2D Fourier Transform

Overview

- Signals as functions (1D, 2D)
  - Tools
- 1D Fourier Transform
  - Summary of definition and properties in the different cases
    - CTFT, CTFS, DTFS, DTFT
    - DFT
- 2D Fourier Transforms
  - Generalities and intuition
  - Examples
  - A bit of theory
- Discrete Fourier Transform (DFT)
- Discrete Cosine Transform (DCT)

Signals as functions

- Continuous functions of real independent variables
  - 1D: $f=f(x)$
  - 2D: $f=f(x,y)$ x,y
  - Real world signals (audio, ECG, images)
- Real valued functions of discrete variables
  - 1D: $f=f[k]$
  - 2D: $f=f[i,j]$
  - *Sampled* signals
- Discrete functions of discrete variables
  - 1D: $f^d=f^d[k]$
  - 2D: $f^d=f^d[i,j]$
  - *Sampled and quantized* signals

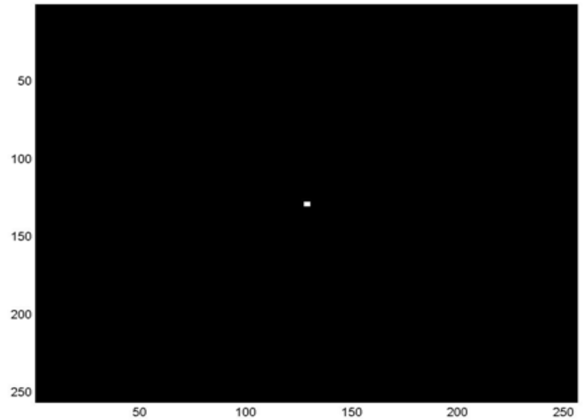
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

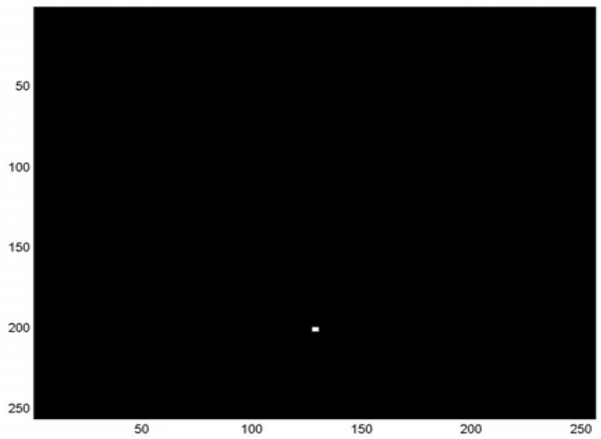
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i,j \neq 0; i \neq j \end{cases}$$



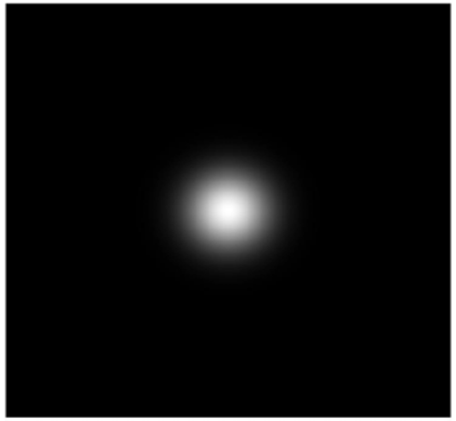
$$\delta[i, j-J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



Example 2: Gaussian

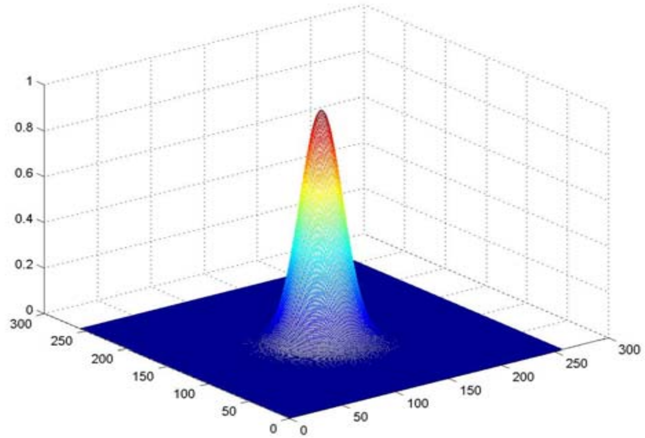
Continuous function

$$f(x,y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{x^2+y^2}{2\sigma^2}}$$

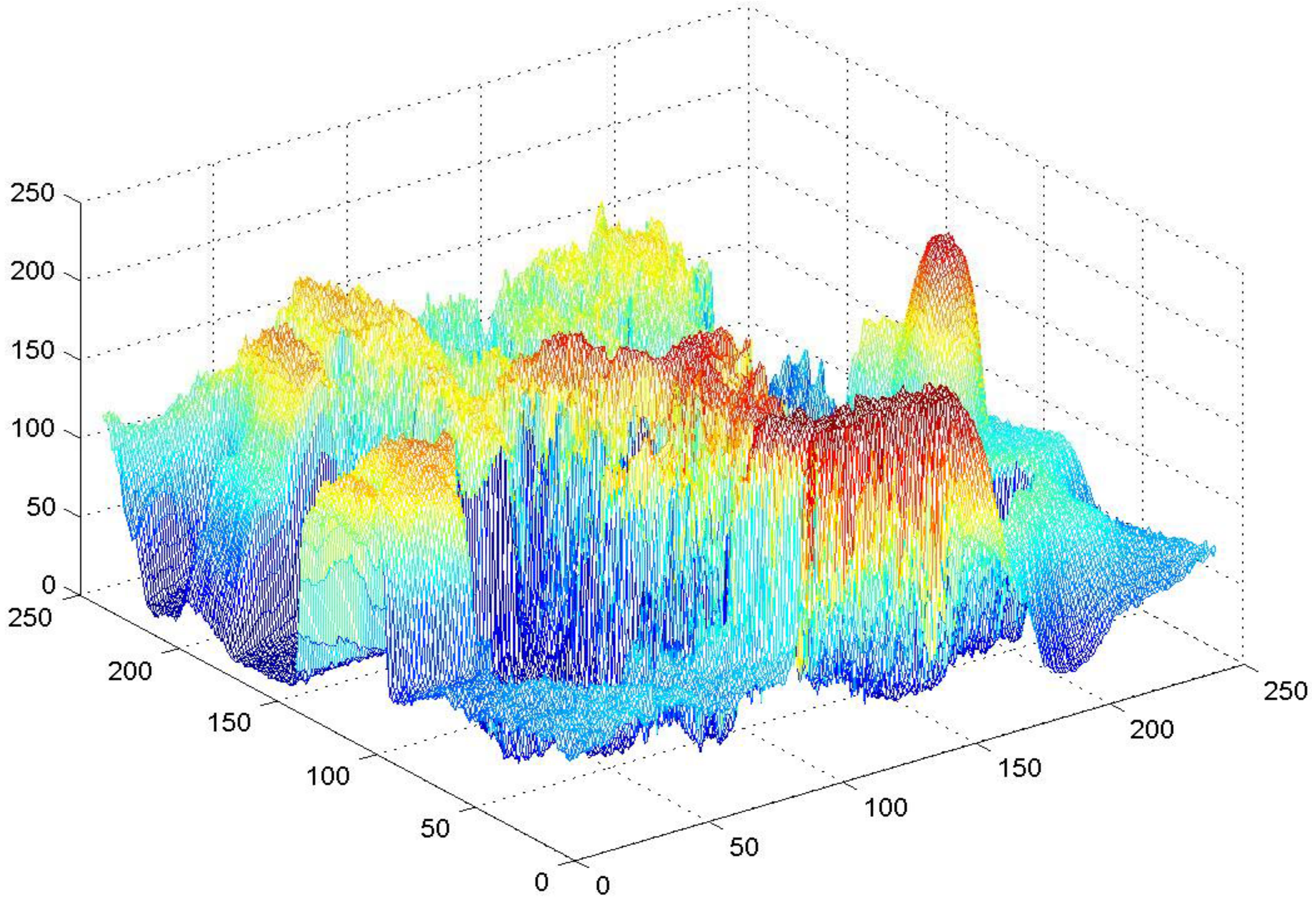


Discrete version

$$f[i,j] = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{i^2+j^2}{2\sigma^2}}$$

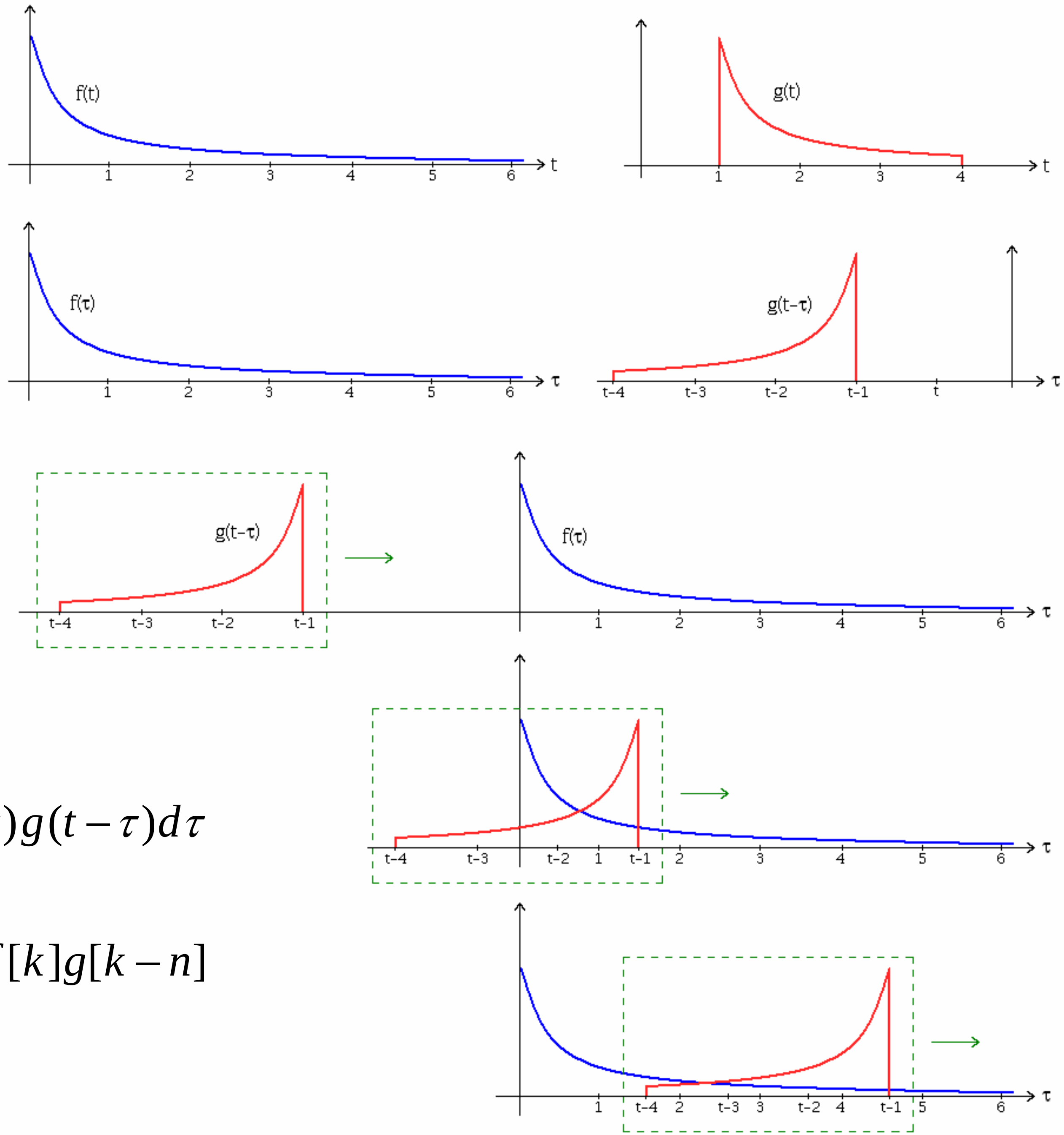


Example 3: Natural image



Example 3: Natural image

Convolution



$$c(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$$

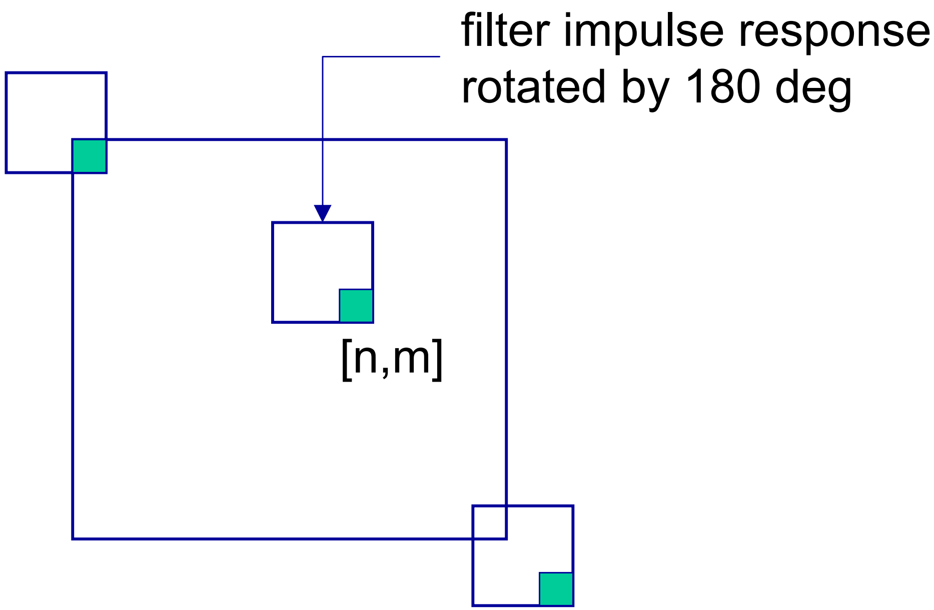
$$c[n] = f[n] * g[n] = \sum_{k=-\infty}^{+\infty} f[k] g[k-n]$$

2D Convolution

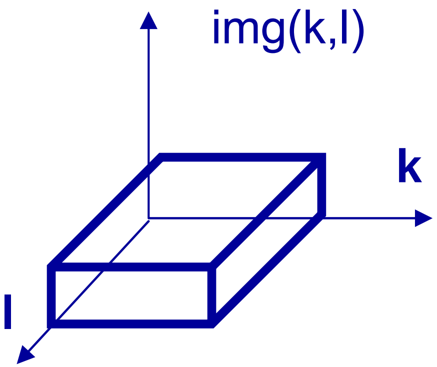
$$c(x,y) = f(x,y) \otimes g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau,\nu)\, g(x-\tau, y-\nu)\, d\tau\, d\nu$$

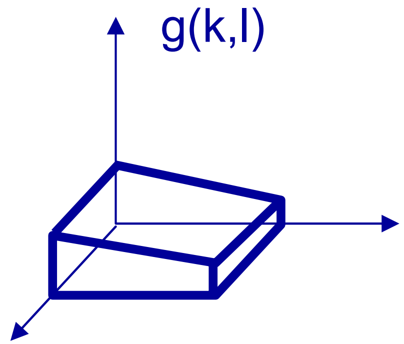
$$c[i,k] = \sum_{k=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} f[n,m]\, g[i-n, k-m]$$

- Associativity
- Commutativity
- Distributivity



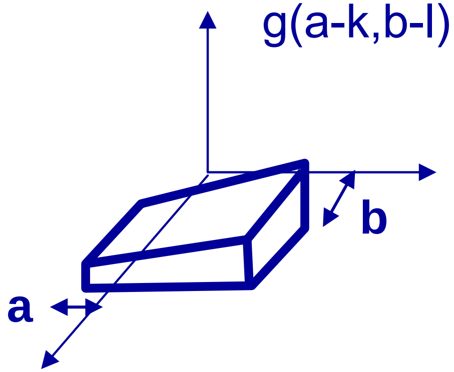
2D Convolution

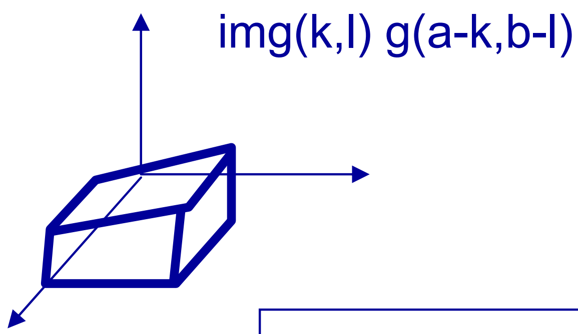




1. fold about origin
2. displace by ‘a’ and ‘b’

3. compute integral
   of the box





Tricky part: borders
- *(zero padding, mirror...)*

Convolution

Filtering with filter h(x,y)

$$f_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(s,t)h(x-s,y-t)dsdt$$

- Convolution with a 2D Dirac pulse

  $f_2(x, y) = f_1(x, y)$ sampling property of the delta function

- Convolution a Dirac pulse shifted by (x_0,y_0)

  $f_2(x, y) = f_1(x - x_0, y - y_0)$

- Fourier transform…

  $F_2(u, v) = F_1(u, v)\ H(u, v)$

- … and vice versa

  $g(x,y) = f_1(x, y)\ f_2(x, y)$ then $G(u,v) = F_1(u, v) * F_2(u, v)$

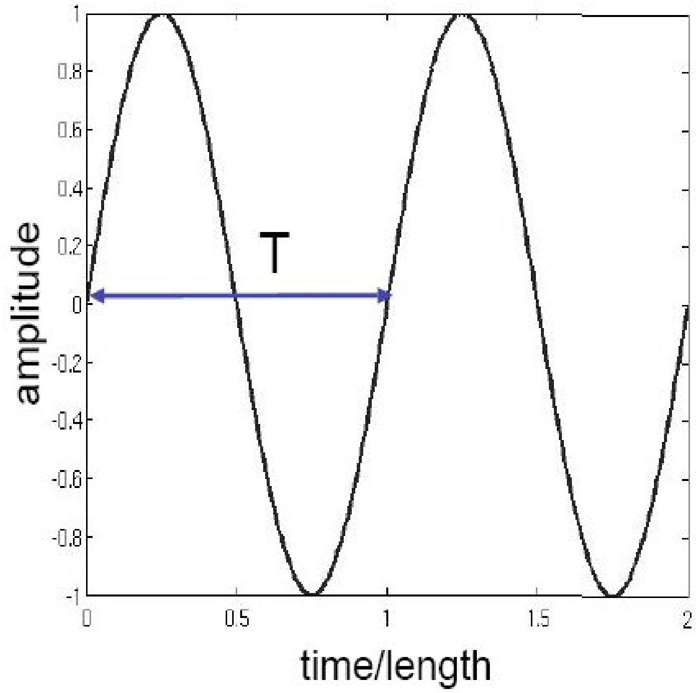
Fourier Transform

- Different formulations for the different classes of signals
  - Summary table: Fourier transforms with various combinations of continuous/discrete time and frequency variables.
  - Notations:
    - CT: continuous time
    - DT: Discrete Time
    - FT: Fourier Transform (integral synthesis)
    - FS: Fourier Series (summation synthesis)
    - P: periodical signals
    - T: sampling period
    - ω_s: sampling frequency ($\omega_s=2\pi/T$)
    - For DTFT: $T=1 \rightarrow \omega_s=2\pi$

1D FT: basics

- Define frequency
  = 1/T
  cycles per unit time
  cycles per unit distance

- Here f = 1



Fourier Transform: Concept

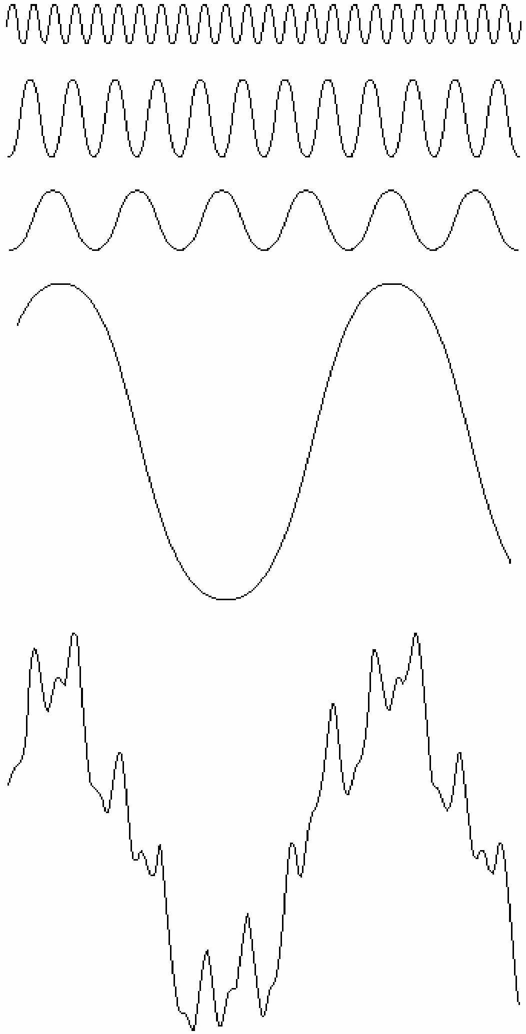


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- A signal can be represented as a weighted sum of sinusoids.

- Fourier Transform is a change of basis, where the basis functions consist of sines and cosines (complex exponentials).

Fourier Transform

- Cosine/sine signals are easy to define and interpret.
- However, it turns out that the analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.

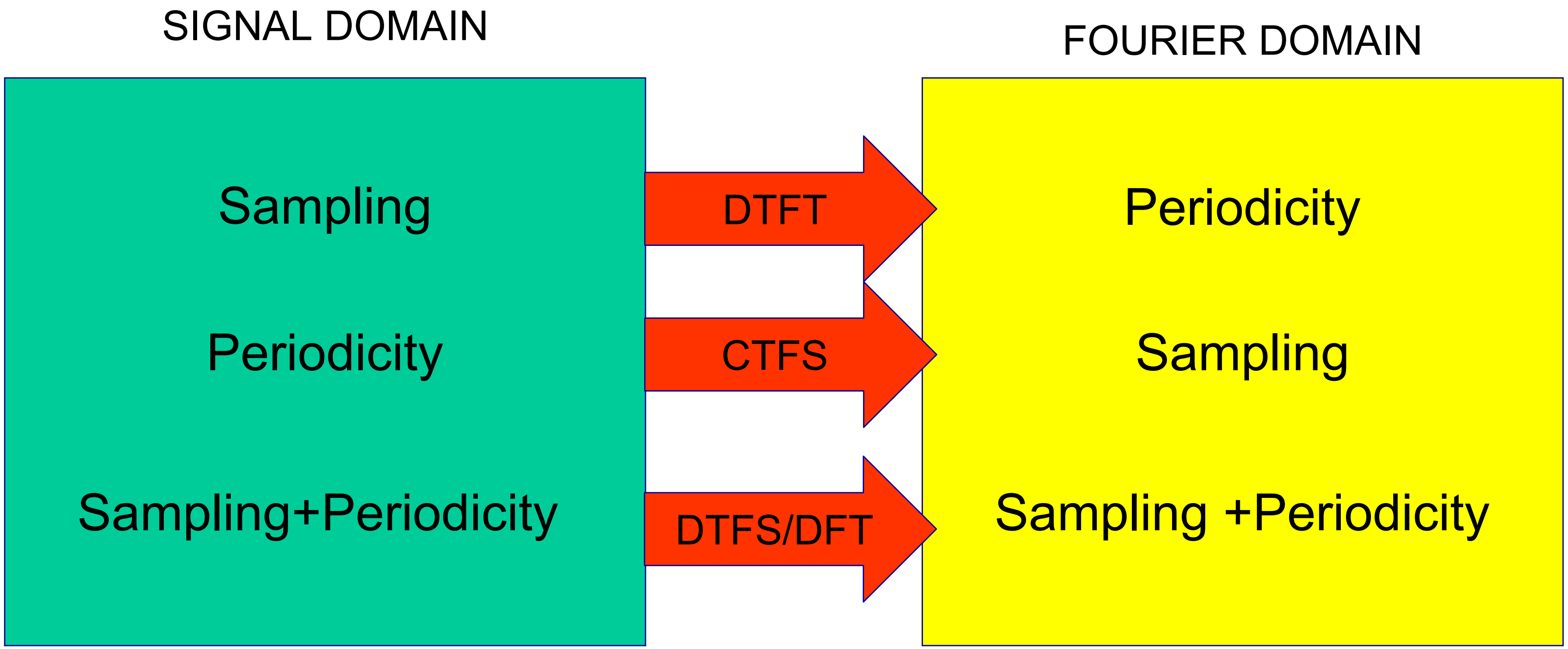
- A complex number has real and imaginary parts: z = x + j*y

- A complex exponential signal: r*exp(j*a) =r*cos(a) + j*r*sin(a)

Overview

Transform	Time	Frequency	Analysis/Synthesis	Duality
(Continuous Time) Fourier Transform (CTFT)	C	C	$F(\omega)=\int_t f(t)e^{-j\omega t}dt$ $f(t)=\frac{1}{2\pi}\int_\omega F(\omega)e^{j\omega t}d\omega$	Self-dual
(Continuous Time) Fourier Series (CTFS)	C P	D	$F[k]=\int_{-T/2}^{T/2} f(t)e^{-j2\pi kt/T}dt$ $f(t)=\sum_k F[k]e^{j2\pi kt/T}$	Dual with DTFT
Discrete Time Fourier Transform (DTFT)	D	C P	$F\left(e^{j\omega t}\right)=\sum_n f[n]e^{-j2\pi\omega n/\omega_s}$ $f[n]=\frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega t}\right)e^{j2\pi\omega n/\omega_s}d\omega$	Dual with CTFS
Discrete Time Fourier Series (DTFS)	D P	D P	$F[k]=\frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/T}$ $f[n]=\sum_{n=0}^{N-1} F[k]e^{j2\pi kn/T}$	Self dual

Dualities



SIGNAL DOMAIN
FOURIER DOMAIN
Sampling
Periodicity
Sampling+Periodicity
DTFT
CTFS
DTFS/DFT
Periodicity
Sampling
Sampling +Periodicity

Discrete time signals

- Sequences of samples
- f[k]: sample values
- Assumes a unitary spacing among samples (T_s=1)
- Normalized frequency Ω
- Transform
  - DTFT for NON periodic sequences
  - CTFS for periodic sequences
  - DFT for *periodized* sequences
- All transforms are 2π periodic

$$\Omega = \omega T_s$$

- *Sampled* signals
- $f(kT_s)$: sample values
- The sampling interval (or period) is T_s
- Non normalized frequency ω
- Transform
  - DTFT
  - CSTF
  - DFT
  - BUT accounting for the fact that the sequence values have been generated by sampling a real signal → $f_k=f(kT_s)$
- All transforms are periodic with period ω_s

CTFT

- Continuous Time Fourier Transform
- Continuous time *a-periodic* signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency ω is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

Fourier *integral*

$$F(\omega) = \int_t f(t) e^{-j\omega t} dt$$ analysis

$$f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$$ synthesis

CTFT: change of notations

- Fourier Transform of a 1D continuous signal

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

"Euler's formula" $$e^{-j\omega x} = \cos(\omega x) - j\sin(\omega x)$$

- Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega$$

Change of notations:

$$\omega \to 2\pi u$$

$$\begin{cases} \omega_x \to 2\pi u \\ \omega_y \to 2\pi v \end{cases}$$

Then CTFT becomes

- Fourier Transform of a 1D continuous signal

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

"Euler's formula" $$e^{-j2\pi ux} = \cos(2\pi ux) - j\sin(2\pi ux)$$

- Inverse Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

CTFS

- Continuous Time Fourier Series
- Continuous time periodic signals
  - The signal is periodic with period T_0
  - The transform is “sampled” (it is a series)

our notations

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f_{T_o}(t)e^{-jn\omega_0 t}dt$$

$$f_{T_0}(t) = \sum_n D_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

fundamental frequency

table notations

$$F[k] = \int_{-T/2}^{T/2} f(t)e^{-j2\pi kt/T}dt$$

$$f(t) = \sum_k F[k]e^{j2\pi kt/T}$$

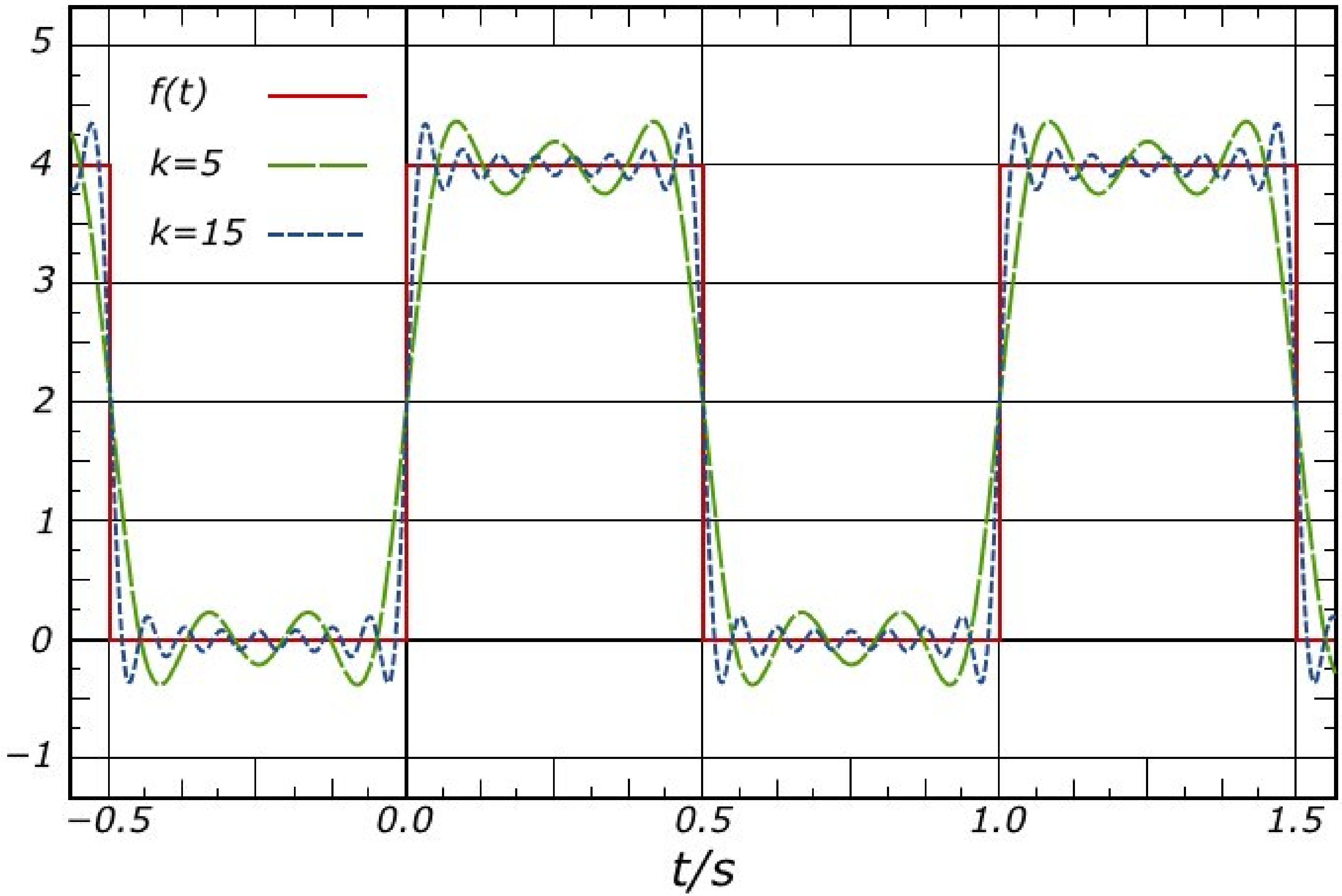
$T_0 \leftrightarrow T$

$D_n \leftrightarrow F[k]$

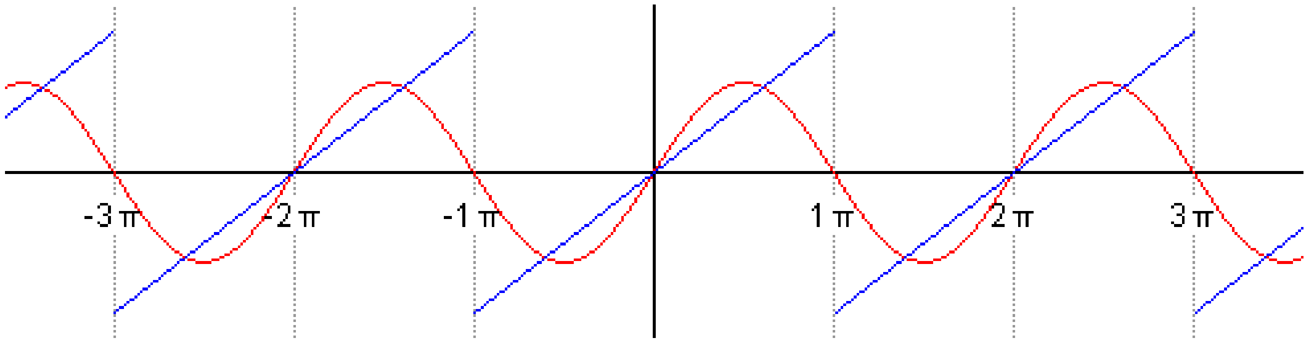
CTFS

- Representation of a continuous time signal as a sum of orthogonal components in a complete orthogonal signal space
  - The exponentials are the basis functions
- Fourier series are periodic with period equal to the fundamental in the set ($2\pi/T_0$)
- Properties
  - even symmetry → only cosinusoidal components
  - odd symmetry → only sinusoidal components

CTFS: example 1



CTFS: example 2



From sequences to discrete time signals

- Looking at the sequence as to a set of samples obtained by sampling a real signal with frequency ω_s we can still use the formulas for calculating the transforms as derived for the sequences by
  - Stratching the time axis (and thus squeezing the frequency axis if T_s>1)

$$\Omega = \omega T_s$$

$$2\pi \to \omega_s = \frac{2\pi}{T_s}$$

  - Enclosing the sampling interval T_s in the value of the sequence samples (DFT)

$$f_k = T_s f\left(kT_s\right)$$

DTFT

- Discrete Time Fourier Transform
- *Discrete time a-periodic* signal
- The transform is periodic and continuous with period $\Omega_0 = 2\pi$

our notations

$$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j\Omega k}$$

$$f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}\,d\Omega$$

normalized frequency

table notations

$$F\left(e^{j\omega t}\right) = \sum_{n} f[n]e^{-j2\pi\omega n/\omega_s}$$

$$f[n] = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega t}\right)e^{j2\pi\omega n/\omega_s}\,d\omega$$

non normalized frequency

$$F(\Omega) = \bar{F}_c\left(\frac{\Omega}{T_s}\right)$$

$$\Omega = \omega T_s$$

$$T_s = 2\pi/\omega_s$$

Discrete Time Fourier Transform (DTFT)

- F(Ω) can be obtained from $\bar{F}_c(\omega)$ by replacing ω with Ω/T_s. Thus F(Ω) is identical to $\bar{F}_c(\omega)$ frequency scaled by a factor $1/T_s$
  - T_s is the sampling interval in time domain
- Notations

$$F(\Omega) = \bar{F}_c\left(\frac{\Omega}{T_s}\right)$$

$$\omega_s = \frac{2\pi}{T_s} \rightarrow T_s = \frac{2\pi}{\omega_s}$$ periodicity of the spectrum

$$\omega = \frac{\Omega}{T_s} \rightarrow \Omega = \omega T_s$$ normalized frequency (the spectrum is 2π-periodic)

$$F(\Omega) \rightarrow F(\omega T_s) = F(2\pi\omega/\omega_s)$$

$$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j\Omega k} \rightarrow F(\omega T_s) = F(\omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j2k\pi\omega/\omega_s}$$

DTFT: *unitary* frequency

$$\Omega = 2\pi u \qquad (\omega = 2\pi f)$$

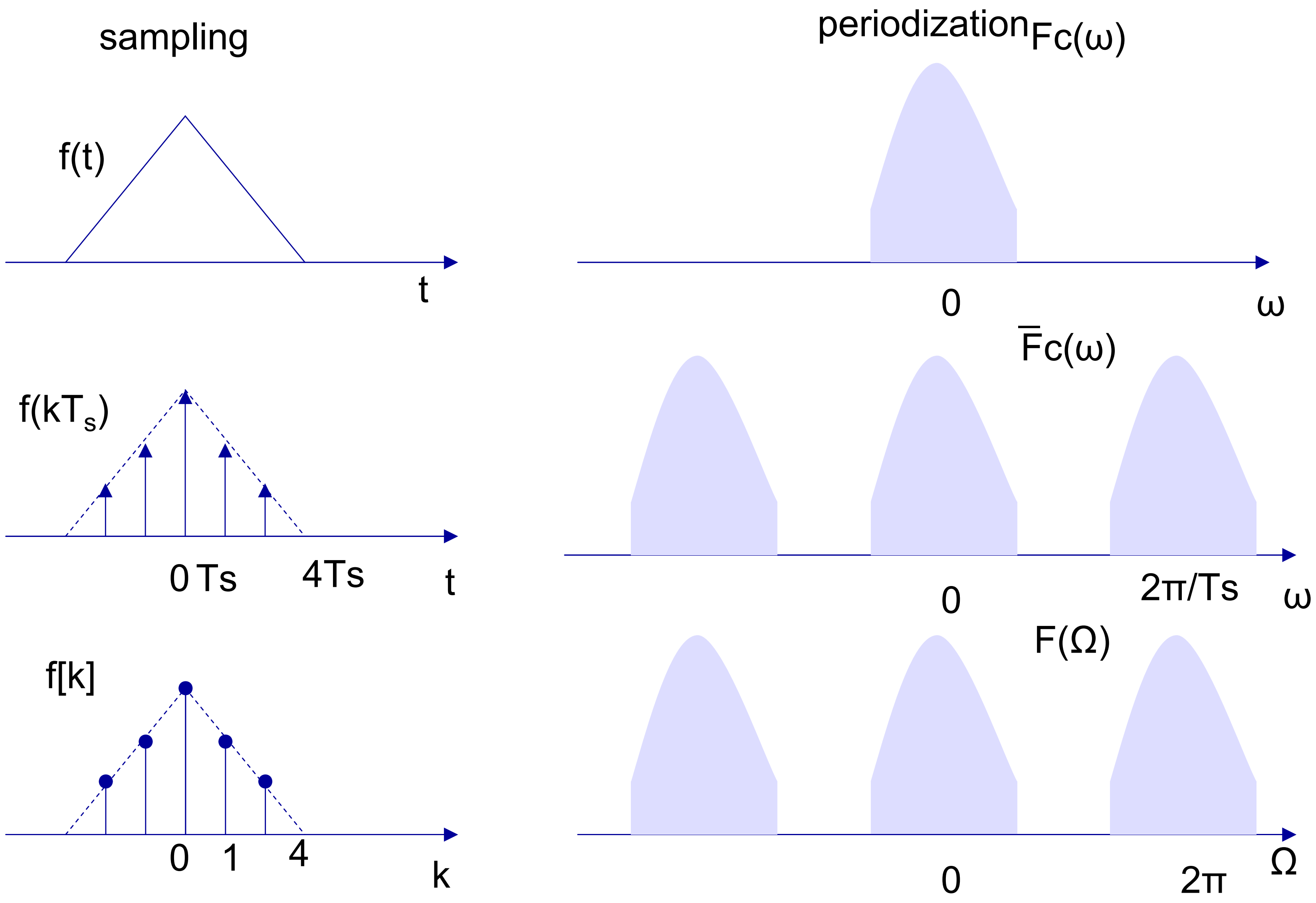
$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \rightarrow F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku}$$

$$f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}d\Omega \rightarrow f[k] = \int_{1} F(u)e^{j2\pi ku}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi ku}du$$

$$\begin{cases} F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku} \\ f[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi ku}du \end{cases}$$

NOTE: when T_s=1, Ω=ω and the spectrum is 2π-periodic. The unitary frequency u=2π/ Ω corresponds to the signal frequency f=2π/ω. This could give a better intuition of the transform properties.

Connection DTFT-CTFT



Differences DTFT-CTFT

- The DTFT is periodic with period $\Omega_s=2\pi$ (or $\omega_s=2\pi/T_s$)
- The discrete-time exponential $e^{j\Omega k}$ has a unique waveform only for values of Ω in a continuous interval of 2π
- *Numerical computations can be conveniently performed with the Discrete Fourier Transform (DFT)*

DTFS

- Discrete Time Fourier Series
- Discrete time periodic sequences of period N_0
  - Fundamental frequency

$$\Omega_0 = 2\pi / N_0$$

our notations

$$D_r = \frac{1}{N_0} \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}$$

$$f[k] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 k}$$

table notations

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/T}$$

$$f[k] = \sum_{n=0}^{N-1} F[k] e^{j2\pi kn/T}$$

Discrete Fourier Transform (DFT)

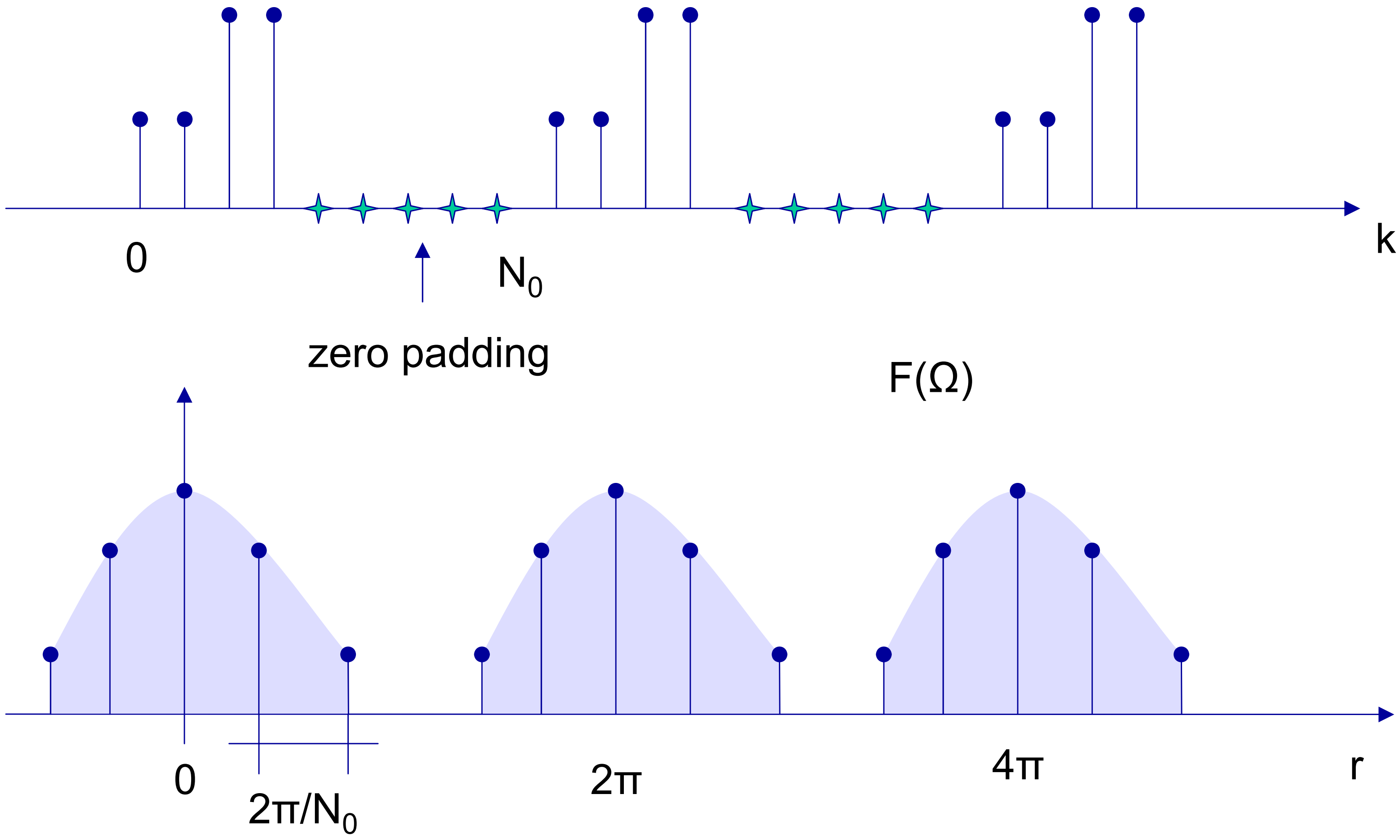
$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-jr\Omega_0 k} = \sum_{k=0}^{N_0-1} f_k e^{-j\frac{2\pi}{N_0}rk}$$

$$f_k = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F_r e^{jr\Omega_0 k} = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F_r e^{jr\frac{2\pi}{N_0}k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

- The DFT transforms N_0 samples of a discrete-time signal to the same number of discrete frequency samples
- The DFT and IDFT are a *self-contained*, one-to-one transform pair for a length-N_0 discrete-time signal (that is, the DFT is not merely an approximation to the DTFT as discussed next)
- However, the DFT is very often used as a practical approximation to the DTFT

DFT



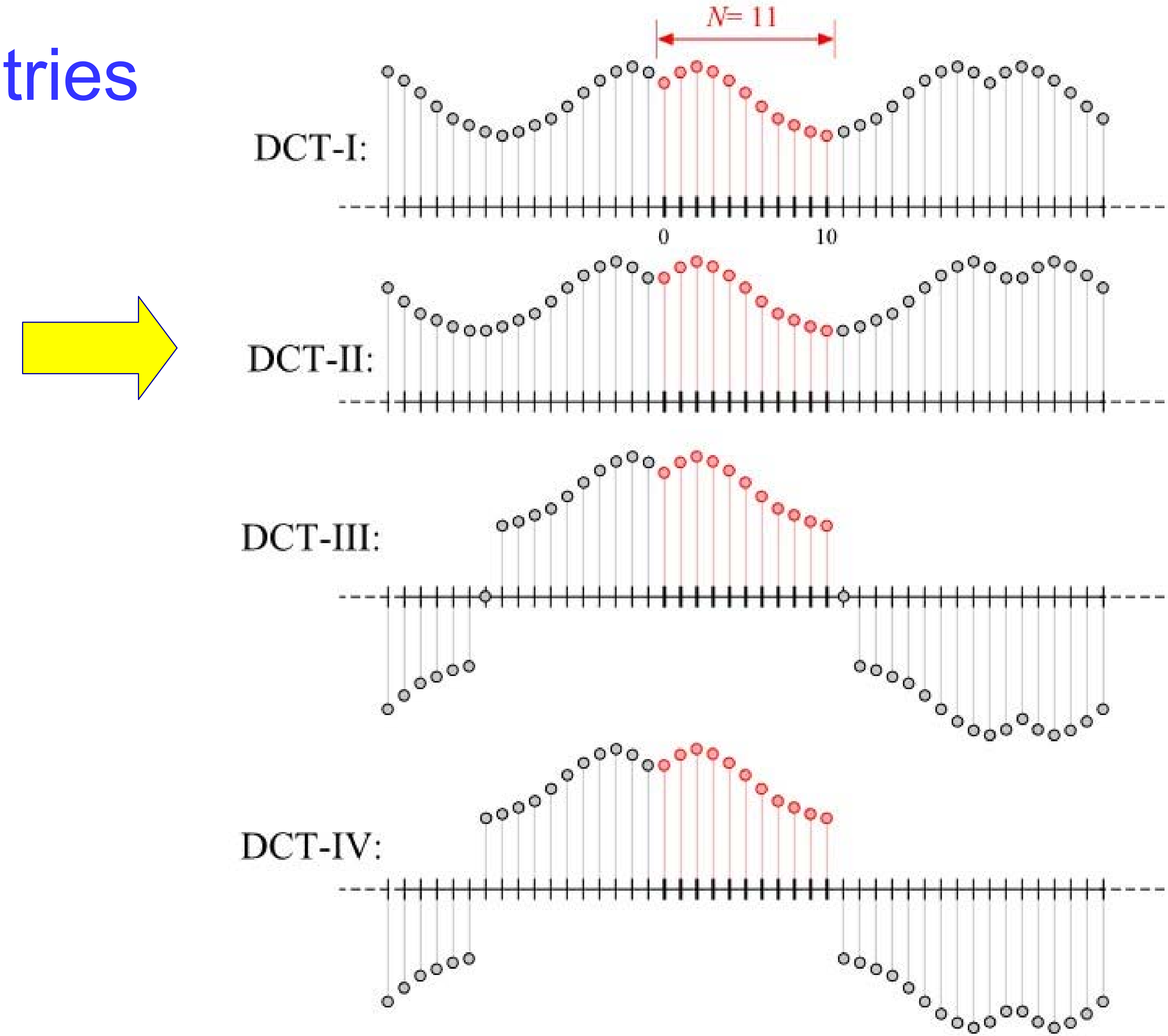
Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, of which four are common
- Strong connection with the Karunen-Loeven transform
  - VERY important for signal compression

DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is even about the sample *a*, in which case the even extension is ***dcbabcd***, or the data is even about the point *halfway* between *a* and the previous point, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

Symmetries



DCT

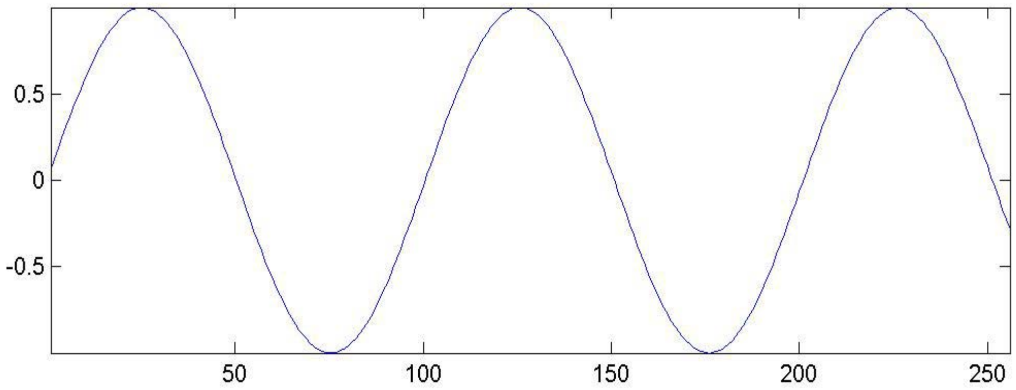
$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots., N_0 - 1$$

$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

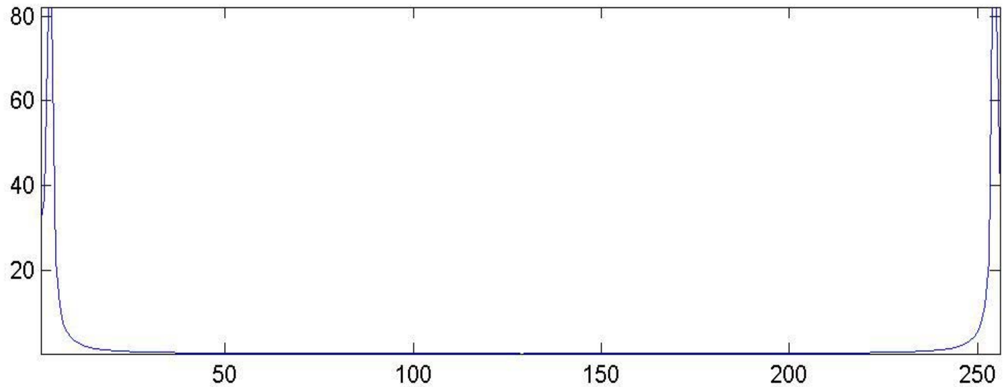
- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of √2 (see above), this can be used to make the transform matrix orthogonal.

Sinusoids

- Frequency domain characterization of signals $F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$

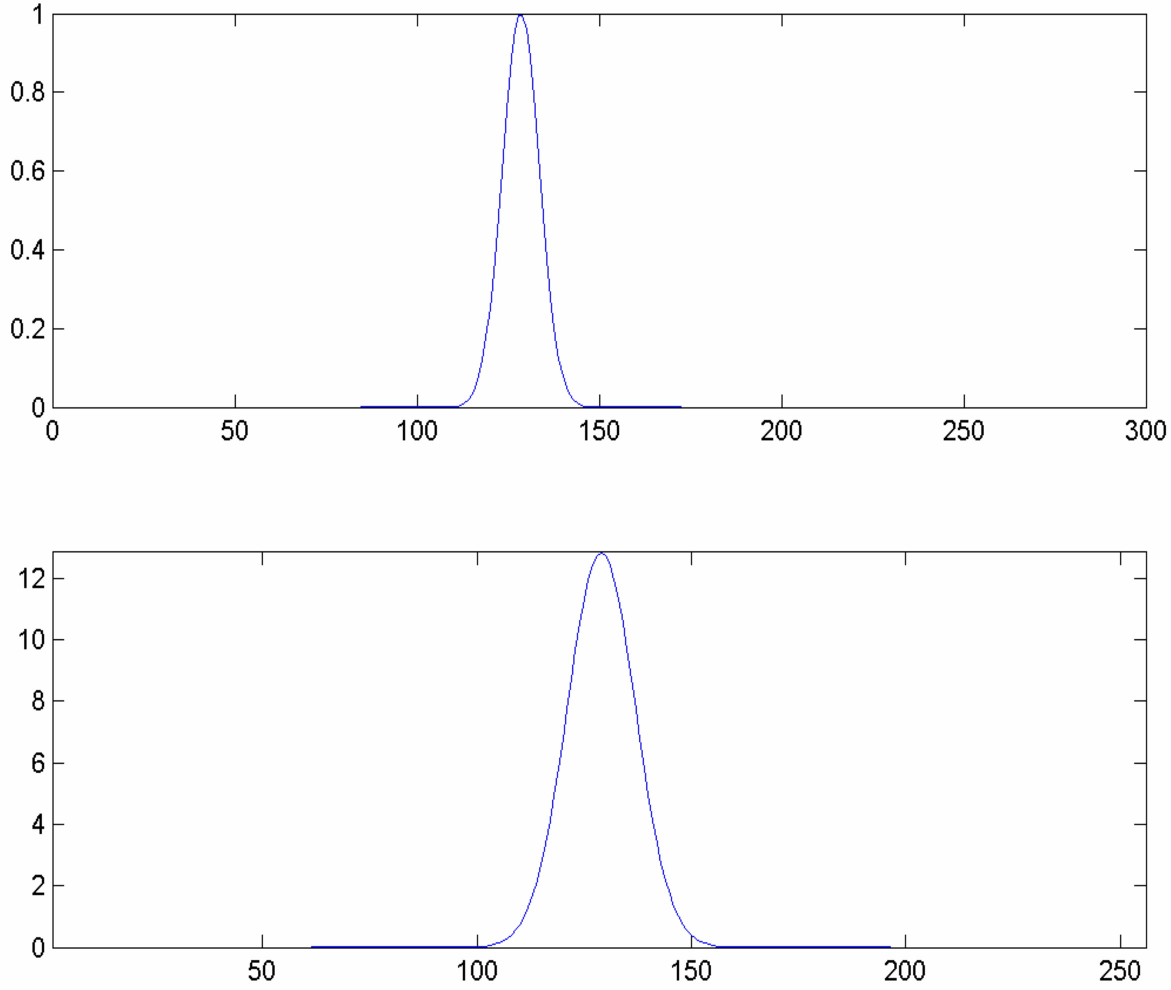


Signal domain



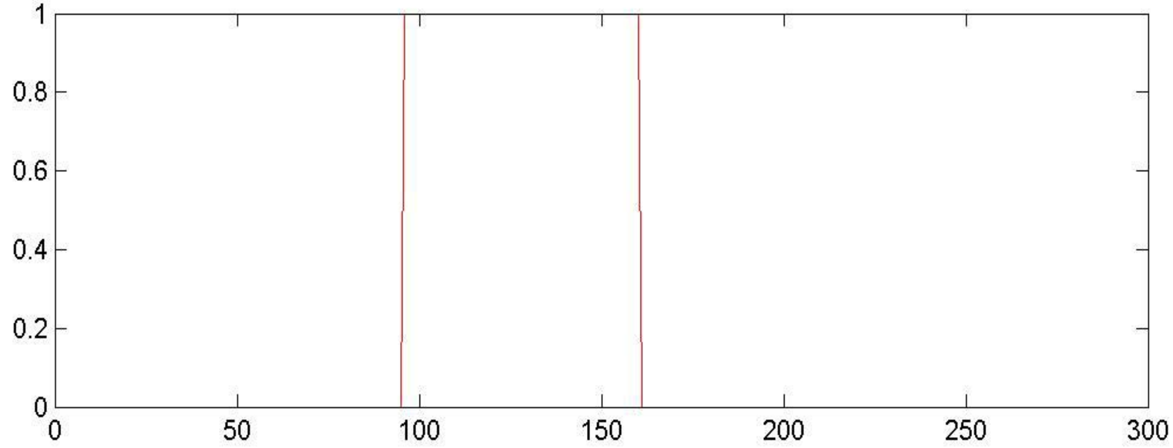
Frequency domain

Gaussian

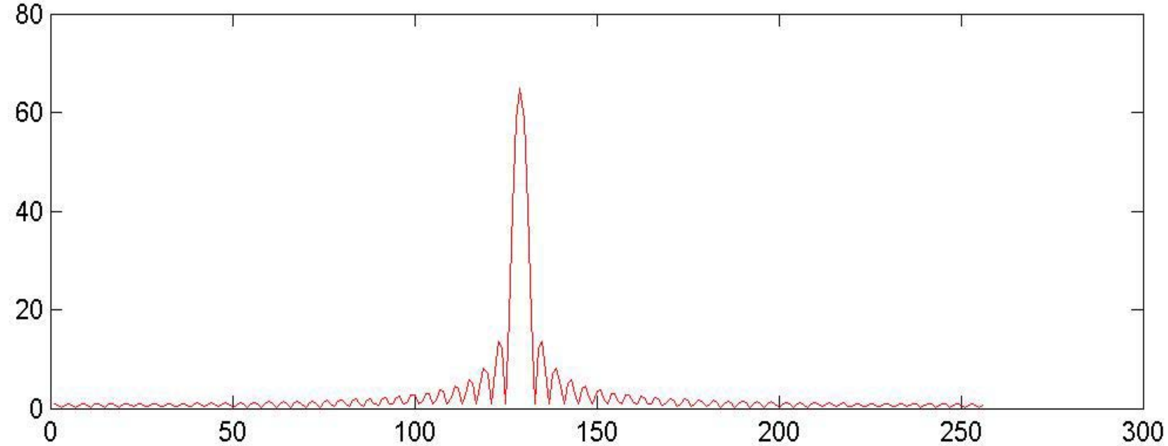


1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
12
10
8
6
4
2
0
50
100
150
200
250

rect



sinc function



Images vs Signals

1D

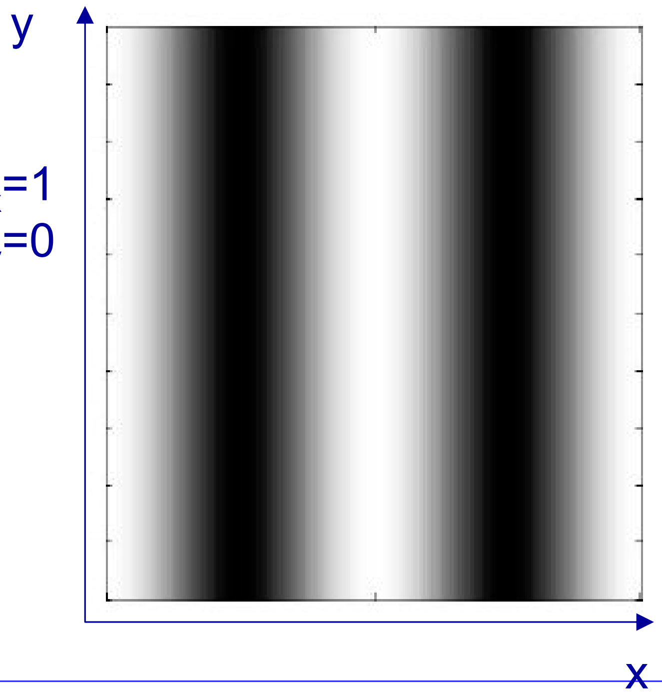
- Signals
- Frequency
  - Temporal
  - Spatial
- Time (space) frequency characterization of signals
- Reference space for
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ….

2D

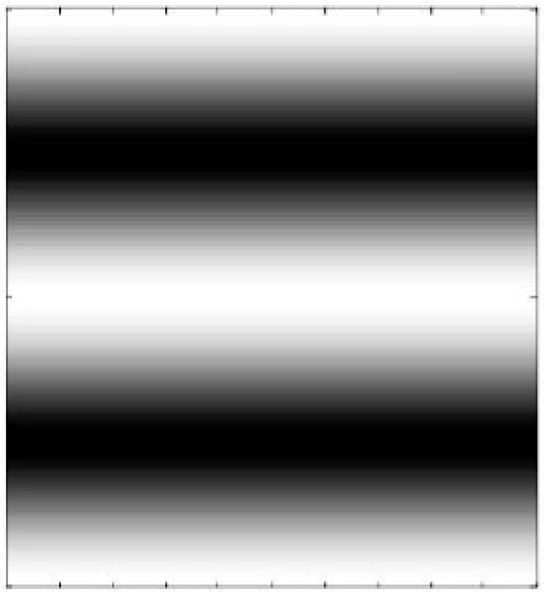
- Images
- Frequency
  - Spatial
- Space/frequency characterization of 2D signals
- Reference space for
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ….

2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies

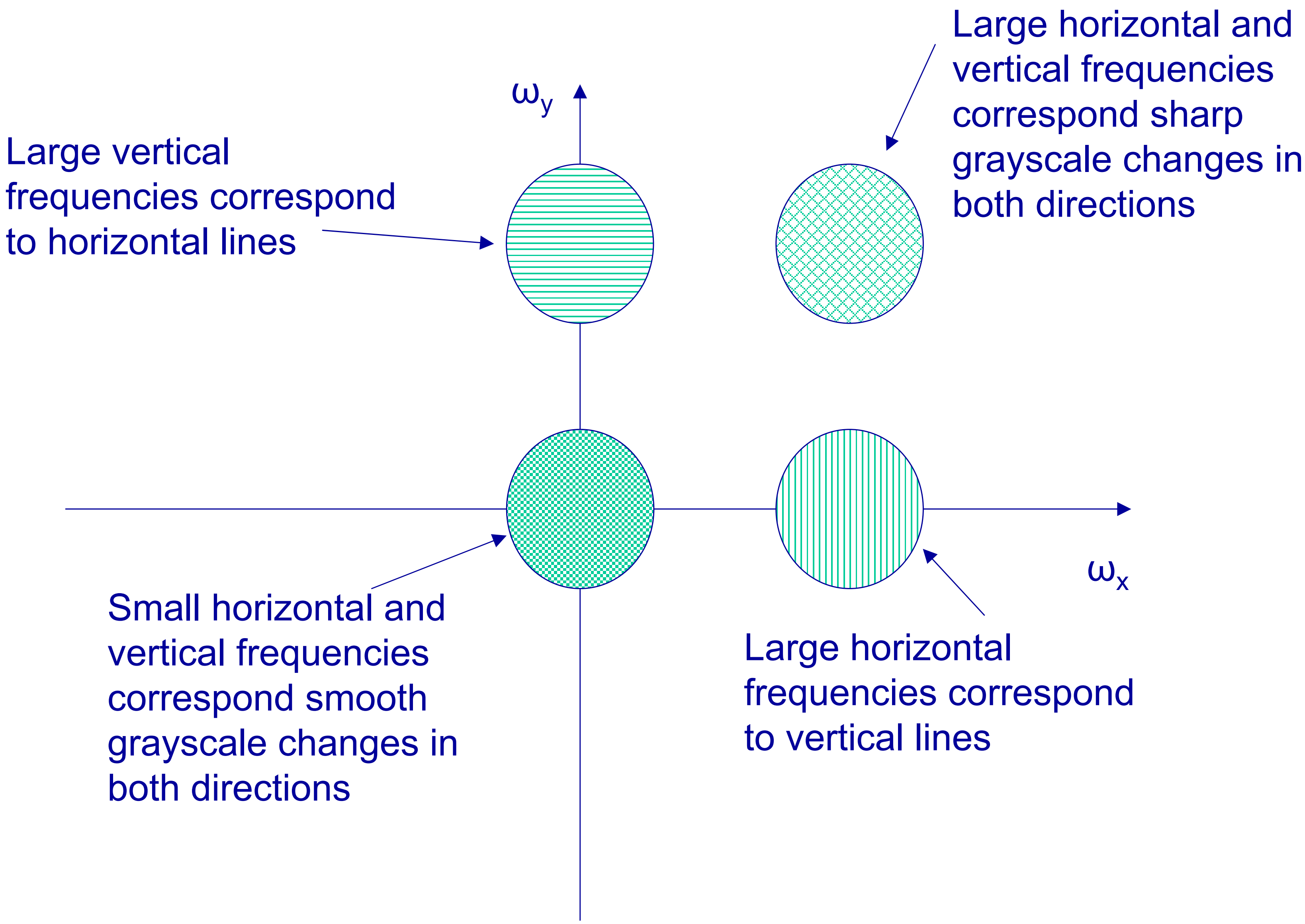


$\omega_x=1$
$\omega_y=0$

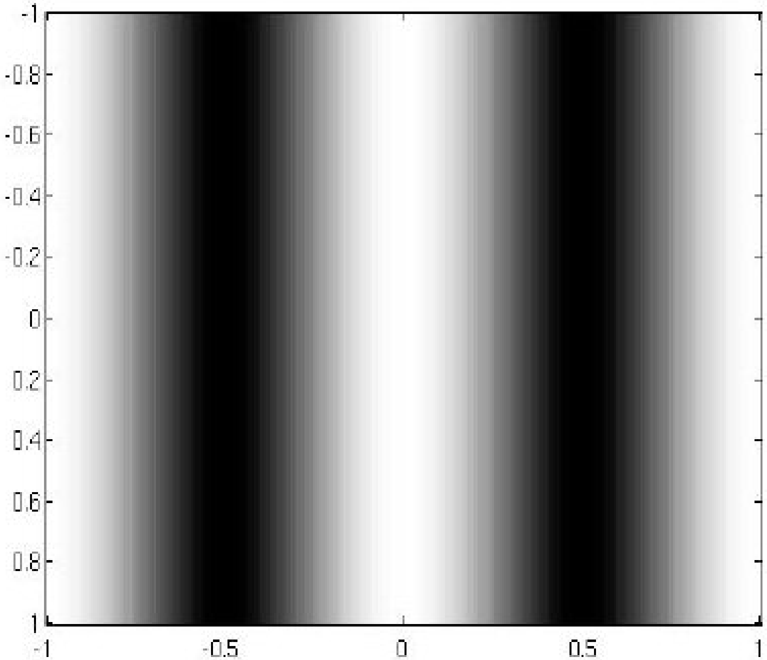


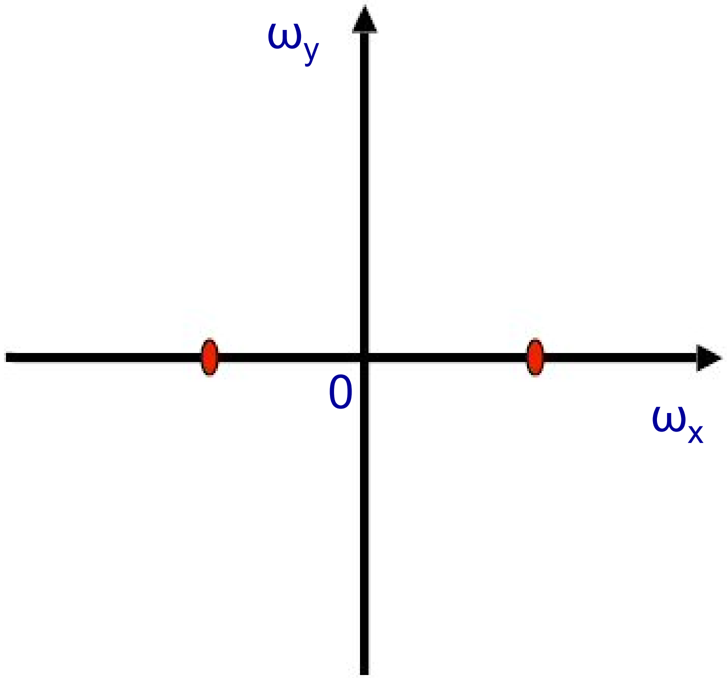
$\omega_x=0$
$\omega_y=1$

2D Frequency domain

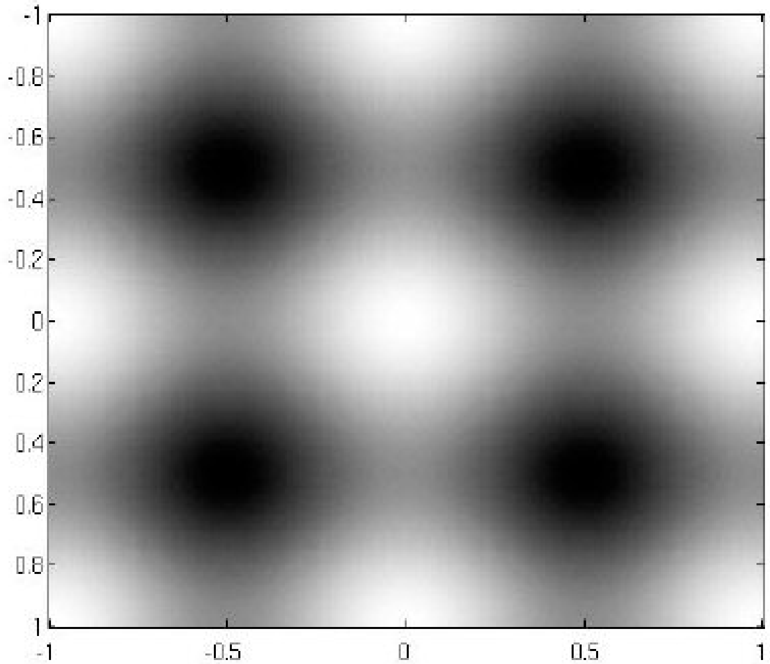


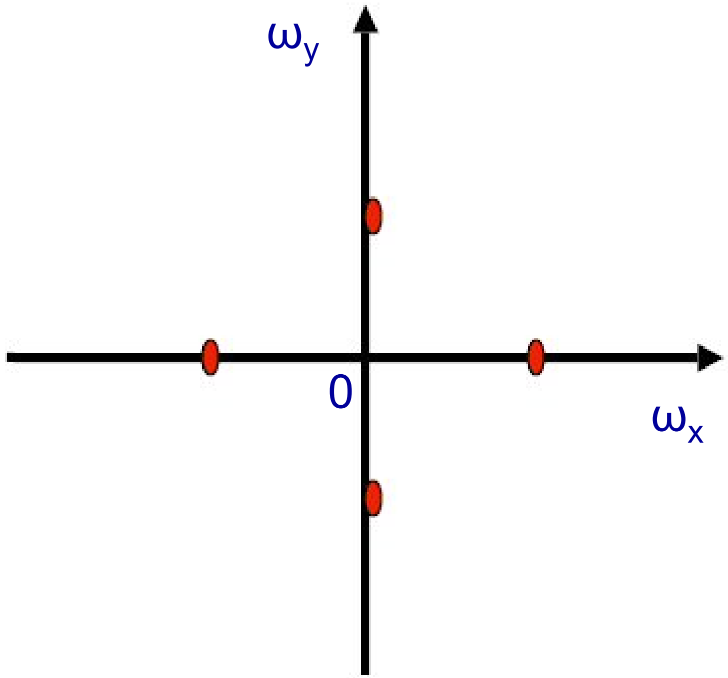
Vertical grating



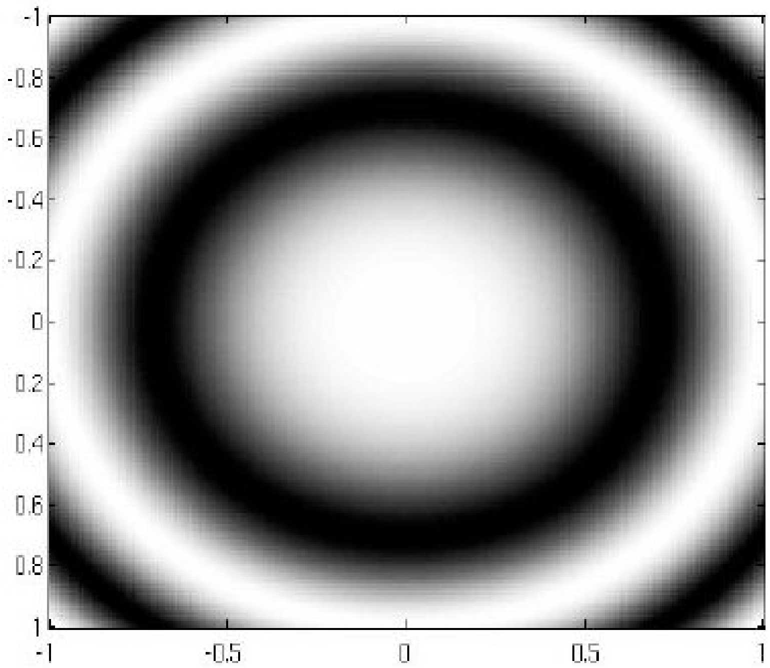


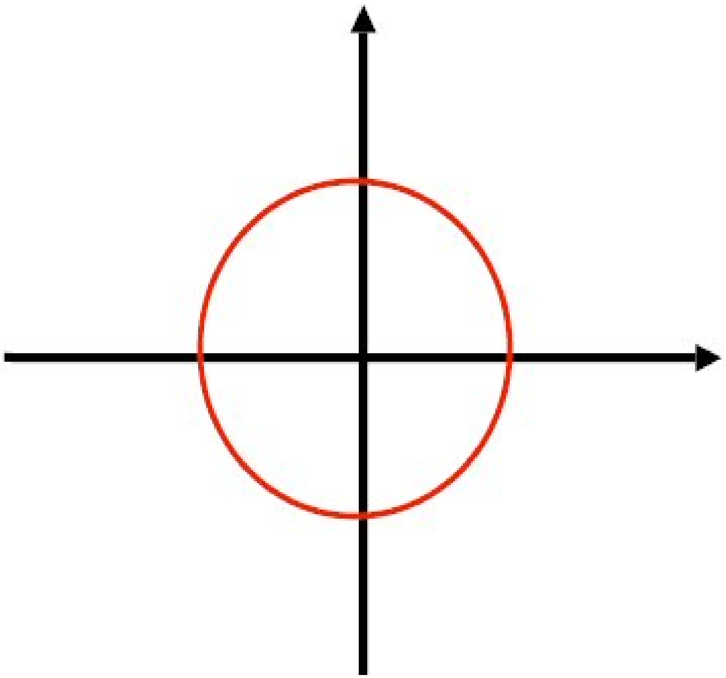
Double grating





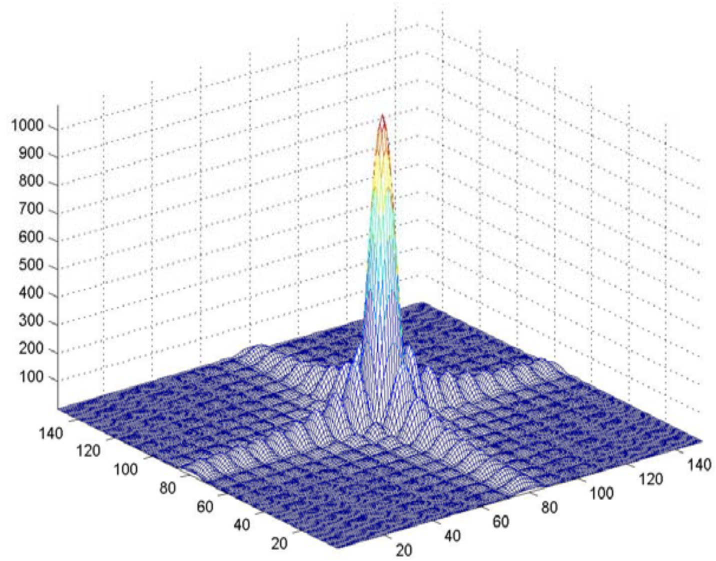
Smooth rings

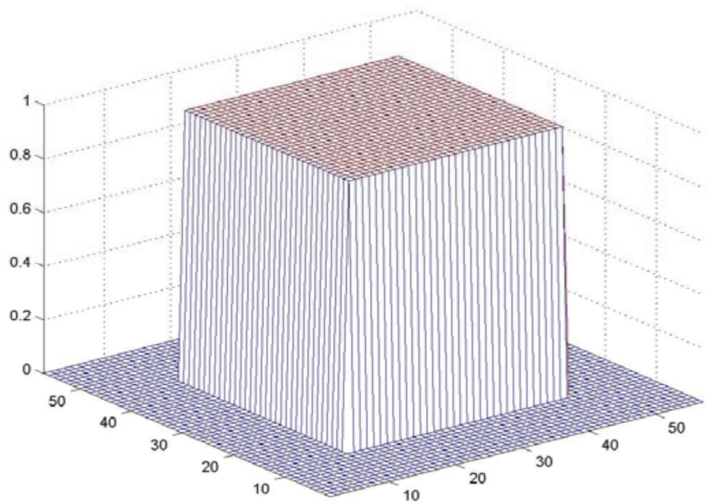


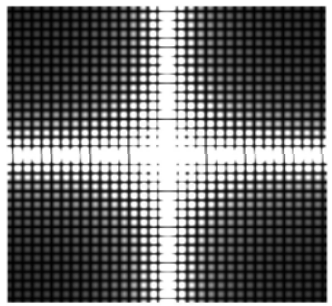


2D box

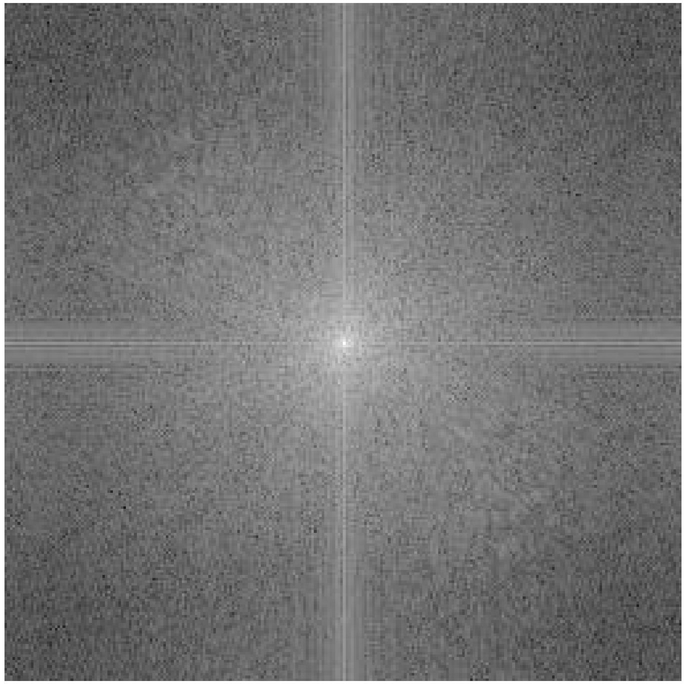
2D sinc



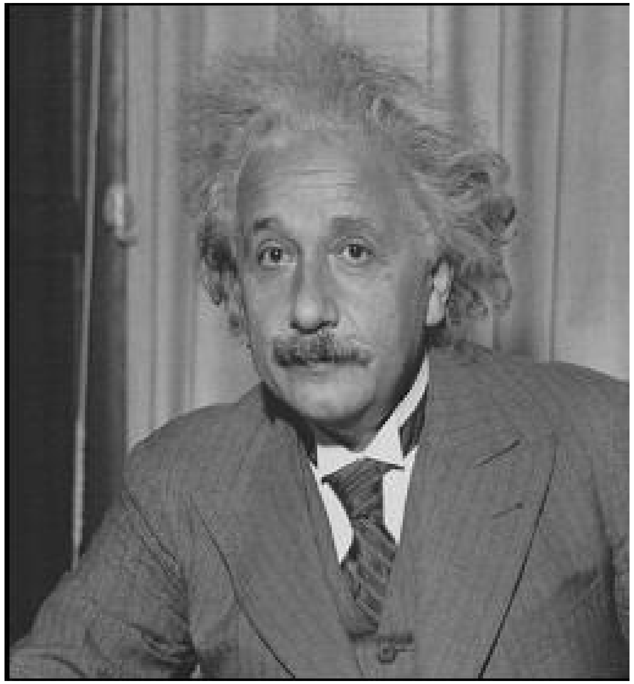


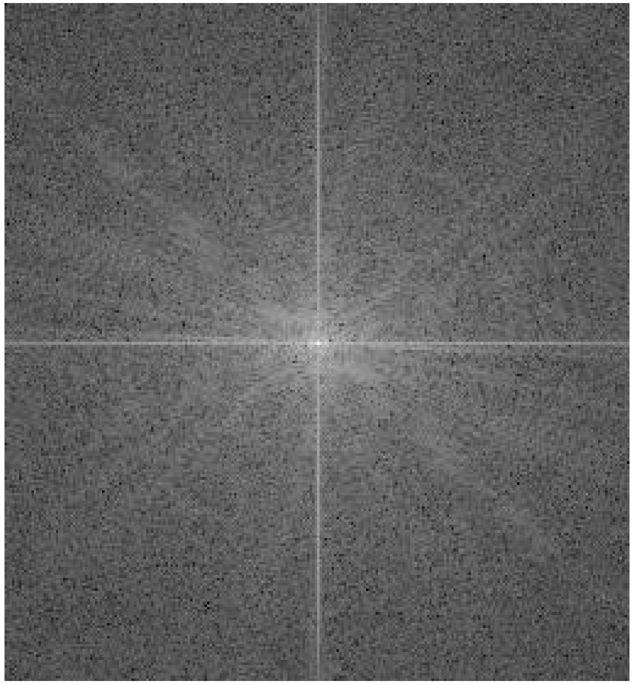


Margherita Hack



Einstein





log amplite of the spectrum

What we are going to analyze

- 2D Fourier Transform of continuous signals (2D-CTFT)

1D $$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt, \; f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} dt$$

- 2D Fourier Transform of discrete signals (2D-DTFT)

1D $$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k] e^{-j\Omega k}, \; f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega k} dt$$

- 2D Discrete Fourier Transform (2D-DFT)

1D $$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}, \; f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}, \Omega_0 = \frac{2\pi}{N_0}$$

2D Continuous Fourier Transform

- Continuous case (x and y are real) – 2D-CTFT (notation 1)

$$\hat{f}\left(\omega_x,\omega_y\right)=\int_{-\infty}^{+\infty} f\left(x,y\right)e^{-j\left(\omega_x x+\omega_y y\right)}dxdy$$

$$f\left(x,y\right)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\hat{f}\left(\omega_x,\omega_y\right)e^{j\left(\omega_x x+\omega_y y\right)}d\omega_x d\omega_y$$

$$\iint f\left(x,y\right)g^*\left(x,y\right)dxdy=\frac{1}{4\pi^2}\iint \hat{f}\left(\omega_x,\omega_y\right)\hat{g}^*\left(\omega_x,\omega_y\right)d\omega_x d\omega_y$$ Parseval formula

$$f=g\rightarrow\iint\left|f\left(x,y\right)\right|^2dxdy=\frac{1}{4\pi^2}\iint\left|\hat{f}\left(\omega_x,\omega_y\right)\right|^2d\omega_x d\omega_y$$ Plancherel equality

2D Continuous Fourier Transform

- Continuous case (x and y are real) – 2D-CTFT

$$\omega_x = 2\pi u$$

$$\omega_y = 2\pi v$$

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 \, dudv =$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 \, dudv$$

2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dxdy$$

$$f(x,y) = \int_{-\infty}^{+\infty} \hat{f}(u,v)\, e^{j2\pi(ux+vy)}\, dudv =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| f(x,y) \right|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \hat{f}(u,v) \right|^2 dudv$$

Plancherel's equality

2D Discrete Fourier Transform

The independent variable (t,x,y) is discrete

$$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}$$

$$f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

$$F[u,v] = \sum_{i=0}^{N_0-1} \sum_{k=0}^{N_0-1} f[i,k] e^{-j\Omega_0 (ui+vk)}$$

$$f_{N_0}[i,k] = \frac{1}{N_0^{\ 2}} \sum_{u=0}^{N_0-1} \sum_{v=0}^{N_0-1} F[u,v] e^{j\Omega_0 (ui+vk)}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

Delta

- Sampling property of the 2D-delta function (Dirac's delta)

$$\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) f(x, y) dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\left(\delta(x, y)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) e^{-j2\pi(ux+vy)} dxdy = 1$$

$$F\left(\delta(x - x_0, y - y_0)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) e^{-j2\pi(ux+vy)} dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

Constant functions

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y)=1 \qquad \forall x,y$$

$$F\{k\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)}dxdy = \delta(u,v)$$

Take the inverse Fourier Transform of the impulse function

$$F^{-1}\{\delta(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u,v)e^{j2\pi(ux+vy)}dudv = e^{j2\pi(0x+v0)} = 1$$

Trigonometric functions

- Cosinusoidal function oscillating along the x axis
  - Constant along the y axis

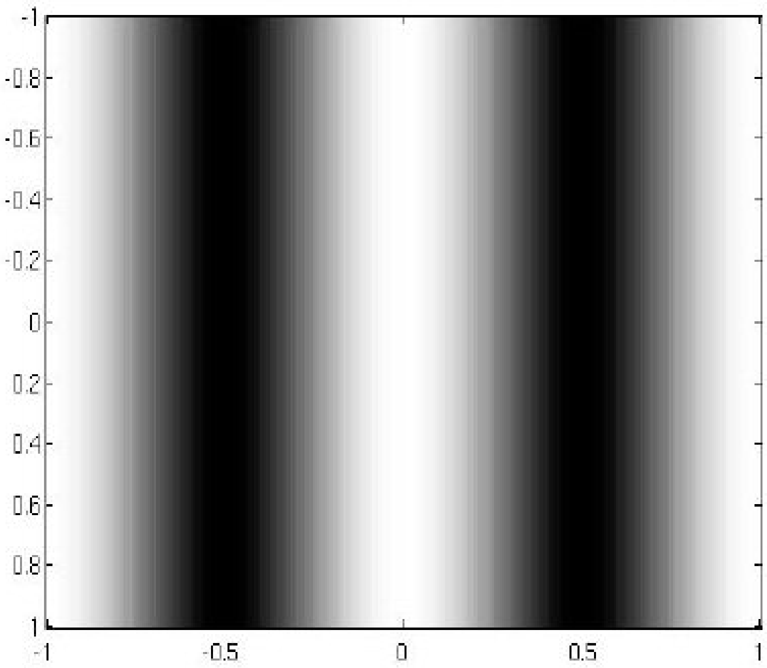
$$s(x,y) = \cos(2\pi fx)$$

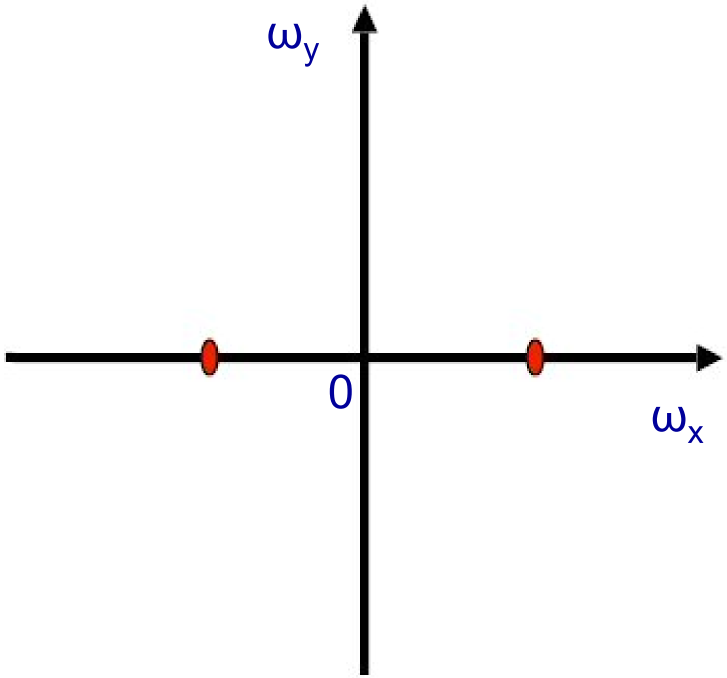
$$F\left\{\cos(2\pi fx)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(2\pi fx) e^{-j2\pi(ux+vy)} dxdy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{e^{j2\pi(fx)} + e^{-j2\pi(fx)}}{2}\right] e^{-j2\pi(ux+vy)} dxdy$$

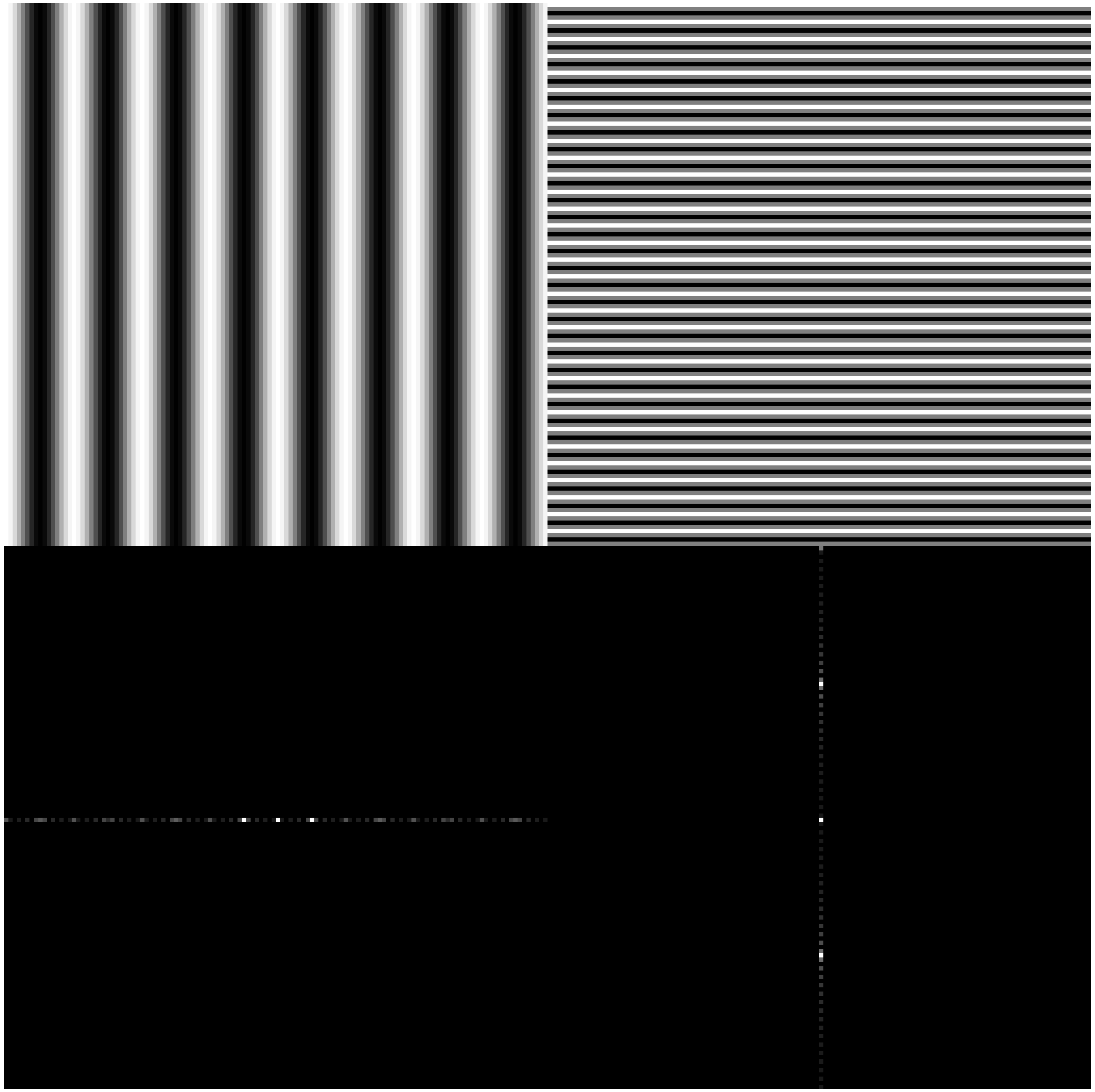
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] dxdy = \frac{1}{2}\left[\delta\left((u-f)\right) + \delta\left((u+f)\right)\right]$$

Vertical grating

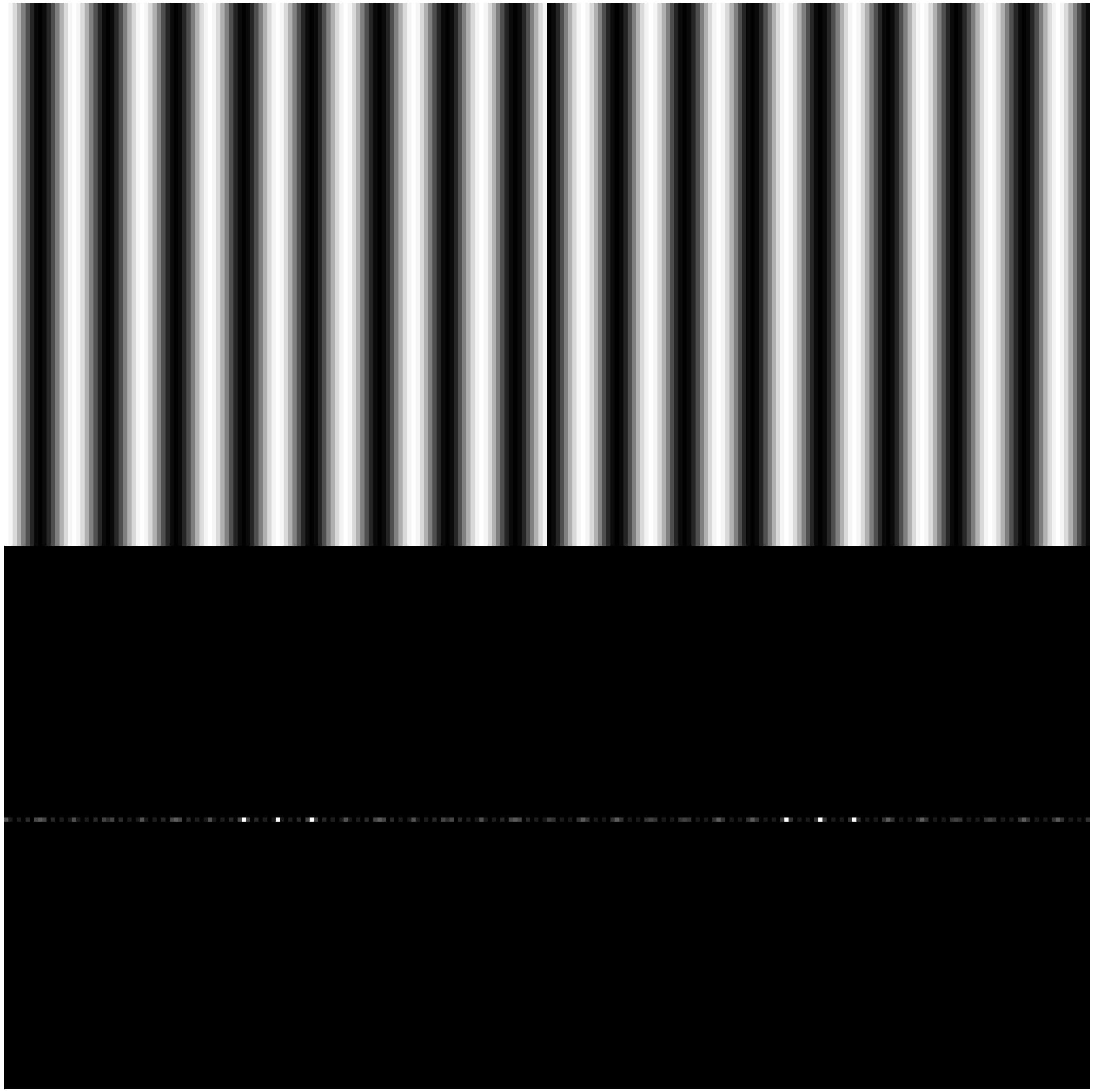




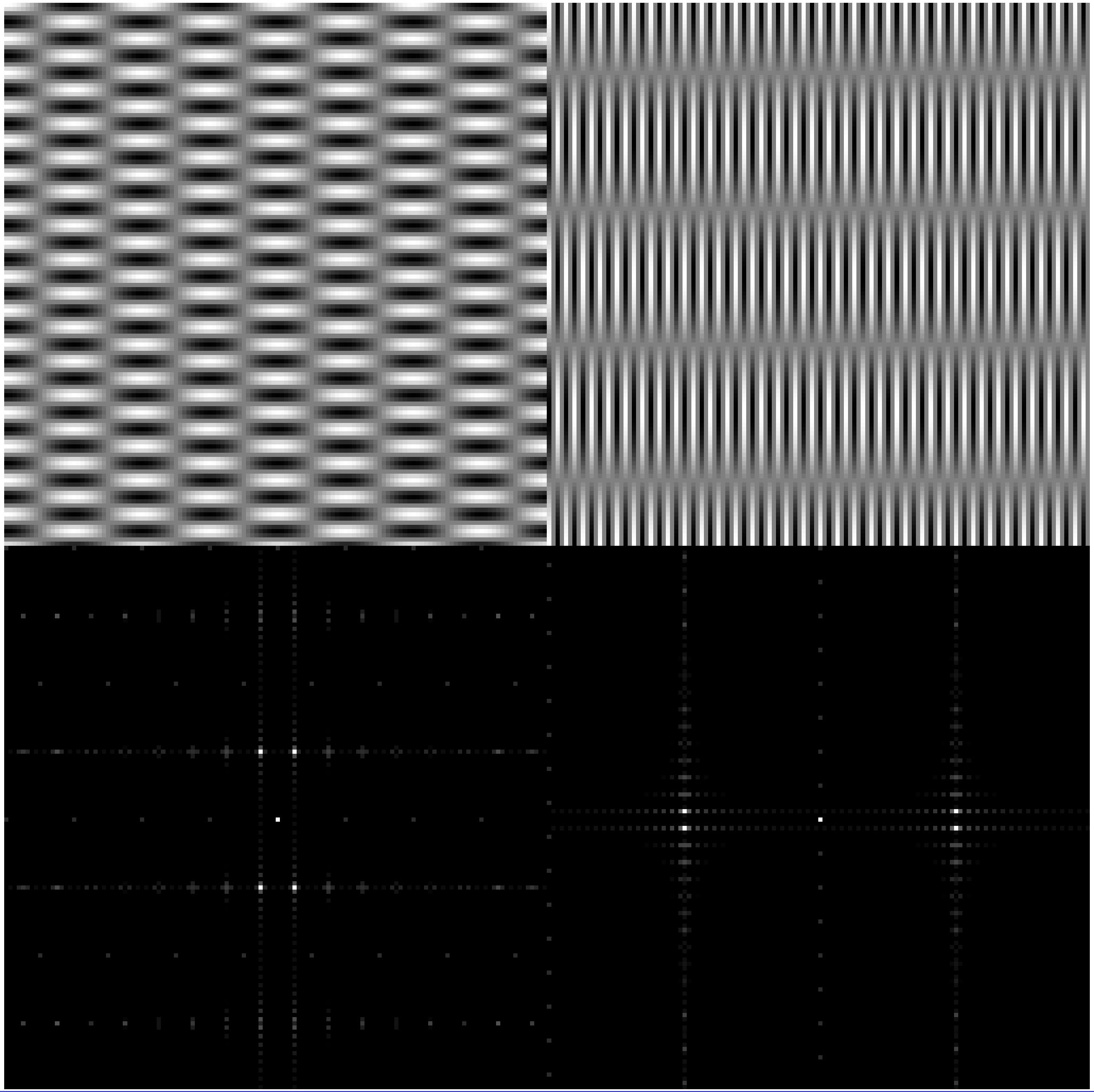
Ex. 1



Ex. 2

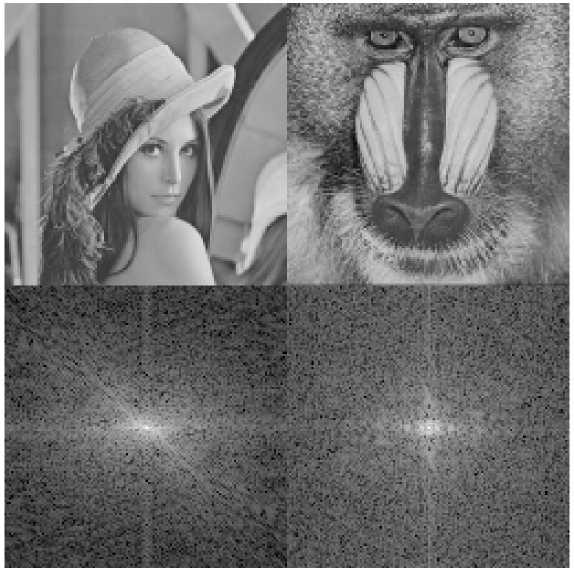
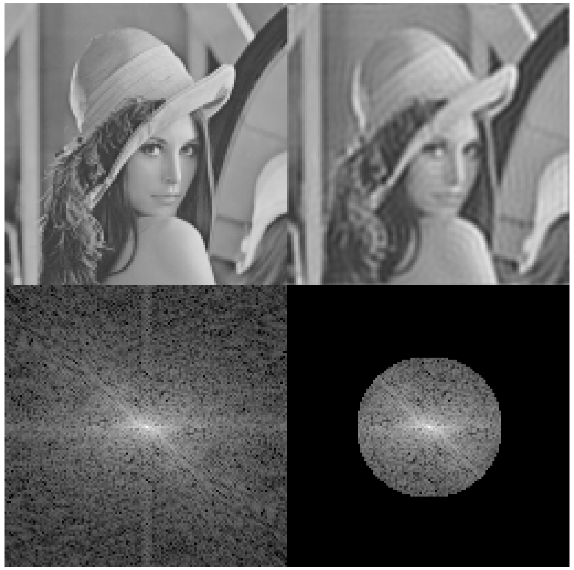


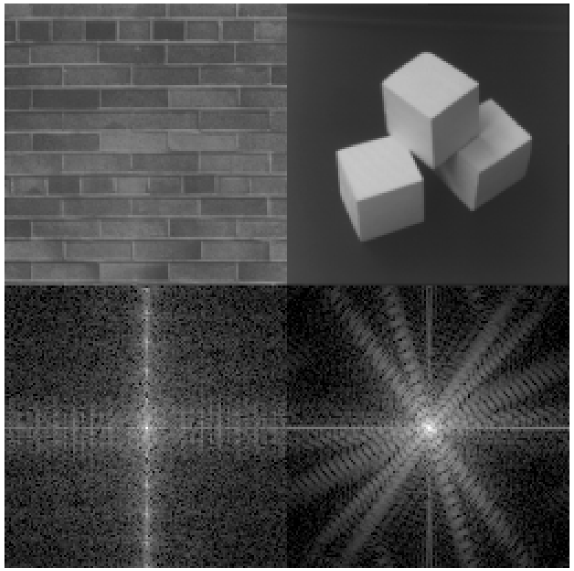
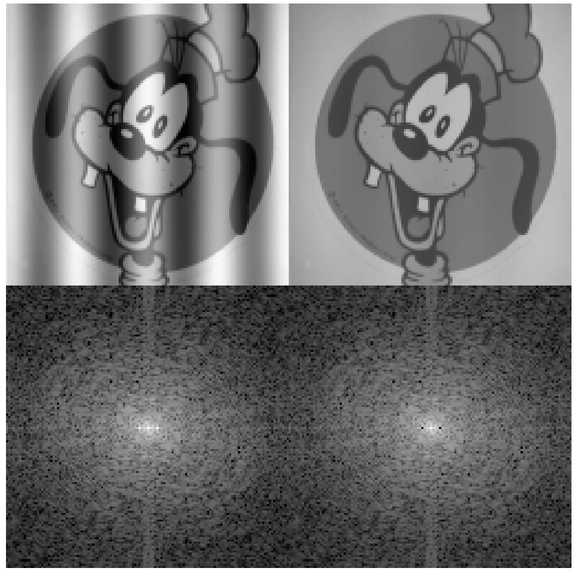
Ex. 3



Magnitudes

Examples

Properties

- Linearity $$af(x,y)+bg(x,y)\Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f(x-x_0,y-x_0)\Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0x+v_0y)}f(x,y)\Leftrightarrow F(u-u_0,v-v_0)$$
- Convolution $$f(x,y)*g(x,y)\Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f(x,y)g(x,y)\Leftrightarrow F(u,v)*G(u,v)$$
- Separability $$f(x,y)=f(x)f(y)\Leftrightarrow F(u,v)=F(u)F(v)$$

Separability

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx\right]e^{-j2\pi vy}dy$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy$$

2D Fourier Transform can be implemented as a sequence of 1D Fourier Transform operations performed independently along the two axis

2D Fourier Transform of a Discrete function

- Fourier Transform of a 2D a-periodic signal defined over a 2D discrete grid
  - The grid can be thought of as a 2D brush used for sampling the continuous signal with a given spatial resolution (T_x,T_y)

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}, \; f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}dt$$

$$F(\Omega_x, \Omega_y) = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} f[k_1, k_2]e^{-j\left(k_1\Omega_x + k_2\Omega_y\right)}$$

$$f[k] = \frac{1}{4\pi^2}\int_{2\pi}\int_{2\pi} F(\Omega_x, \Omega_y)e^{j\left(k_1\Omega_x + k_2\Omega_y\right)}d\Omega_x\Omega_y$$

Unitary frequency notations

$$\begin{cases} \Omega_x = 2\pi u \\ \Omega_y = 2\pi v \end{cases}$$

$$F(u,v) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1,k_2] e^{-j2\pi(k_1 u + k_2 v)}$$

$$f[k_1,k_2] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{-j2\pi(k_1 u + k_2 v)} du dv$$

- The integration interval for the inverse transform has width=1 instead of 2π
  - It is quite common to choose

$$\frac{-1}{2} \leq u,v < \frac{1}{2}$$

Properties

- Periodicity: 2D Fourier Transform of a discrete a-periodic signal is periodic with period
  - The period is 1 for the unitary frequency notations and 2π for normalized frequency notations. Referring to the firsts:

$$F(u+k, v+l) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left(um+vn\right)} e^{-j2\pi km} e^{-j2\pi ln}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

$$= F(u,v)$$

Arbitrary integers

1 1

Properties

- Linearity
- shifting
- modulation
- convolution
- multiplication
- separability
- energy conservation properties also exist for the 2D Fourier Transform of discrete signals.
- NOTE: in what follows, (k_1,k_2) is replaced by (m,n)

Fourier Transform: Properties

- Linearity $af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$
- Shifting $f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$
- Modulation $e^{j2\pi(u_0m+v_0n)} f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$
- Convolution $f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$
- Separable functions $f[m,n]= f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$
- Energy conservation $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|f[m,n]\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$

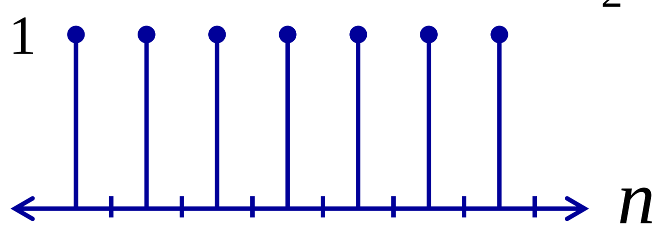
Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

where M and N are integers

$comb_2[n]$



2D-DTFT: delta

- Define *Kronecker delta function*

$$\delta[m,n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Fourier Transform of the Kronecker delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\delta[m,n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1$$

Fourier Transform: (piecewise) constant

- Fourier Transform of 1

$$f[m,n]=1$$

$$F[u,v]=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[1e^{-j2\pi(um+vn)}\right]=$$

$$=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(u-k,v-l)$$

To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.

Impulse Train

$$comb_{M,N}[m,n] \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

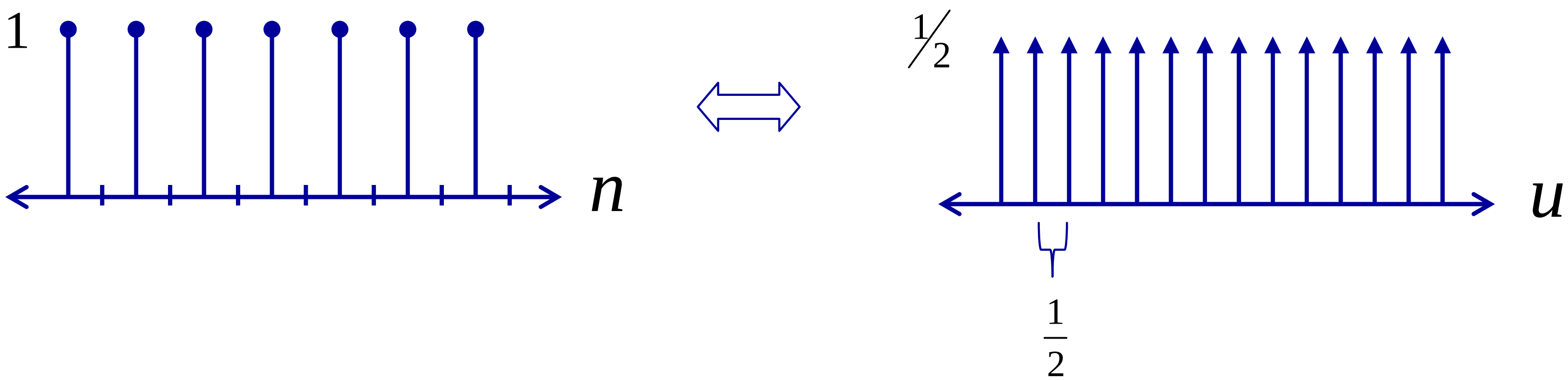
$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(x-kM, y-lN)$$

- Fourier Transform of an impulse train is also an impulse train:

$$\underbrace{\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]}_{comb_{M,N}[m,n]} \Leftrightarrow \underbrace{\frac{1}{MN} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(u-\frac{k}{M}, v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

Impulse Train

$$comb_2[n] \quad \Longleftrightarrow \quad \frac{1}{2} comb_{\frac{1}{2}}(u)$$



Impulse Train

$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM, y-lN\right)$$

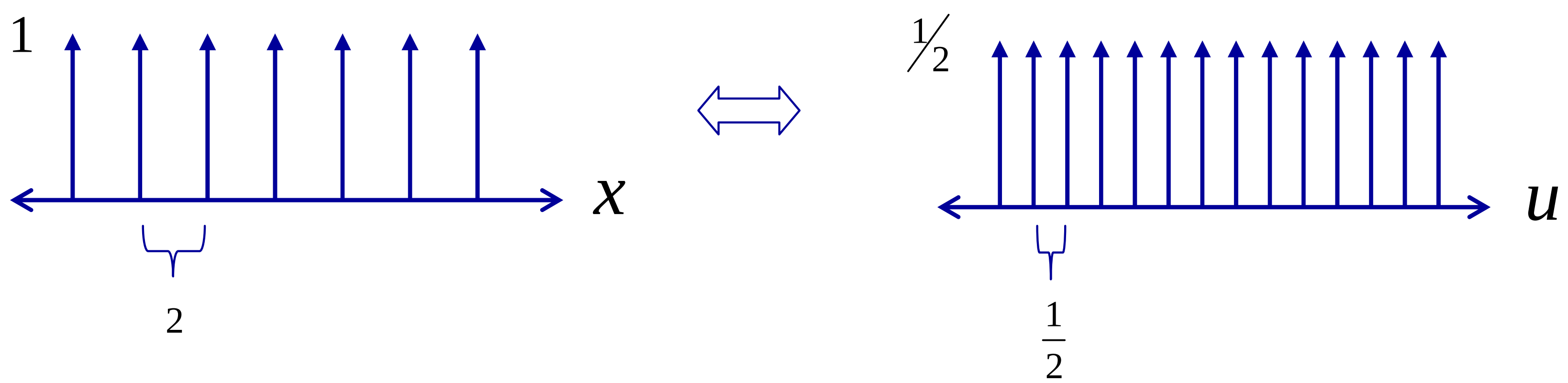
- In the case of continuous signals:

$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM, y-lN\right)}_{comb_{M,N}(x,y)} \Leftrightarrow \frac{1}{MN}\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M}, v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

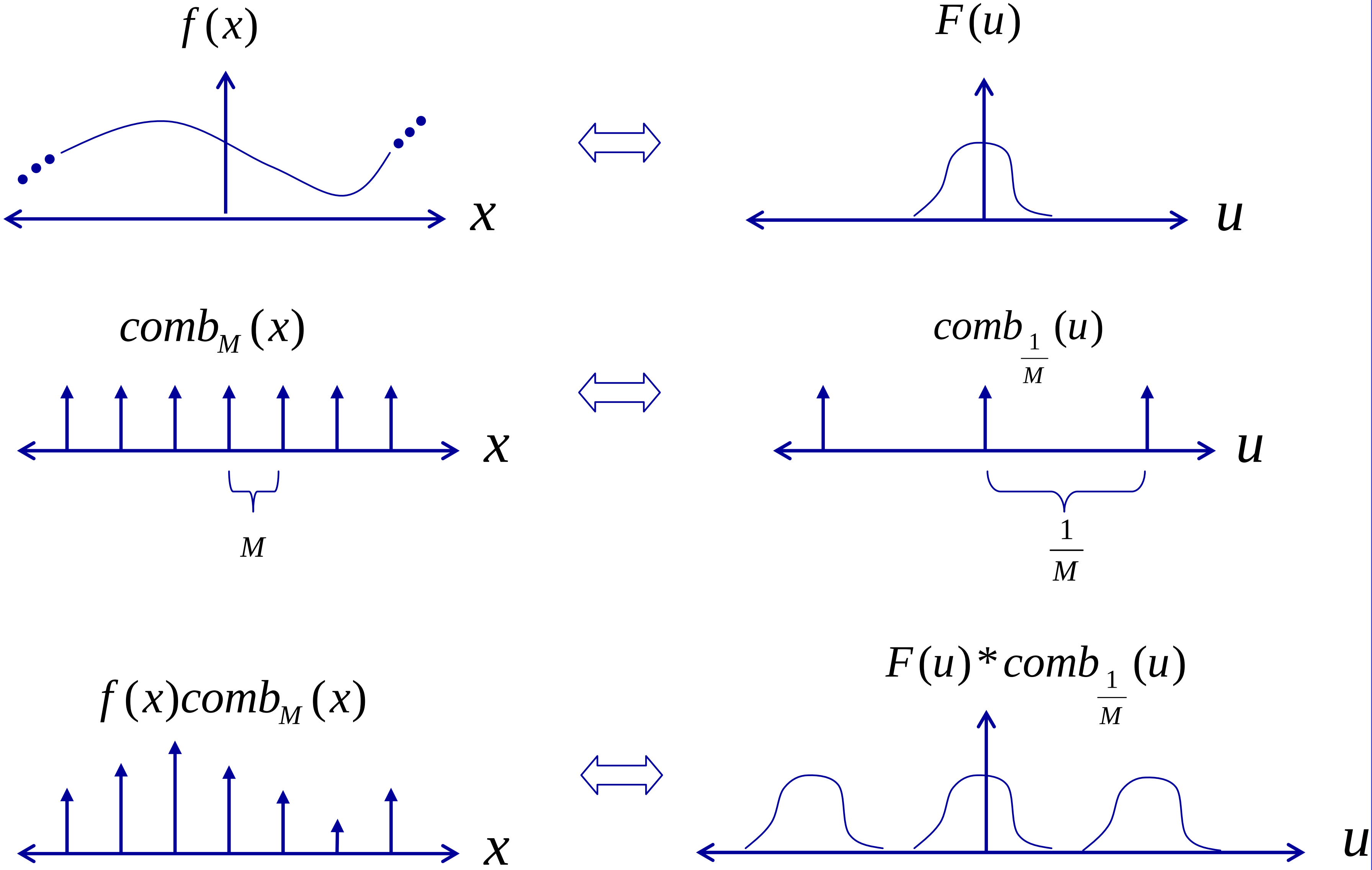
Impulse Train

$comb_2(x)$

$\frac{1}{2}comb_{\frac{1}{2}}(u)$

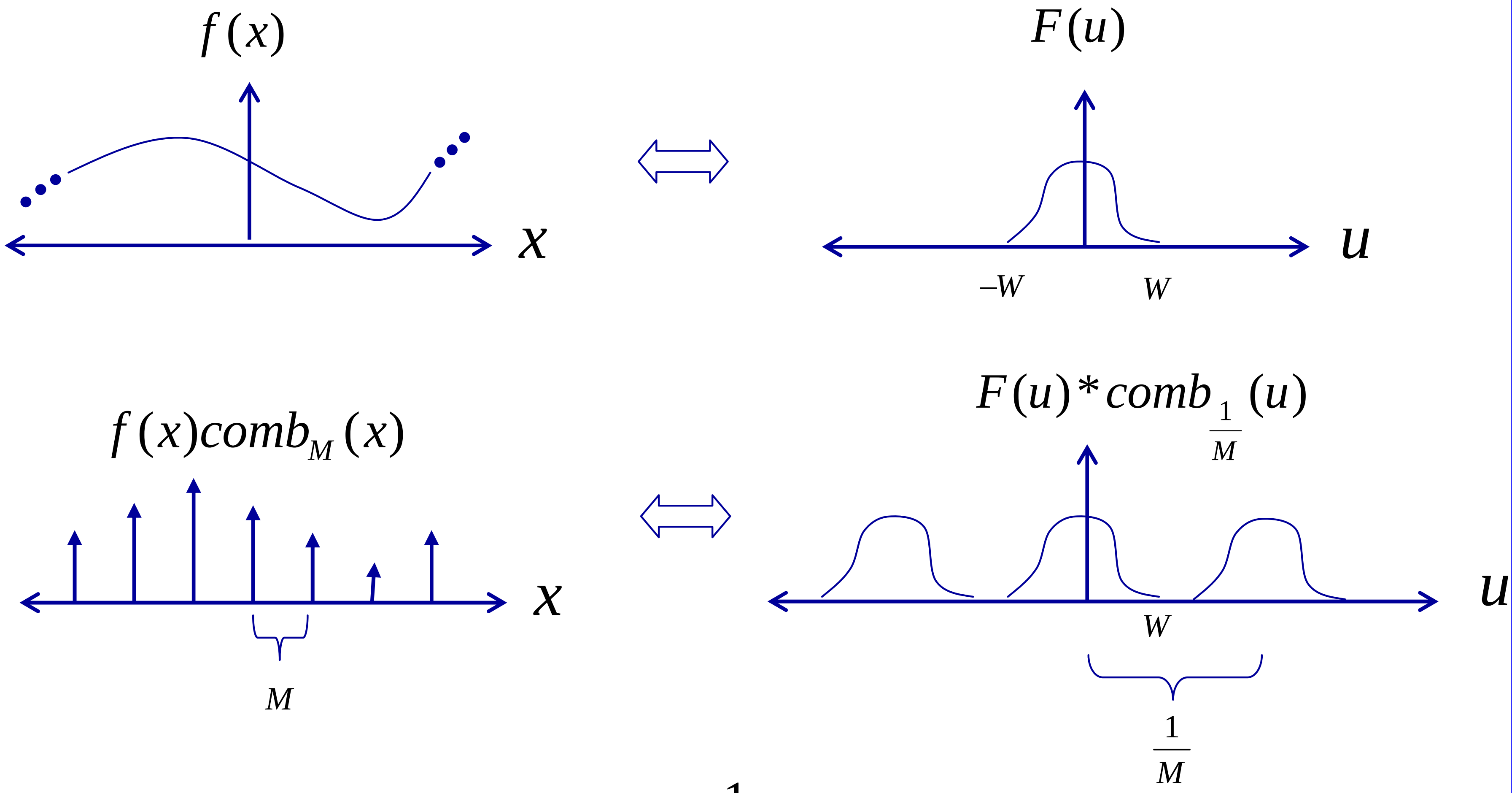


Sampling



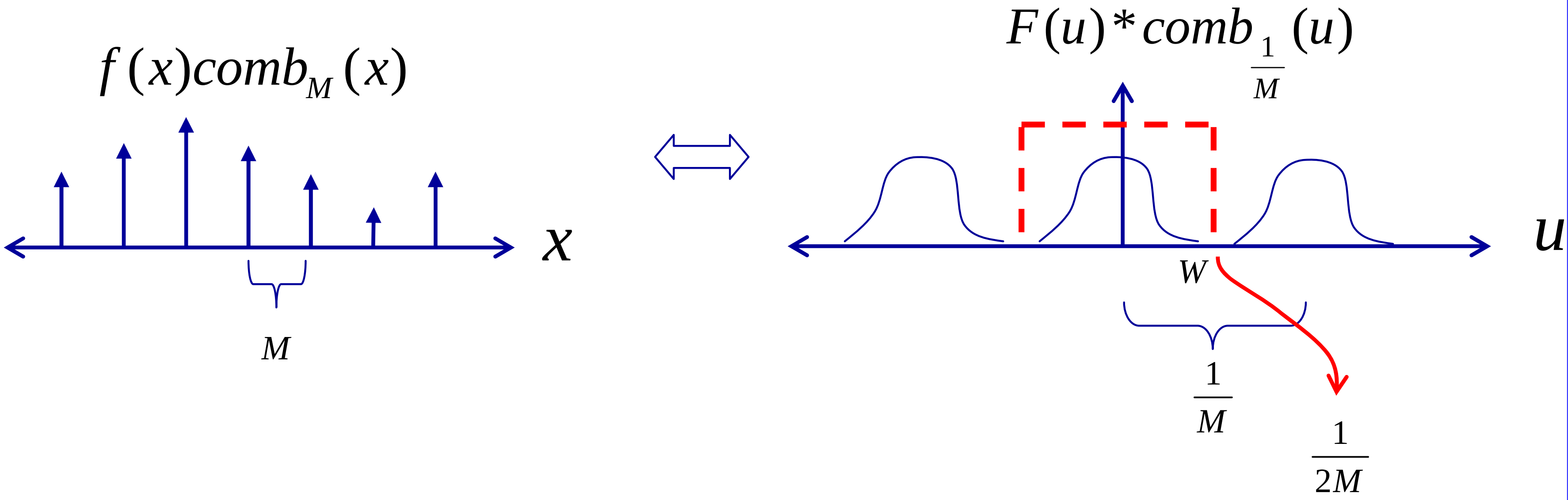
$f(x)$
x
$F(u)$
u
$comb_M(x)$
x
M
$comb_{\frac{1}{M}}(u)$
u
$\frac{1}{M}$
$f(x)comb_M(x)$
x
$F(u) * comb_{\frac{1}{M}}(u)$
u

Sampling revisitation



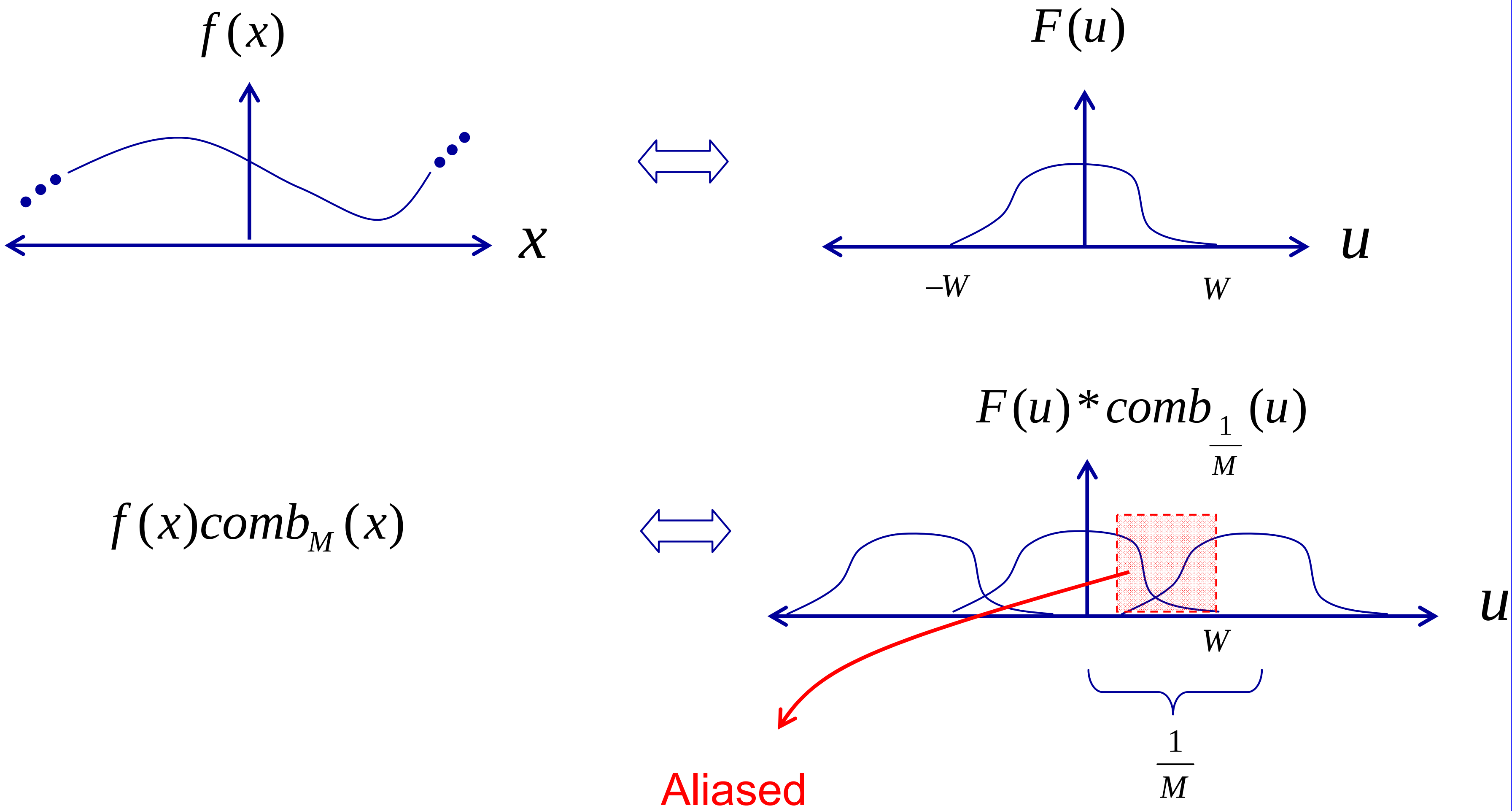
No aliasing if $\frac{1}{M} > 2W$

Sampling and aliasing

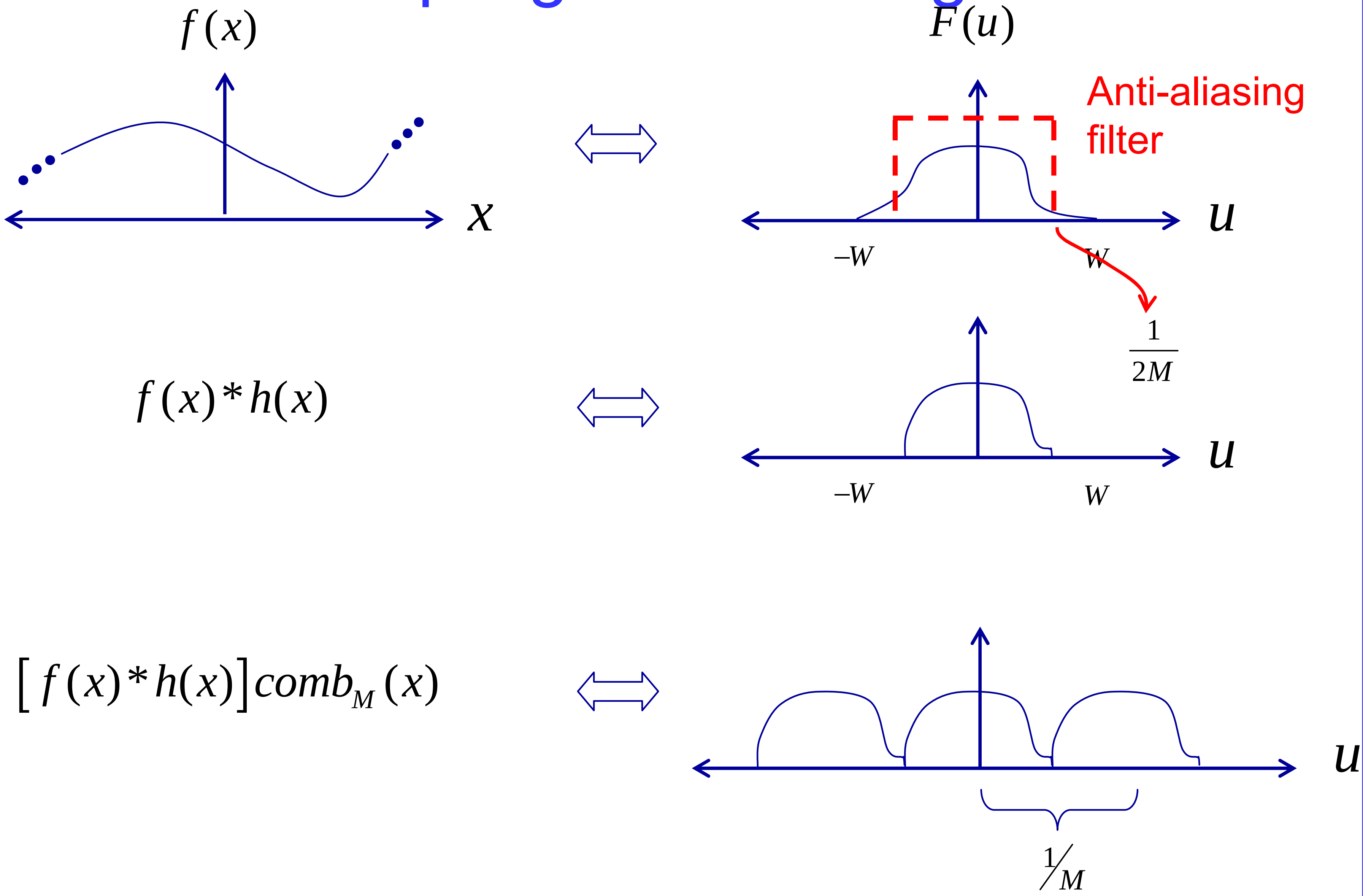


If there is no aliasing, the original signal can be recovered from its samples by low-pass filtering.

Sampling and aliasing



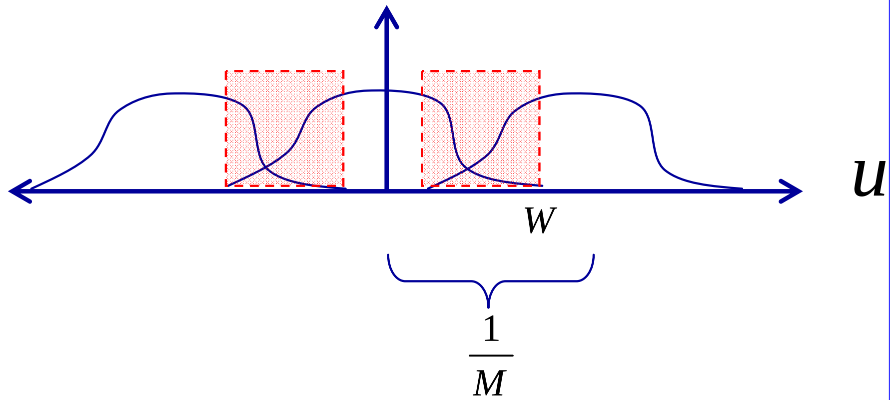
Sampling and aliasing



Sampling and aliasing

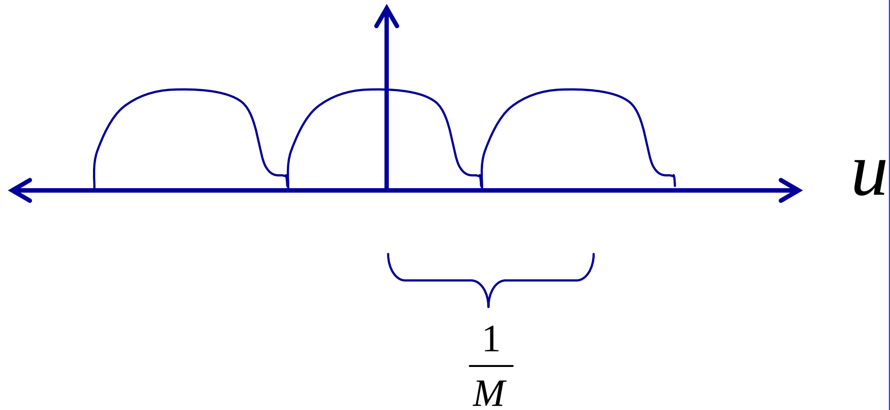
- Without anti-aliasing filter:

$$f(x)comb_M(x) \Longleftrightarrow$$



- With anti-aliasing filter:

$$[f(x)*h(x)]comb_M(x) \Longleftrightarrow$$



Aliasing in images

- Without the anti-aliasing filter the recovered image (subsampling+upsampling) is different from the original.
- With anti-aliasing filter (low-pass), the *smoothed* version of the original image can be recovered by interpolation

Anti-Aliasing

Original image

Anti-Aliasing

```
a=imread('barbara.tif');
b=imresize(a,0.25);
c=imresize(b,4);
```

Downsampling without anti-aliasing followed by interpolation

No anti-aliasing,
bicubic
interpolation

WITH anti-aliasing,
bicubic interpolation

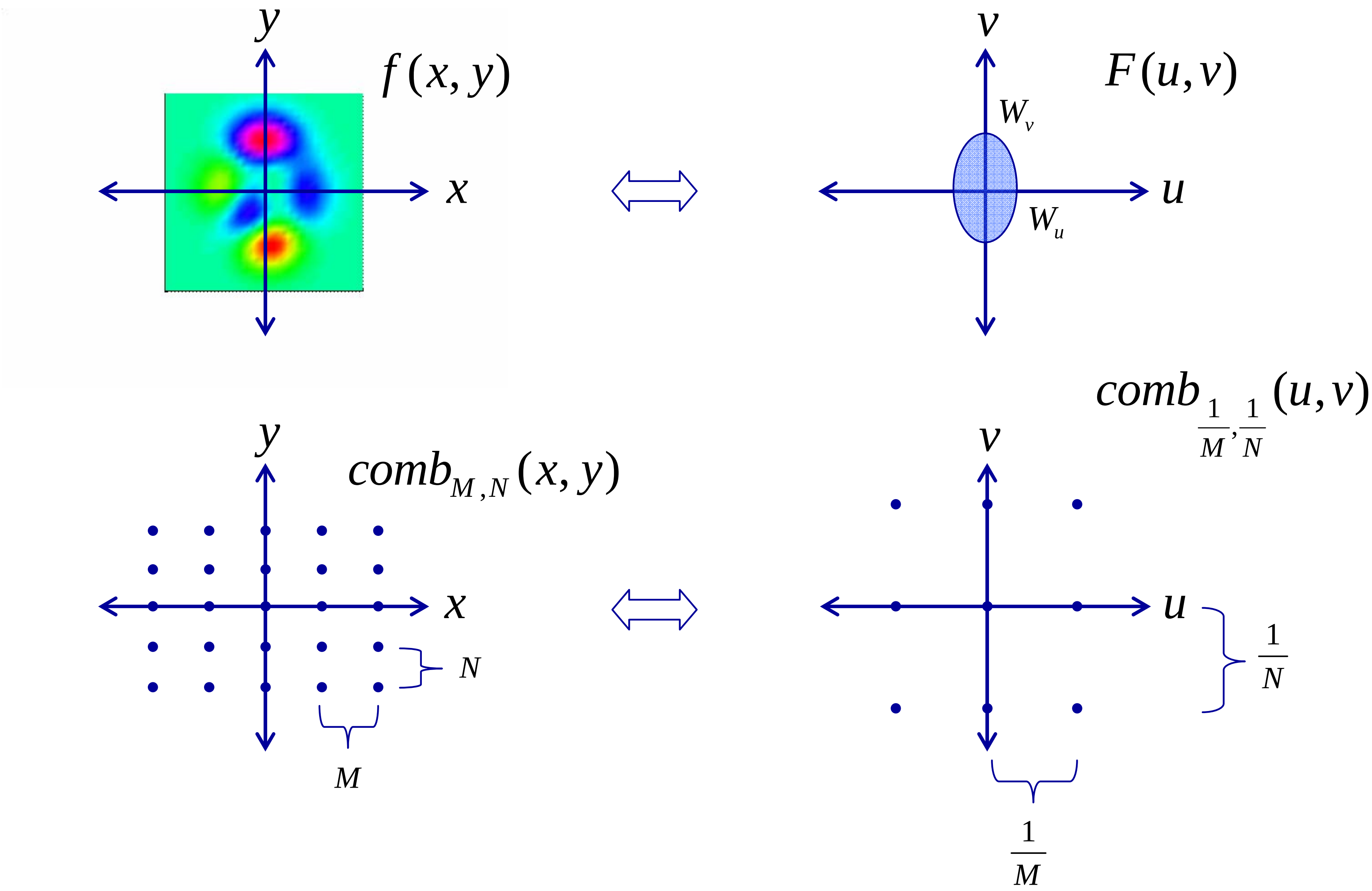
Anti-Aliasing

```
a=imread('barbara.tif');
H=zeros(512,512);
H(256-64:256+64, 256-
64:256+64)=1;

Da=fft2(a);
Da=fftshift(Da);
Dd=Da.*H;
Dd=fftshift(Dd);
d=real(ifft2(Dd));
```

WITH anti-aliasing by low-pass filtering in frequency domain (box filter in image space)

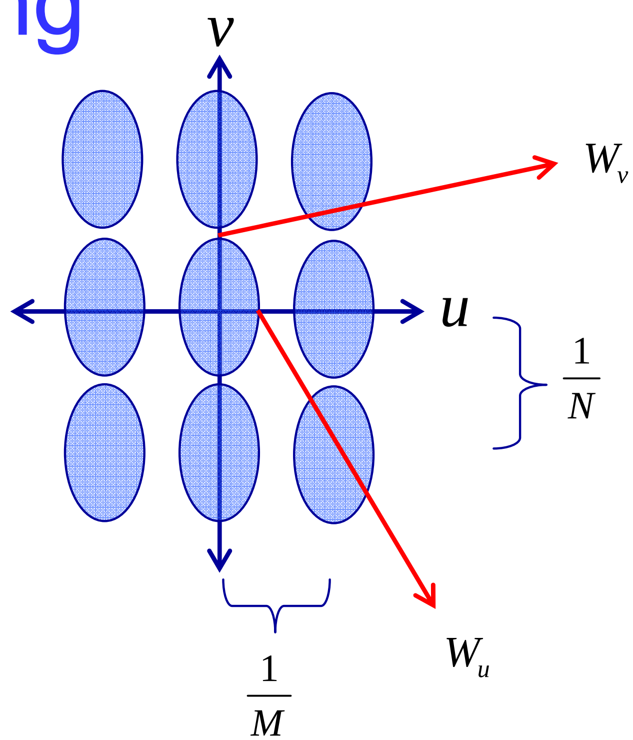
Sampling



y
x
f(x,y)
v
u
F(u,v)
W_v
W_u
y
x
$comb_{M,N}(x,y)$
N
M
v
u
$comb_{\frac{1}{M},\frac{1}{N}}(u,v)$
$\frac{1}{N}$
$\frac{1}{M}$

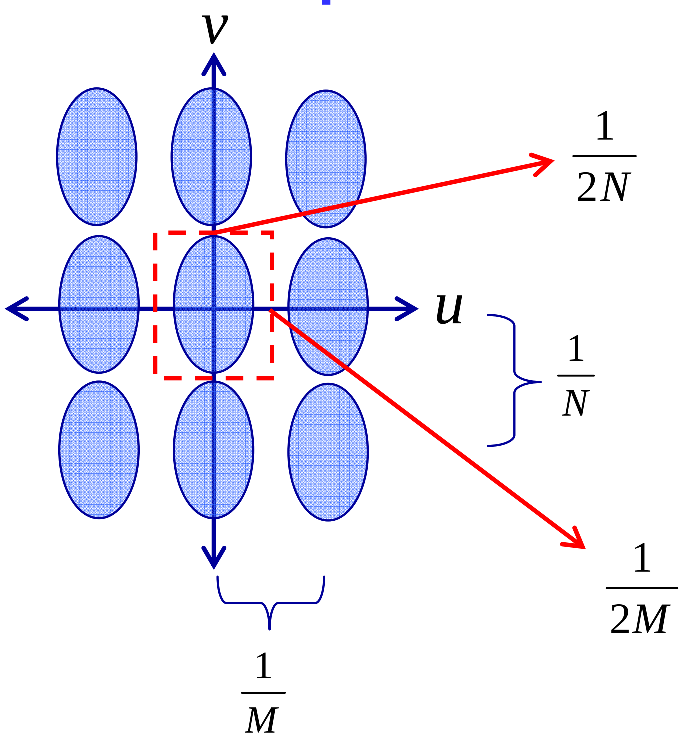
Sampling

$$f(x,y)comb_{M,N}(x,y) \quad \Leftrightarrow$$



No aliasing if $\frac{1}{M} > 2W_u$ *and* $\frac{1}{N} > 2W_v$

Interpolation



Ideal reconstruction filter:

$$H(u,v) = \begin{cases} MN, \text{ for } u \le \frac{1}{2M} \text{ and v} \le \frac{1}{2N} \\ 0, \text{ otherwise} \end{cases}$$

Ideal Reconstruction Filter

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(u,v)e^{j2\pi(ux+vy)}dudv=\int_{\frac{1}{2N}}^{\frac{1}{2N}}\int_{\frac{-1}{2M}}^{\frac{1}{2M}}MNe^{j2\pi(ux+vy)}dudv$$

$$=\int_{\frac{-1}{2M}}^{\frac{1}{2M}}Me^{j2\pi ux}du\int_{\frac{-1}{2N}}^{\frac{1}{2N}}Ne^{j2\pi vy}dv$$

$$=M\frac{1}{j2\pi x}\left(e^{j2\pi x\frac{1}{2M}}-e^{-j2\pi x\frac{1}{2M}}\right)N\frac{1}{j2\pi y}\left(e^{j2\pi y\frac{1}{2N}}-e^{-j2\pi y\frac{1}{2N}}\right)$$

$$=\frac{\sin\left(\frac{\pi}{M}x\right)}{\frac{\pi}{M}x}\frac{\sin\left(\frac{\pi}{N}y\right)}{\frac{\pi}{N}y}$$

$$\sin(x)=\frac{1}{2j}\left(e^{jx}-e^{-jx}\right)$$