

# Lectures on

## DIFFERENTIAL GEOMETRY AND TOPOLOGY

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### Lecture III

Tensor product spaces p. 1

(p,q) tensors p. 4

Examples p. 5

#### \* Tensor products

Let  $V, W$  (finite dimensional) vector spaces over a field  $K$  ( $\dim_K V = n, \dim_K W = m$ );

their tensor product  $V \otimes_K W$  (or  $V \otimes W$ , if no confusion arises) is defined as follows

$$V \otimes_K W := \left\{ \pi : \begin{matrix} V^* \times W^* \\ \downarrow \quad \downarrow \\ V \times W \end{matrix} \longrightarrow K / \pi \text{ bilinear} \right\}$$

It is naturally a vector space, generated by  $\pi$ 's of the form

$$\text{decomposable vectors in } V \otimes W \quad \text{vectors in } V \quad \text{vectors in } W$$
$$\pi : (v^*, w^*) \longmapsto v^*(v) \cdot w^*(w)$$

$v^* \in V^*, w^* \in W^*$        $v \in V, w \in W$        $\pi \in K$       product in  $K$

Notice that  $\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$

$\forall \alpha \in K, v \in V, w \in W$ , and

$$(\alpha v_1 + \beta v_2) \otimes w = \alpha(v_1 \otimes w) + \beta(v_2 \otimes w) \quad \text{etc.}$$

(as maps).

Given, as usual, bases  $e = (e_1, \dots, e_n)$ ,  $f = (f_1, \dots, f_m)$  in  $V$  and  $W$ , resp., it is easily checked that

$e \otimes f = (e_i \otimes f_j)_{\substack{i=1 \dots n \\ j=1 \dots m}}$  yields a basis for

$V \otimes W$ , i.e., any  $\pi \in V \otimes W$  can be written as

$$\pi = \sum_{i,j} c_{ij} e_i \otimes f_j, \text{ for uniquely determined } c_{ij} \in K.$$

Hence  $\dim V \otimes W = \dim V \cdot \dim W = n \cdot m$

$$\begin{aligned} \text{Notice, in particular, } & (e_i \otimes f_j)(e_k^*, f_l^*) = \\ & = e_k^*(e_i) \cdot f_l^*(f_j) = \delta_{ik} \cdot \delta_{lj} \end{aligned}$$

and that, if  $v = \sum_i \alpha_i e_i$ ,  $w = \sum_j \beta_j f_j$ ,

then

$$v \otimes w = \sum_{i,j} \alpha_i \beta_j e_i \otimes f_j$$

Important remark

$$\text{Hom}(V, W) \cong W \otimes V^*$$

canonically

$$\text{This comes from } (w \otimes v^*)(x) = v^*(x) w$$

*leave it as it stands*

Concretely, choosing bases

$$w = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \quad m \times 1 \quad w^* = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \quad 1 \times n$$

$$w \otimes w^* = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \quad m \times 1 \quad \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \quad 1 \times n = \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \quad m \times n \rightarrow \quad m \times n - \text{matrix}$$

matrix product

$$\boxed{w \otimes w^* = \left\{ \sum_{i,j} a_{ij} (f_i \otimes e_j^*) \right\}}$$

$$\sum_{i,j} a_{ij} (f_i \otimes e_j^*)(\cdot, e_k) = \sum_{i,j} a_{ij} e_j^*(e_k) f_i$$

$$= \sum_i a_{ik} f_i, \text{ i.e. } e_{12} \mapsto \sum_i a_{ik} f_i$$

$\rightarrow \begin{pmatrix} \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} & \text{R} \end{pmatrix}$

$\downarrow$  R-column

of the matrix representation  
of a homomorphism  $T$

$$A \oplus \cdots \oplus A \oplus \cdots \oplus A \in (\mathbb{K}, \dots, \mathbb{K}, 0, \dots, 0)$$

One can naturally define  $v_1 \otimes v_2 \otimes v_3$  etc.

either directly or via  $(v_1 \otimes v_2) \otimes v_3$

(canonically isomorphic to  $v_1 \otimes (v_2 \otimes v_3)$ ) and concluding by induction.

We shall stick to the case in which  $\bar{V}_i = V$  or  $V^*$

Define the space of

$\uparrow$

$\downarrow$

contravariance  
index

covariance  
index

notice

↑ notice

$$\mathcal{G}_{p,q} = \underbrace{V \otimes \dots \otimes V}_{q} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{p}$$

$$\mathcal{G}_{p,q} = \{ \pi : V \times V^* \times \dots \times V^* \times V \times \dots \times V^* \rightarrow K \mid \pi$$

multilinear,  
i.e. linear in  
each argument

$$V \otimes V^* \cong V^* \otimes V$$

canonically

$$V \otimes V \otimes V^* \cong$$

$$V^* \otimes V \otimes V \cong$$

$$V \otimes V^* \otimes V$$

etc.

$T \in \mathcal{G}_{p,q}$  is  $p$ -times contravariant  
 $q$ -times covariant

Fixing a basis  $e = (e_1, \dots, e_m)$  in  $V$ ,

together with the dual basis's

$$e^* = (e_1^*, \dots, e_m^*)$$

, we have

$$T = \sum_{I,J} T_{j_1 \dots j_q}^{i_1 \dots i_p} e^{j_1} \otimes e^{j_2} \dots \otimes e^{j_q} \quad \dots \quad e^{i_1} \otimes e^{i_2} \dots \otimes e^{i_p}$$

$$I = (i_1, \dots, i_p)$$

$p$ -multindex..

$$J = (j_1, \dots, j_q)$$

$q$ -multindex

Components of  $T$  with respect to

The basis  $(e^{j_1} \otimes e^{j_2} \dots \otimes e^{j_q} \otimes e_{i_1} \dots \otimes e_{i_p})$

Schematically:

$$T = T_J^I e^J \otimes e_I$$

↑ notice has

$e_j^*$  instead of  $e_j$

↑ Einstein's convention



this notation  
would be

more appropriate

\* Examples  $V, V^*, \text{End}(V) \cong V \otimes V^* \cong V^* \otimes V$   
endomorphisms

are easily recovered.

$$V = \mathcal{Z}_{1,0}, \quad V^* = \mathcal{Z}_{0,1}$$

$$\text{End}(V) \cong \mathcal{Z}_{1,1}$$

$$T = (a_j^i) \quad N = x^i e_i$$

↑  
Notice this  $i$  is row index  
↓  
Einstein's conv.  
 $a_j^i$  ↑ column index

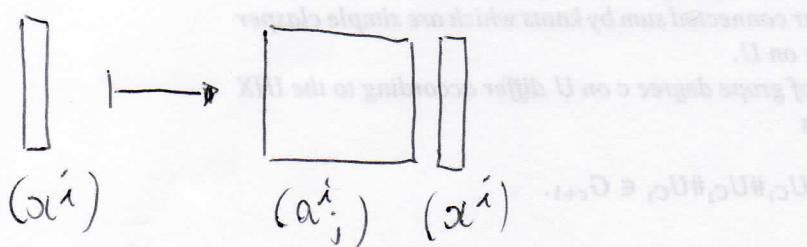
$$T = a_j^i e^j \otimes e_i$$

\* Complete  $T \circ \tau \rightarrow$  using tensor notation (Einstein's convention being employed throughout)

$$(a_j^i e^j \otimes e_i)(x^k e_k) = a_j^i x^k e^j (e_k) e_i = a_j^i x^k \delta_{jk} \cdot e_i$$

Einstein

$$= a_j^i x^j e_i \Rightarrow (x^i) \mapsto (a_j^i x^j)$$



\* Another example (crucial in Riemannian geometry)

$\langle , \rangle$  inner product on a Euclidean vector space  $\mathbb{V}$

$\langle , \rangle$  is a  $(0,2)$ -tensor

$\langle , \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$

bilinear

Upon choosing a basis  $e = (e_1, \dots, e_n)$

(+ symmetric  
+ positive definite)

$$v = x^i e_i$$

$$\langle e_i, e_j \rangle = g_{ij}$$

$$w = y^j e_j$$

$$\langle v, w \rangle = g_{ij} x^i y^j$$

Fig. 3. The configuration space

Theorem 3.16. Suppose that the square  $C$  has an angle  $\pi/2$  at one vertex. Given a mapping of  $C$ , this is equivalent that the mapping of  $C$  has a right-angle at each vertex of  $C$ . Now this means the boundary of  $C$  consists of two straight lines meeting at a right-angle. This is called a  $90^\circ$ -configuration (or  $90^\circ$ -configuration) of  $C$ .

surfaces in all

Suppose  $C$  has a right-angle at one vertex. A picture of the situation for the case of  $n=2$  is shown in Figure 3.10. The basic idea is to find the regions that are obtained by cutting a square along the diagonal. A cut along the diagonal in this case only divides the square into two triangles.

Figure 3.10 shows two such regions, but there's also a third region, which is the intersection of the two triangles. This intersection is the  $90^\circ$ -configuration of  $C$ .