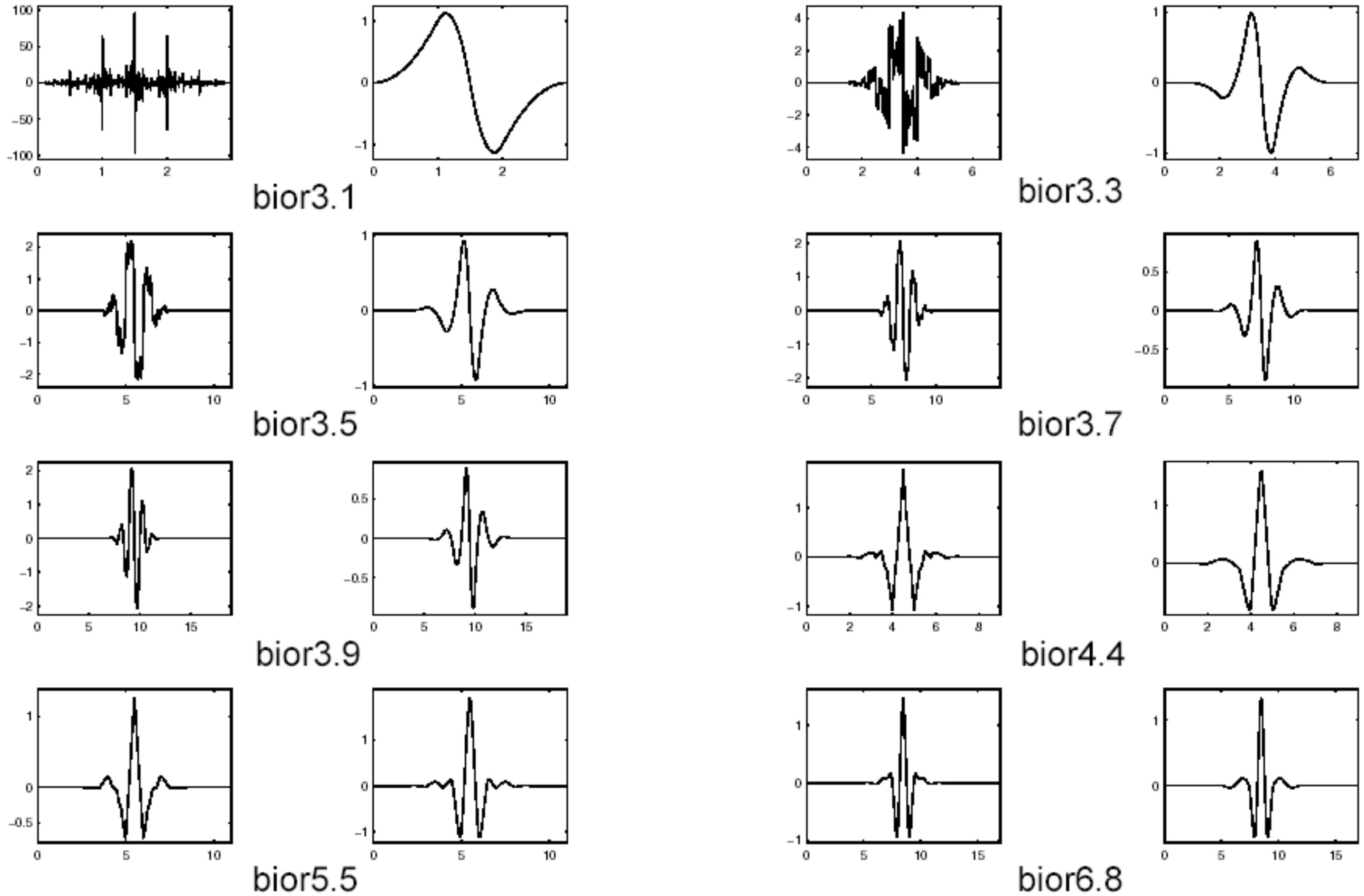
Example: bior3.5



Biorthogonal bases



Biorthogonal bases qui



CMF : orhtogonal filters

- PR filter banks decompose the signals in a basis of *$l^2(Z)$*. This basis is *orthogonal* for *Conjugate Mirror Filters* (CMF).
- [Smith&Barnwell,1984]: Necessary and sufficient condition for PR *orthogonal FIR* filter banks, called CMFs
  - Imposing that *the decomposition filter h is equal to the reconstruction filter h~,* eq. (1) becomes

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2 \quad (1) \rightarrow$$

$$\hat{h}^*(\omega)\hat{h}(\omega)+\hat{h}^*(\omega+\pi)\hat{h}(\omega+\pi)=2 \rightarrow$$

$$\boxed{|\hat{h}(\omega)|^2+|\hat{h}(\omega+\pi)|^2=2}$$

  - Correspondingly

$$\tilde{h}[n]=h[n]$$
$$\tilde{g}[n]=g[n]=(-1)^{1-n}h[1-n]$$

Summary

- PR filter banks decompose the signals in a basis of $l^2(Z)$. This basis is *orthogonal* for *Conjugate Mirror Filters* (CMF).
- [Smith&Barnwell,1984]: Necessary and sufficient condition for PR orthogonal FIR filter banks, called CMFs
  - Imposing that the decomposition filter h is equal to the reconstruction filter $h^{\sim}$, eq. (1) becomes

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2\rightarrow$$
$$\hat{h}^*(\omega)\hat{h}(\omega)+\hat{h}^*(\omega+\pi)\hat{h}(\omega+\pi)=2\rightarrow$$

$$|\hat{h}(\omega)|^2+|\hat{h}(\omega+\pi)|^2=2$$

  - Correspondingly

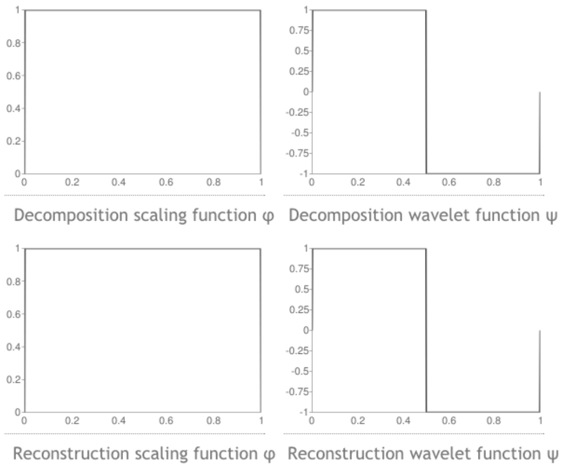
$$\tilde{h}[n]=h[n]$$
$$\tilde{g}[n]=g[n]=(-1)^{1-n}h[1-n]$$

Properties

- Support
  - h, $\tilde{h}$ are FIR → scaling functions and wavelets have compact support
- Vanishing moments
  - The number of vanishing moments of Ψ is equal to the order $\tilde{p}$ of zeros of $\tilde{h}$ in π. Similarly, the number of vanishing moments of $\tilde{\psi}$ is equal to the order p of zeros of h in π.
- Regularity
  - One can show that the regularity of Ψ and φ increases with the number of vanishing moments of $\tilde{\psi}$, thus with the order p of zeros of h in π.
  - Viceversa, the regularity of $\tilde{\psi}$ and $\tilde{\varphi}$ increases with the number of vanishing moments of Ψ, thus with the order $\tilde{p}$ of zeros of $\tilde{h}$ in π.
- Symmetry
  - It is possible to construct both symmetric and anti-symmetric bases using linear phase filters
    - In the orthogonal case only the Haar filter is possible as FIR solution.

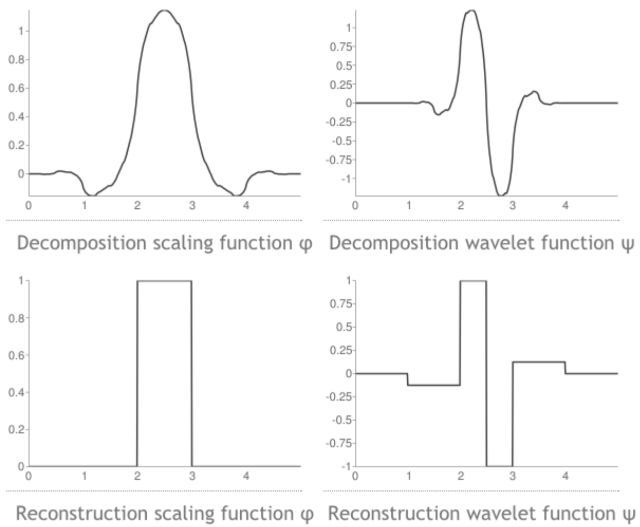
Bior1.1

Wavelet and scaling functions



Bior1.3

Wavelet and scaling functions



Bior1.5

Wavelet and scaling functions