[**Submit your solutions** at the exercise class on March 20, 2019]

In Exercises 1 and 2 we will consider representations of the following quivers:

$$Q_1 : \quad \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \qquad\qquad Q_2 : \quad \bullet \underset{\gamma}{\overset{\delta}{\rightrightarrows}} \bullet$$

Exercise 1.

1. Identify whether each of the following morphisms of representations is a monomorphism, an epimorphism or an isomorphism. We will fix an arbitrary field k and $0 \neq \lambda \in k$.

   Note: A morphism of representations $(h_i\colon V_i \to V_i')_{i\in Q_0}$ is a monomorphism/ epimorphism/ isomorphism if and only if h_i is a monomorphism/ epimorphism/ isomorphism for each $i \in Q_0$.

   (a) The morphism $u = (u_1, u_2, u_3)$ between representations of Q_1 defined as follows:

$$\begin{tikzcd} 0 \arrow[r, "f_\alpha=0"] \arrow[d, "u_1"] & \mathbb{R} \arrow[r, "f_\beta=2"] \arrow[d, "u_2"] & \mathbb{R} \arrow[d, "u_3"] \\ \mathbb{R} \arrow[r, "g_\alpha=\begin{pmatrix}1\\0\\0\end{pmatrix}"'] & \mathbb{R}^3 \arrow[r, "g_\beta=\begin{pmatrix}1&0&0\\0&0&4\end{pmatrix}"'] & \mathbb{R}^2 \end{tikzcd}$$

   where $u_1 = 0$, $u_2 = \begin{pmatrix}0\\2\\1\end{pmatrix}$ and $u_3 = \begin{pmatrix}0\\2\end{pmatrix}$.

   (b) The morphism $v = (v_1, v_2)$ between representations of Q_2 defined as follows:

$$\begin{tikzcd} k^3 \arrow[r, shift left, "f_\delta=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}"] \arrow[r, shift right, "f_\gamma=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}"'] \arrow[d, "v_1"'] & k^4 \arrow[d, "v_2"] \\ k^2 \arrow[r, shift left, "g_\delta=\begin{pmatrix}1&0\end{pmatrix}"] \arrow[r, shift right, "g_\gamma=\begin{pmatrix}0&1\end{pmatrix}"'] & k \end{tikzcd}$$

   where $v_1 = \begin{pmatrix}1&\lambda&0\\0&1&\lambda\end{pmatrix}$ and $v_2 = \begin{pmatrix}0&1&\lambda&0\end{pmatrix}$.

   (c) The morphism $w = (w_1, w_2)$ between representations of Q_2 defined as follows:

$$\begin{tikzcd} k \arrow[r, shift left, "f_\delta=1"] \arrow[r, shift right, "f_\gamma=\lambda"'] \arrow[d, "w_1"'] & k \arrow[d, "w_2"] \\ k \arrow[r, shift left, "g_\delta=\lambda^{-1}"] \arrow[r, shift right, "g_\gamma=1"'] & k \end{tikzcd}$$

   where $w_1 = 1$ and $w_2 = \lambda^{-1}$.

   (3 points)

2. Let $f \in \mathrm{Hom}_R(M, N)$ be a homomorphism of left R-modules. Show that f is a monomorphism if and only if $fg = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(L, M)$ for any module L. Show f is an epimorphism if and only if $gf = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(N, L)$ for any module L.

   (4 points)

Exercise 2.

1. Let ${}_RL$, ${}_RM$, ${}_RN$ be left R-modules and let $f \in \operatorname{Hom}_R(L, M)$ and $g \in \operatorname{Hom}_R(M, L)$ be such that $gf = \mathrm{id}_L$. Show $M = \operatorname{Im} f \oplus \ker g$.

   (3 points)

2. Let Q be a finite quiver without oriented cycles and let $V = (V_i)_{i \in Q_0}$ and $V' = (V'_i)_{i \in Q_0}$ be representations of Q over a field k. We say that V is a *subrepresentation* of V' if V_i is a k-subspace of V'_i for all $i \in Q_0$ and the canonical embeddings $(\iota_i \colon V_i \to V'_i)_{i \in Q_0}$ give rise to a morphism of representations.

   Consider the following representation of Q_1:

$$
V: \qquad \begin{tikzcd} \mathbb{R} \arrow[r, "g_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"'] & \mathbb{R}^3 \arrow[r, "g_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}"'] & \mathbb{R}^2 \end{tikzcd}
$$

   Which of the following collections of subspaces (shown with their canonical embeddings) is not a subrepresentation of V? Why?

   (a)

$$
\begin{tikzcd} 0 \arrow[r, "f_\alpha = 0"] \arrow[d, "0"] & \mathbb{R} \arrow[r, "f_\beta = 4"] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"] & \mathbb{R} \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \\ \mathbb{R} \arrow[r, "g_\alpha"'] & \mathbb{R}^3 \arrow[r, "g_\beta"'] & \mathbb{R}^2 \end{tikzcd}
$$

   (b)

$$
\begin{tikzcd} \mathbb{R} \arrow[r, "f_\alpha = 1"] \arrow[d, "1"] & \mathbb{R} \arrow[r, "f_\beta = 0"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & 0 \arrow[d, "\begin{pmatrix} 0 \\ 0 \end{pmatrix}"] \\ \mathbb{R} \arrow[r, "g_\alpha"'] & \mathbb{R}^3 \arrow[r, "g_\beta"'] & \mathbb{R}^2 \end{tikzcd}
$$

   (c)

$$
\begin{tikzcd} \mathbb{R} \arrow[r, "f_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, "1"] & \mathbb{R}^2 \arrow[r, "f_\beta = \begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & \mathbb{R} \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\ \mathbb{R} \arrow[r, "g_\alpha"'] & \mathbb{R}^3 \arrow[r, "g_\beta"'] & \mathbb{R}^2 \end{tikzcd}
$$

   Using Exercise 2.1, show that V is the direct sum of the two subrepresentations.

   Note: Under the correspondence given in Remark 2.8, a direct sum of representations corresponds to a direct sum of modules.

   (5 points)

Exercise *. *Note: this is an optional extra exercise - it will not be marked.*

1. Let ${}_RM$ be a R-module and ${}_RR$ the regular module. Consider the abelian group $\operatorname{Hom}_R(R, M)$ and the map $\varphi : \operatorname{Hom}_R(R, M) \to M$, $f \mapsto f(1)$. Verify that $\operatorname{Hom}_R(R, M)$ is a left R-module and φ is an isomorphism of R-modules.

   (Bonus exercise)

2. Let ${}_RL, {}_RN \leq {}_RM$. Show that M is the direct sum of L and N if and only if $L + N = M$ and $L \cap N = 0$. Generalise this statement for more than two summands?

   (Bonus exercise)