SCHEMAS AND NOTES ON KINEMATICS

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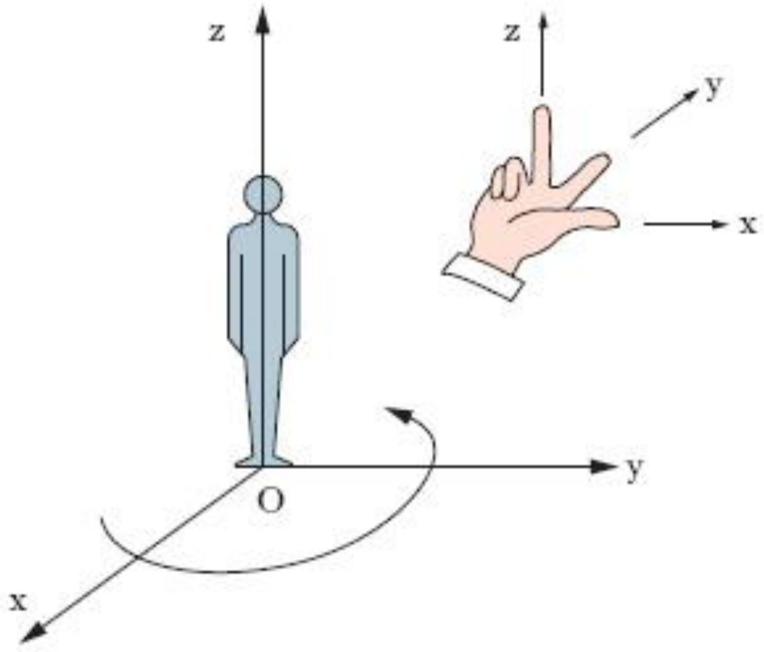
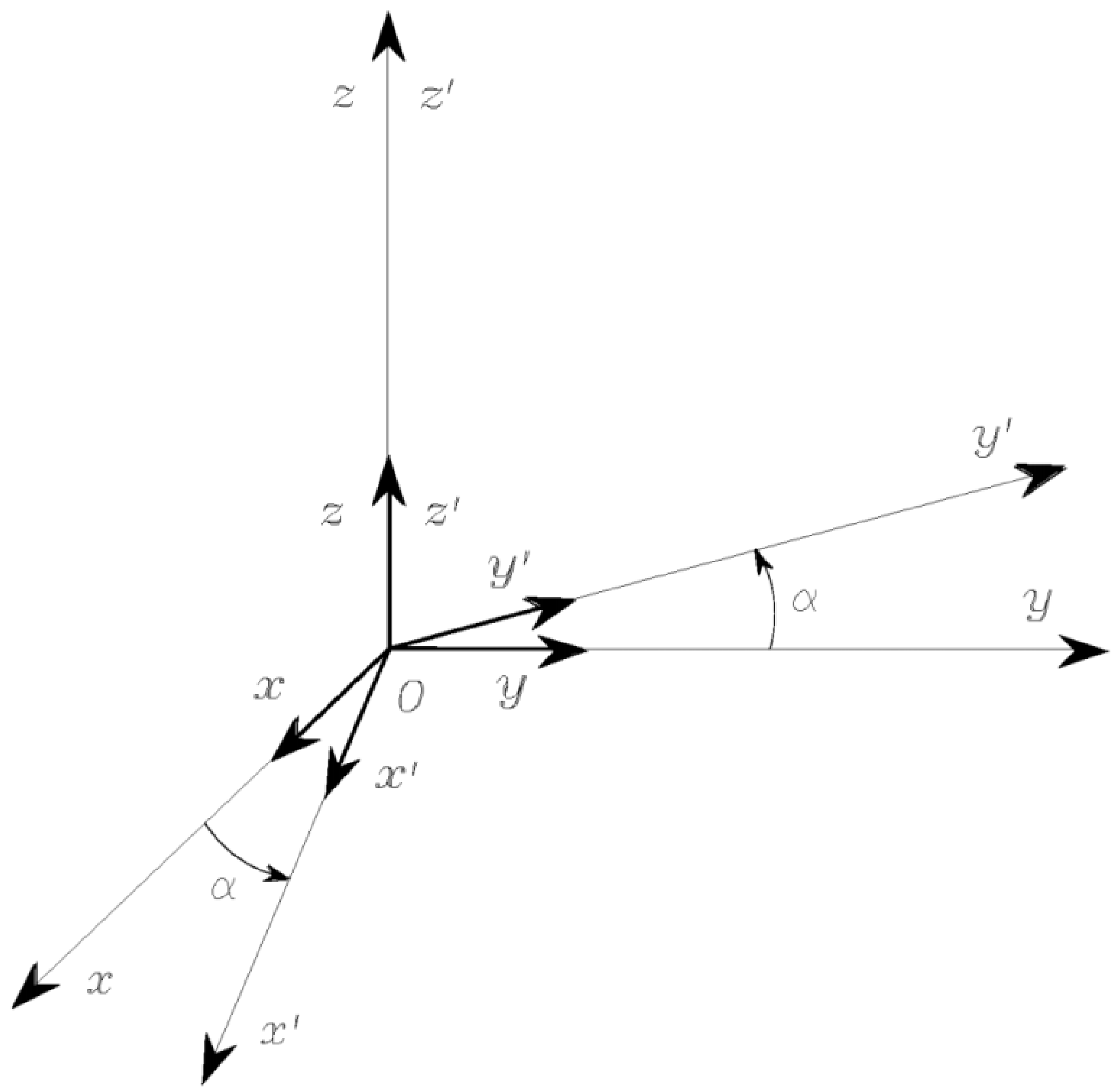
Summary

- Rotations
- Forward Kinematics
- Denavit-Hartenberg parameters
- Inverse Kinematics





z
z'
z
z'
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y'
α
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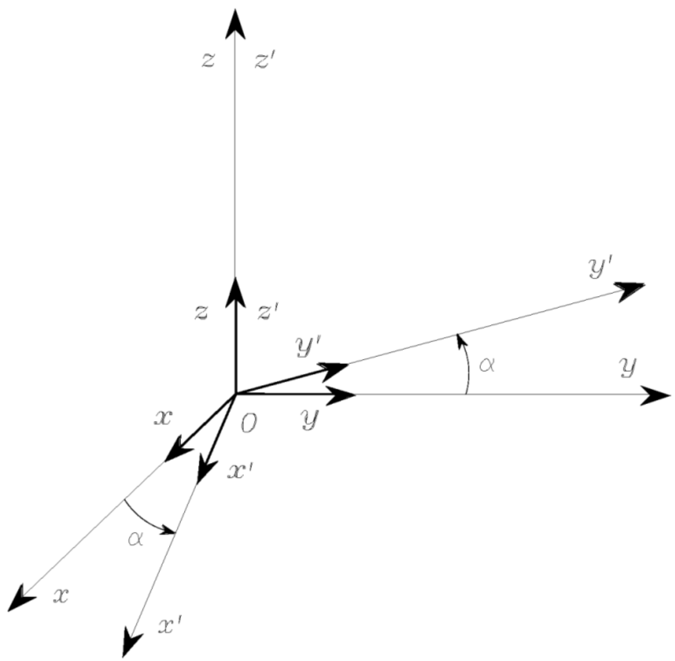


$$\boldsymbol{x}' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{z}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

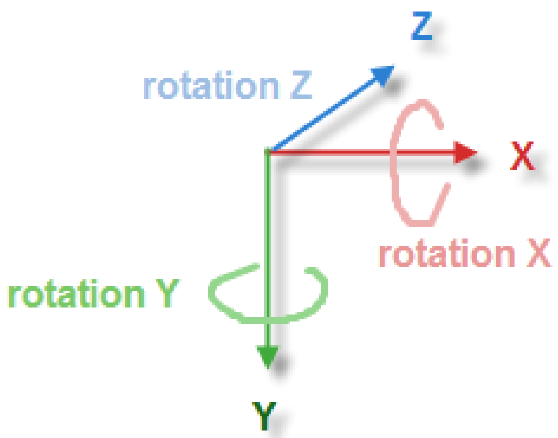
$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\boldsymbol{p} = p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'$$

ElementaryRotations



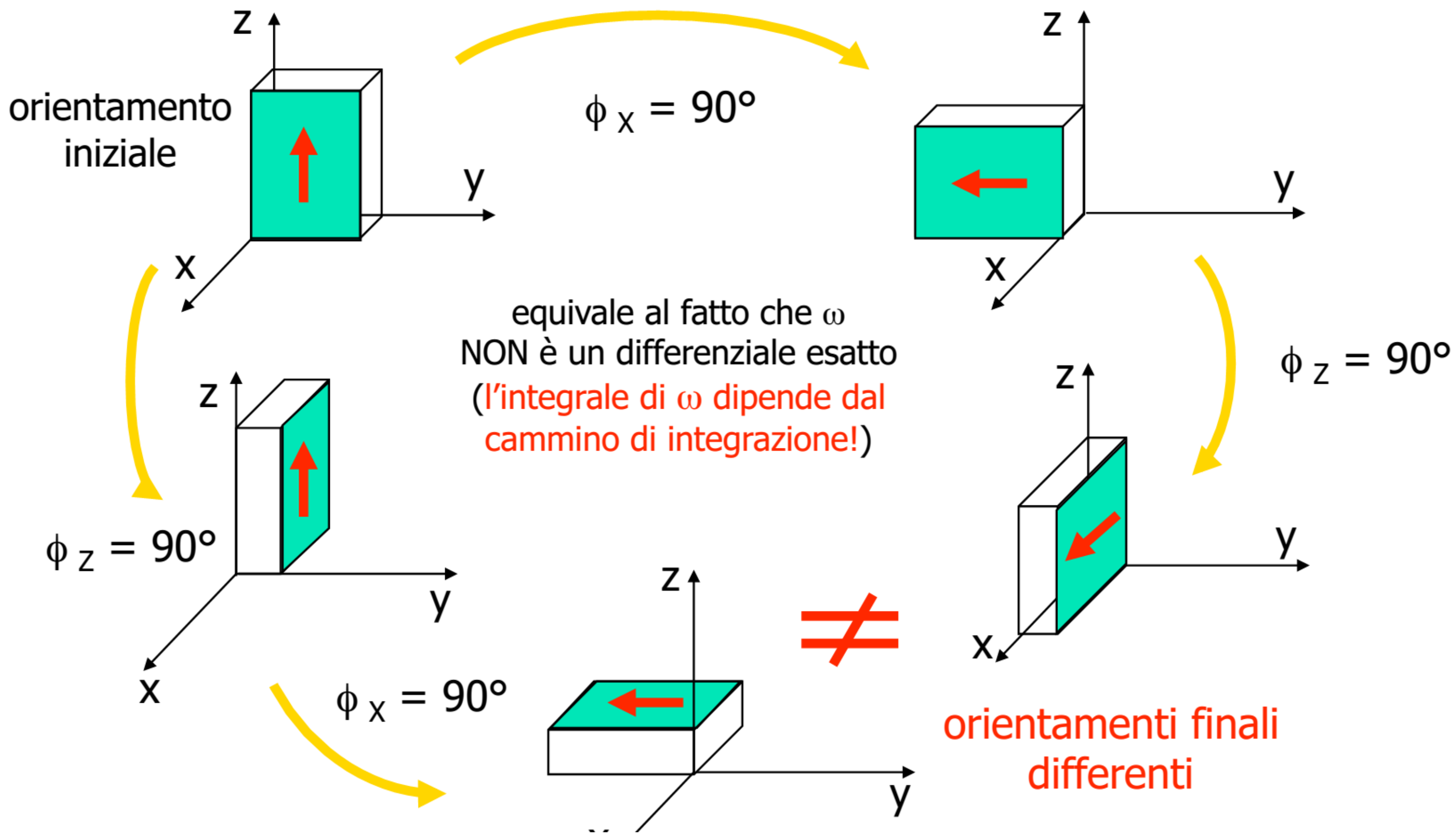
$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta)$$

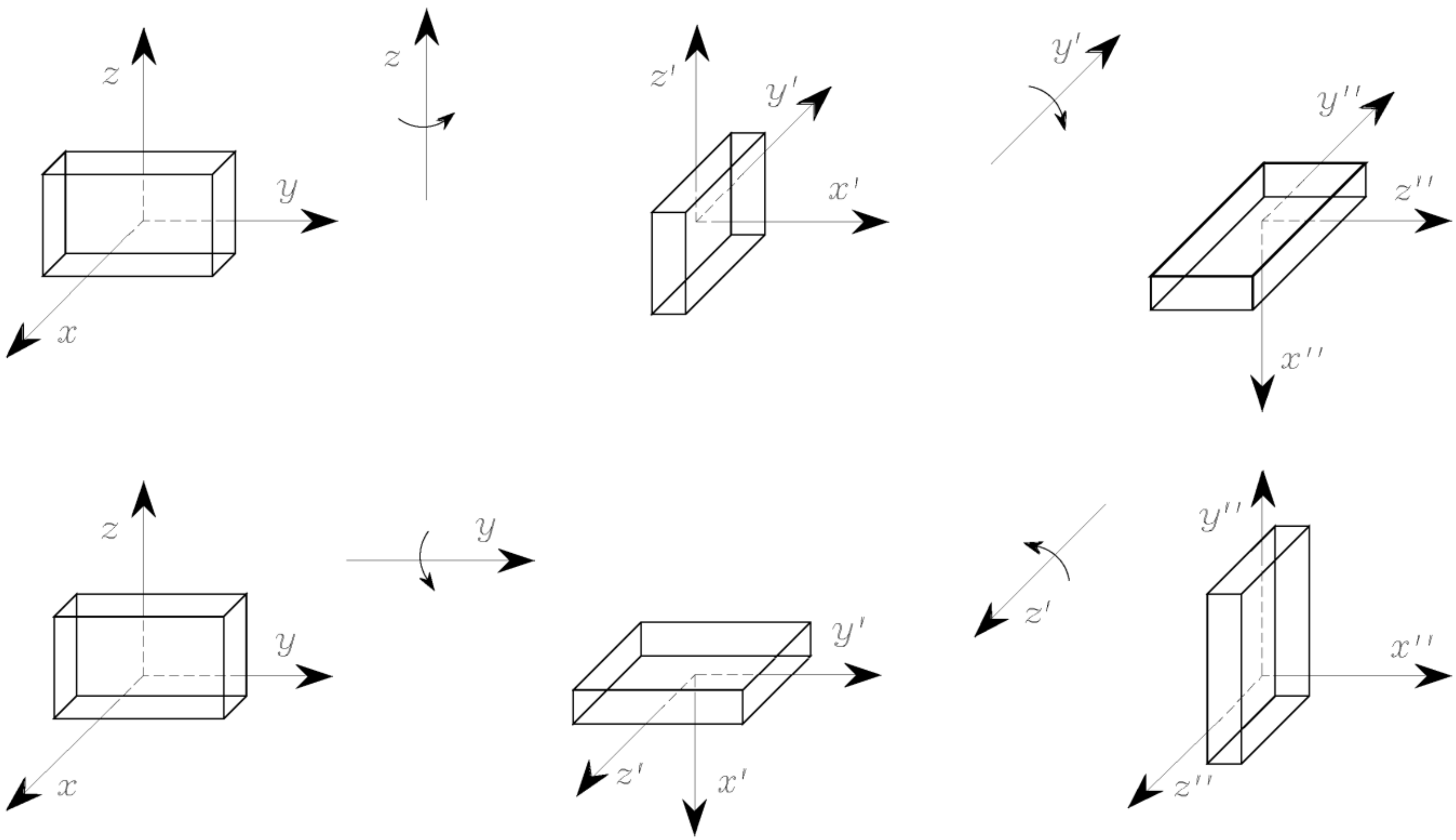
$$\left.\begin{aligned} \boldsymbol{p}^0 &= \boldsymbol{R}_1^0 \boldsymbol{p}^1 \\ \boldsymbol{p}^1 &= \boldsymbol{R}_2^1 \boldsymbol{p}^2 \end{aligned}\right\} \quad \boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2 \qquad \boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$

Rotation (Fixed Reference)



Courtesy of Prof. Alessandro De Luca

Rotations (mobile reference)



Rotations – Euler Angles

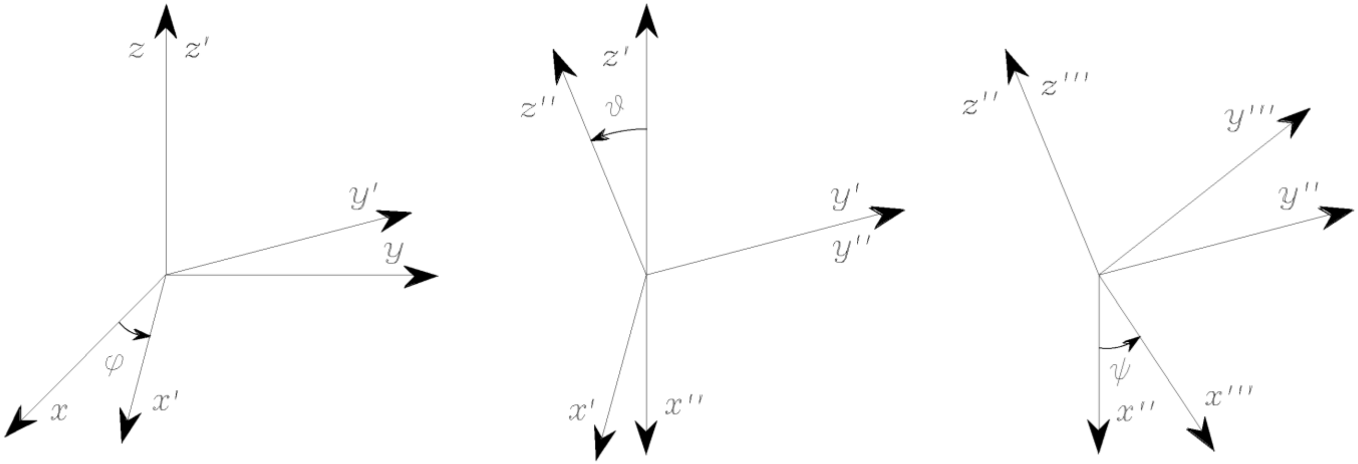


Figure 2.8 Representation of Euler angles ZYZ.

$$
\begin{aligned}
\boldsymbol{R}(\boldsymbol{\phi}) &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi) \\
&= \begin{bmatrix}
c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\
s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\
-s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta
\end{bmatrix}
\end{aligned}
$$

Rotations – R to Euler Angles

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\varphi = \mathrm{Atan2}(r_{23}, r_{13})$$

$$\vartheta = \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \mathrm{Atan2}(r_{32}, -r_{31}).$$

Rotations – RPY

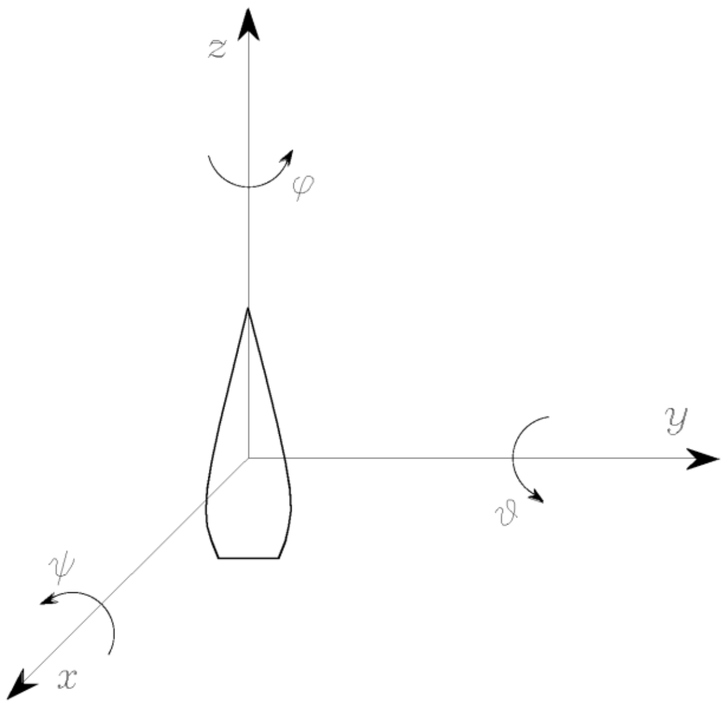


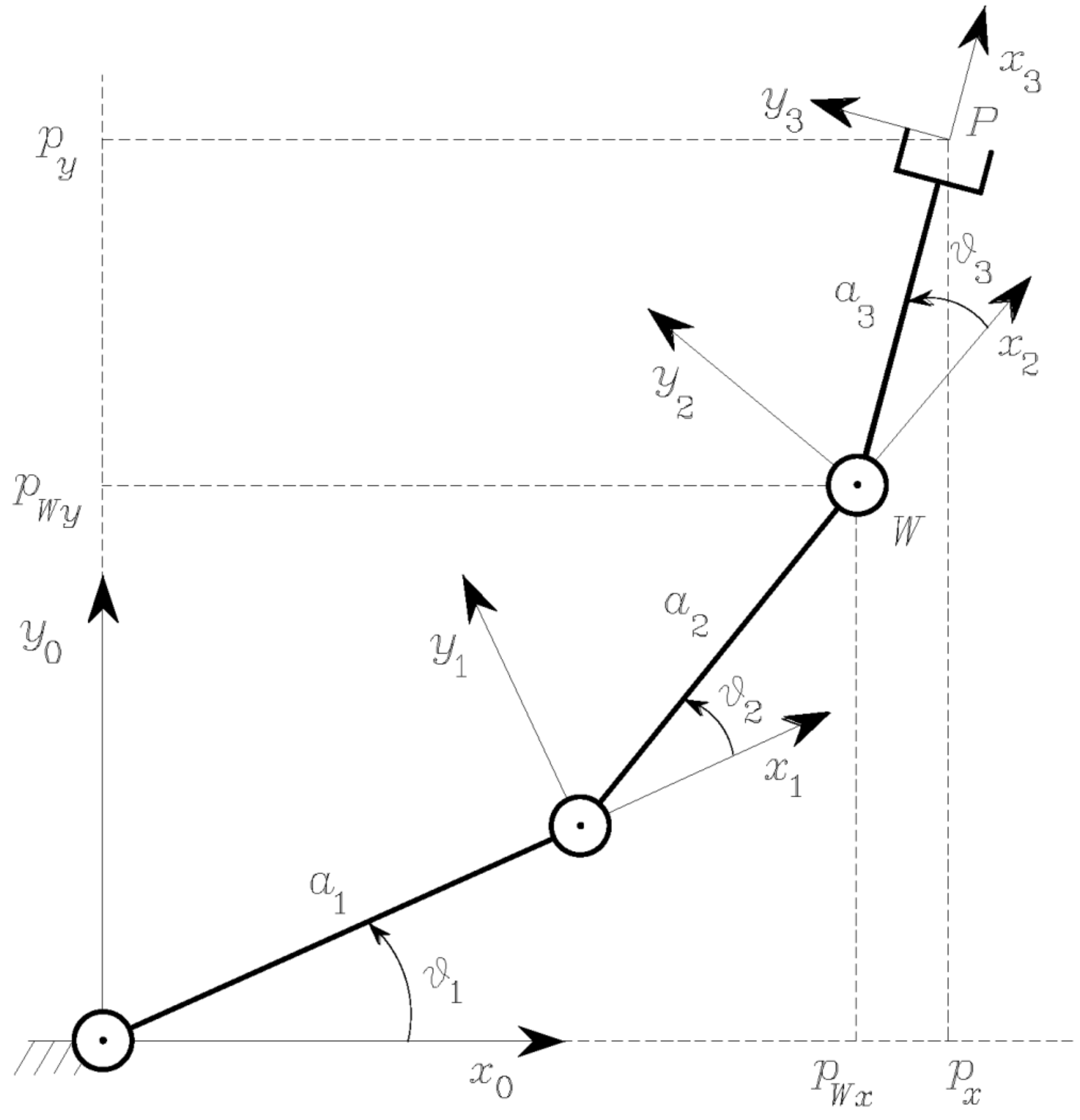
Figure 2.9 Representation of Roll–Pitch–Yaw angles.

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$

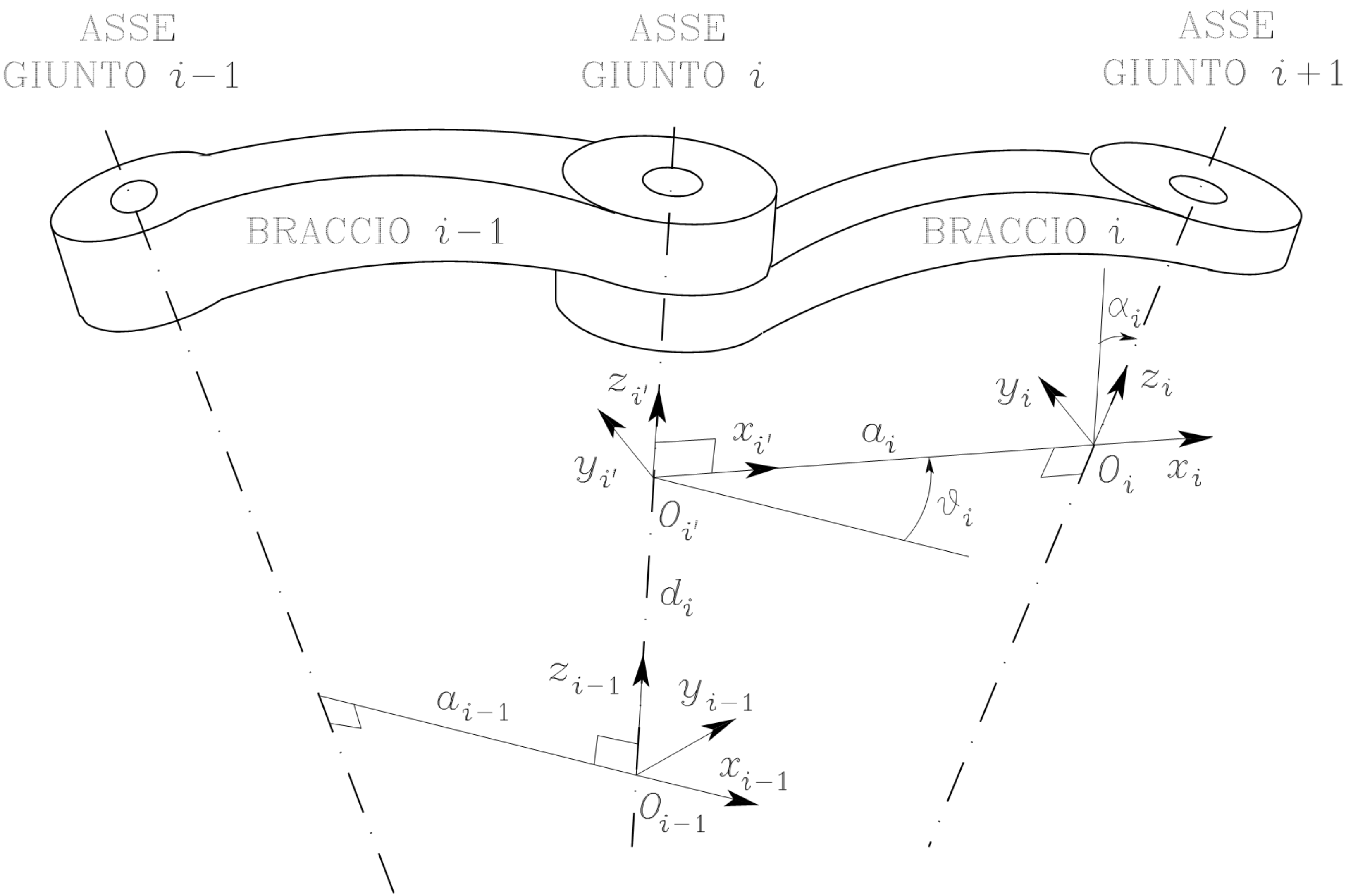
$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

Exercise

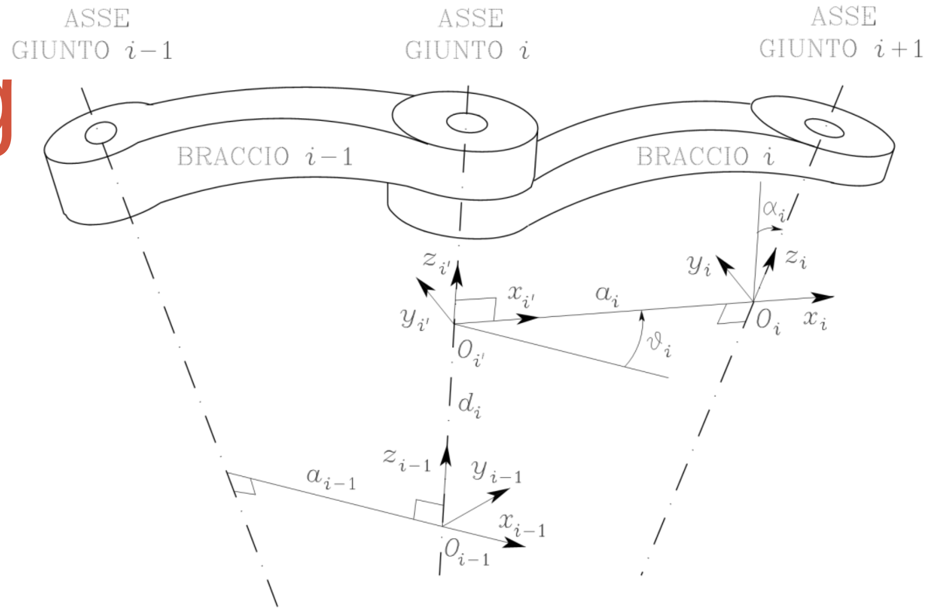
- Three links manipulator



Denavit-Hartenberg Parameters

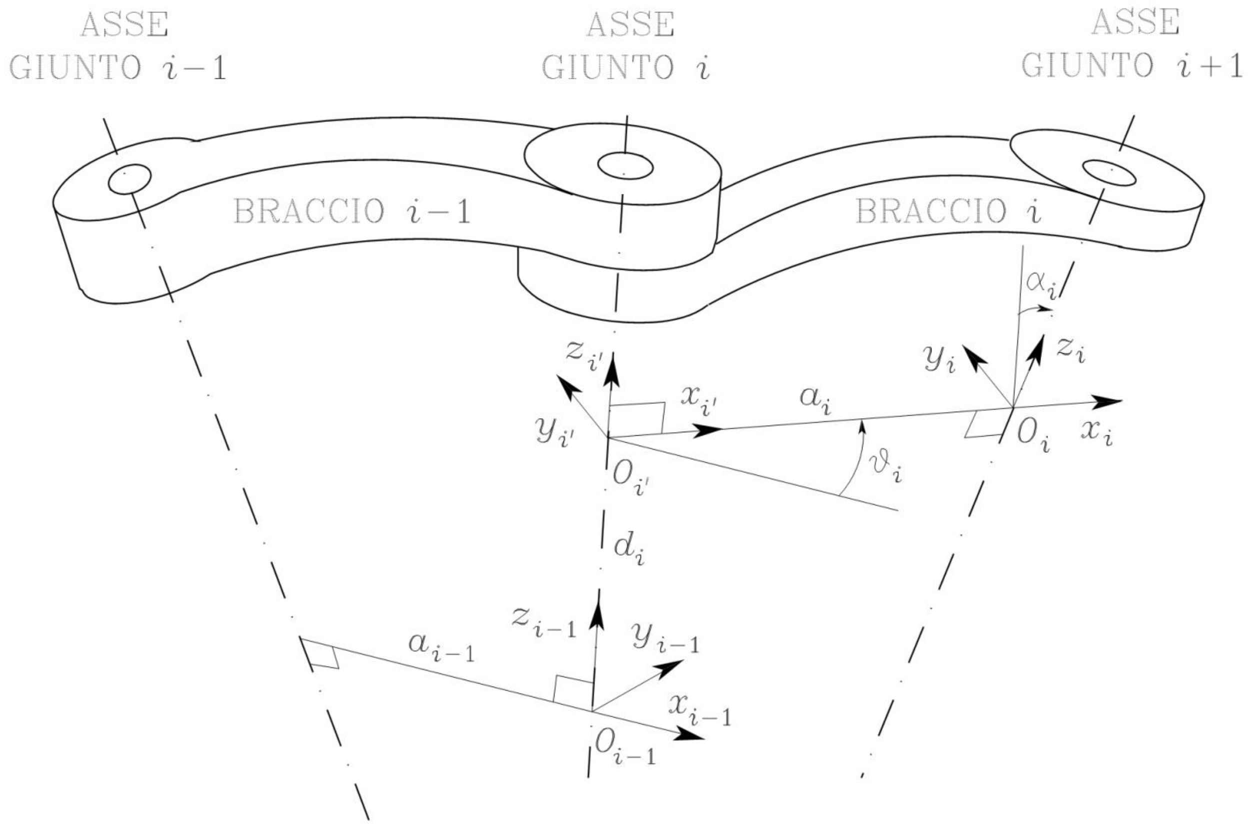


Denavit-Hartenberg



- si sceglie l'asse z_i giacente lungo l'asse del giunto $i+1$
- si individua O_i all'intersezione dell'asse z_i con la normale comune agli assi z_{i-1} e z_i, e con O'_i si indica l'intersezione della normale comune con z_{i-1}
- si assume l'asse x_i diretto lungo la normale comune agli assi z_{i-1} e z_i con verso positivo dal giunto i al giunto $i+1$
- si sceglie l'asse y_i in modo da completare una terna levogira

Denavit-Hartenberg Parameters



a_i distanza di O_i da O'_i;

d_i coordinata su z_{i-1} di O'_i;

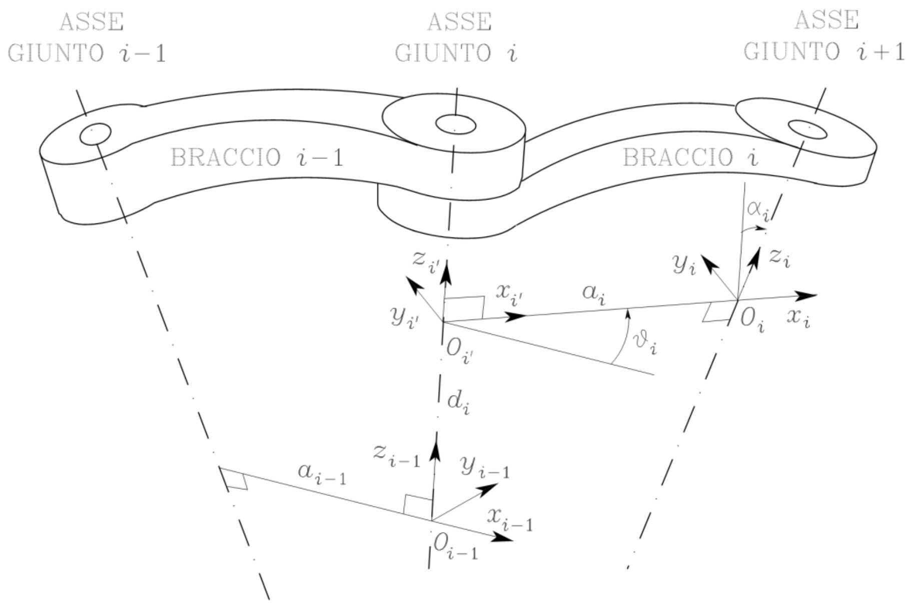
α_i angolo intorno all'asse x_i tra l'asse z_{i-1} e l'asse z_i valutato positivo in senso antiorario;

ϑ_i angolo intorno all'asse z_{i-1} tra l'asse x_{i-1} e l'asse x_i valutato positivo in senso antiorario.

Denavit-Hartenberg

- Definizione non univoca della terna:
  - ⋆ con riferimento alla terna 0, per la quale la sola direzione dell'asse z_0 risulta specificata: si possono quindi scegliere arbitrariamente O_0 ed x_0
  - ⋆ con riferimento alla terna n, per la quale il solo asse x_n risulta soggetto a vincolo (deve essere normale all'asse z_{n-1}): infatti non vi è giunto $n+1$, per cui non è definito z_n e lo si può scegliere arbitrariamente
  - ⋆ quando due assi consecutivi sono paralleli, in quanto la normale comune tra di essi non è univocamente definita
  - ⋆ quando due assi consecutivi si intersecano, in quanto il verso di x_i è arbitrario
  - ⋆ quando il giunto i è prismatico, nel qual caso la sola direzione dell'asse z_{i-1} è determinata

Denavit-Hartenberg Parameters



$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

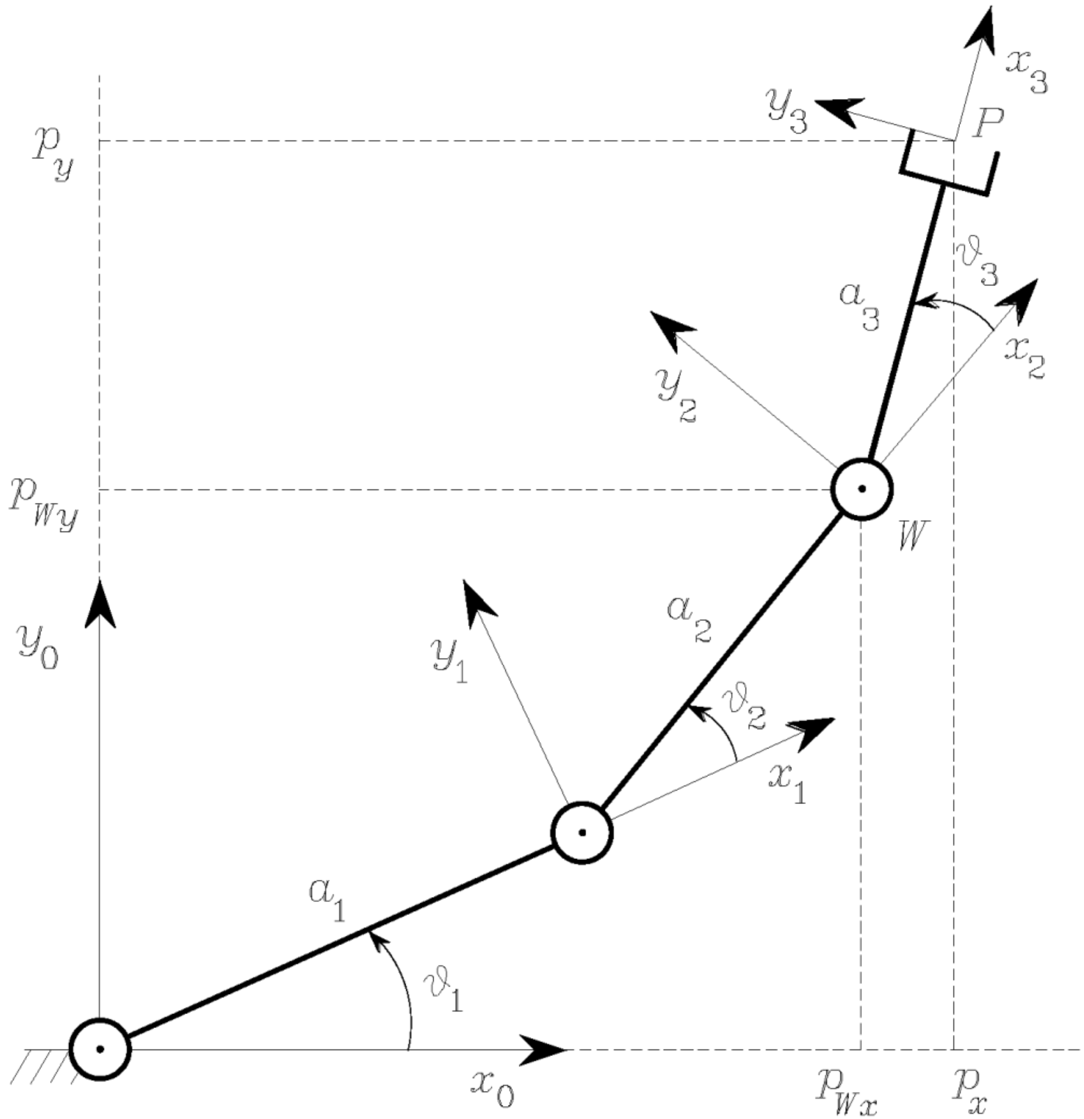
$$\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{i}^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_{i}^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

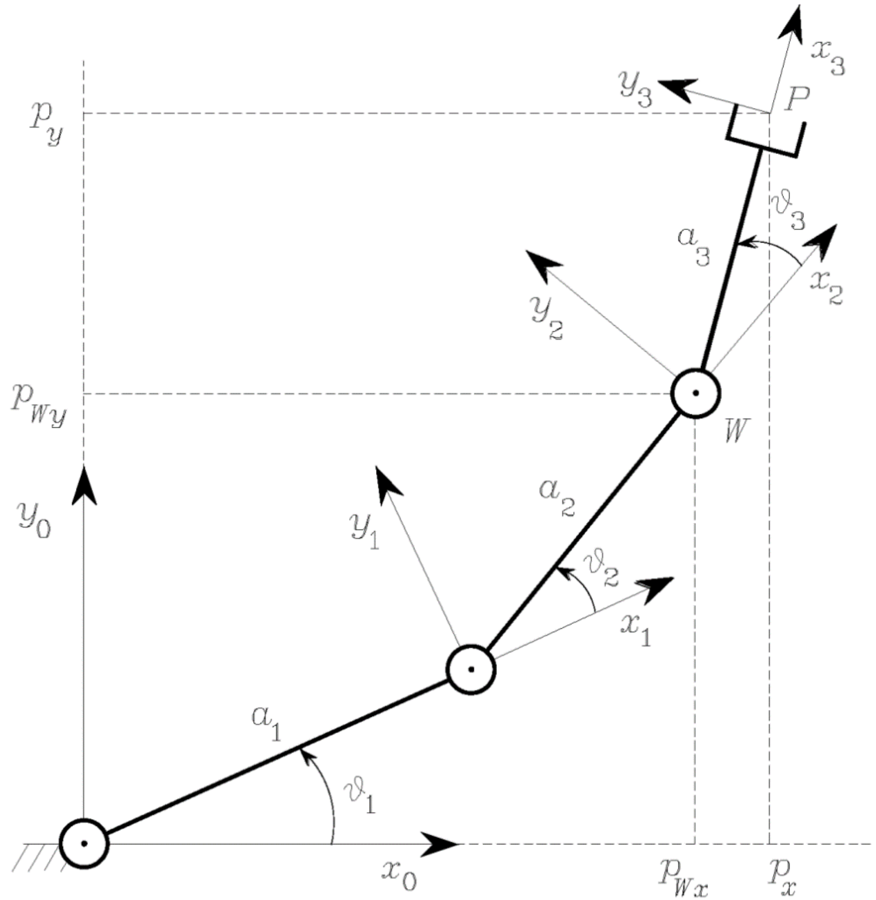
Exercise

- Three links manipulator

Braccio	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3



Exercise

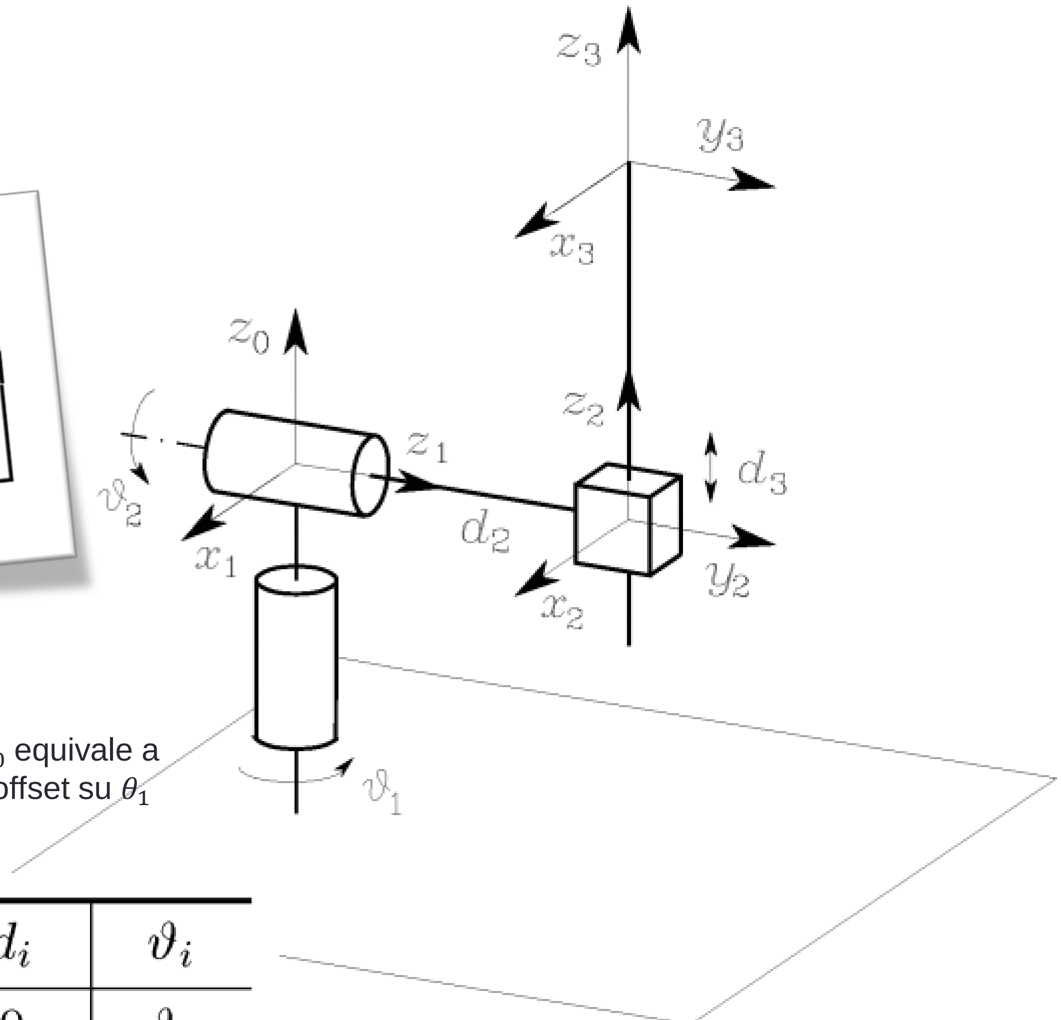


$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise

$$\begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Fissare x_0, y_0 equivale a sceglere un offset su θ_1

Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0

Exercise

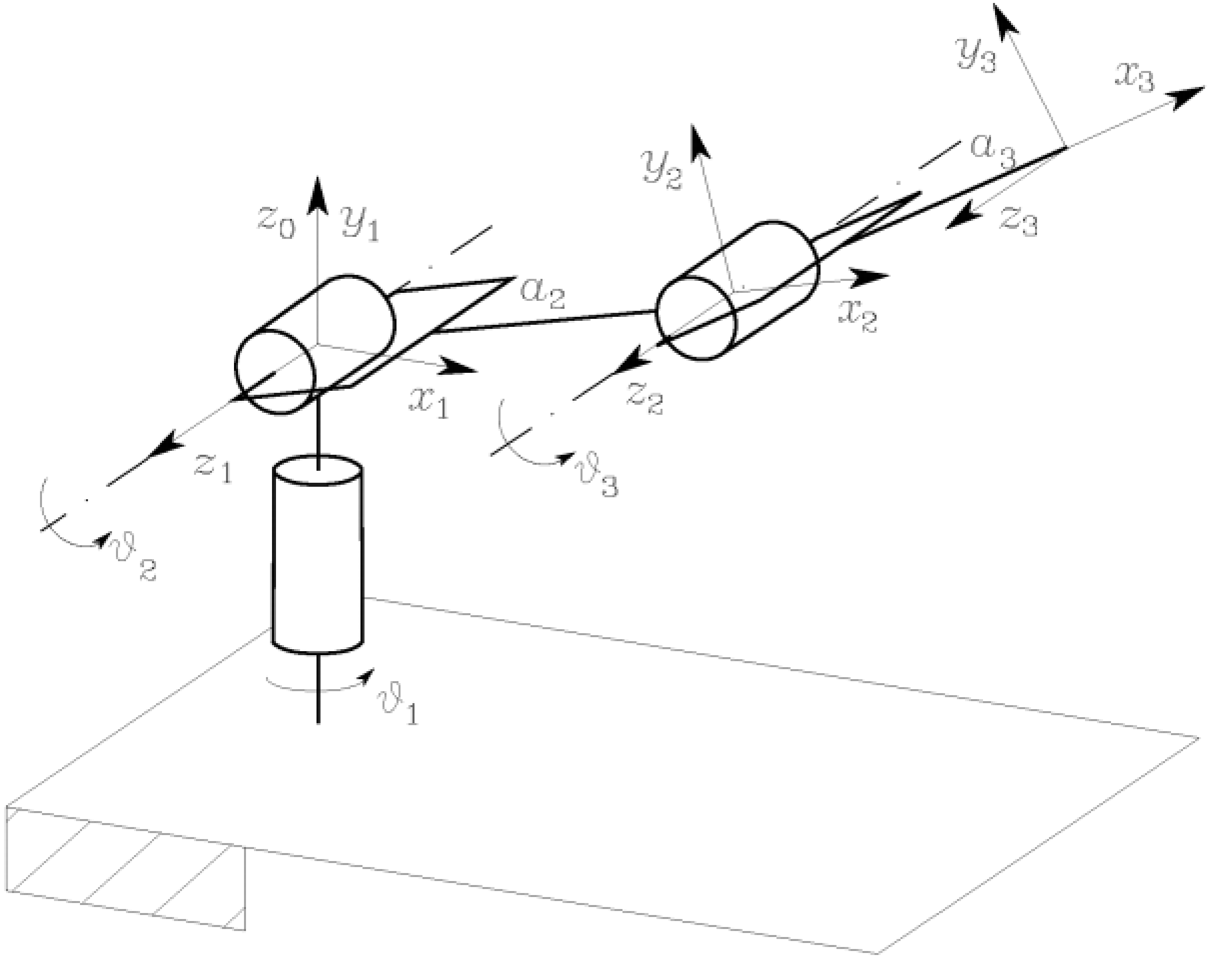
$$
\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_2^1(\vartheta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\boldsymbol{A}_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Computation of the direct kinematics function as in (2.45) yields

$$
\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.61}
$$

Exercise



Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

Exercise

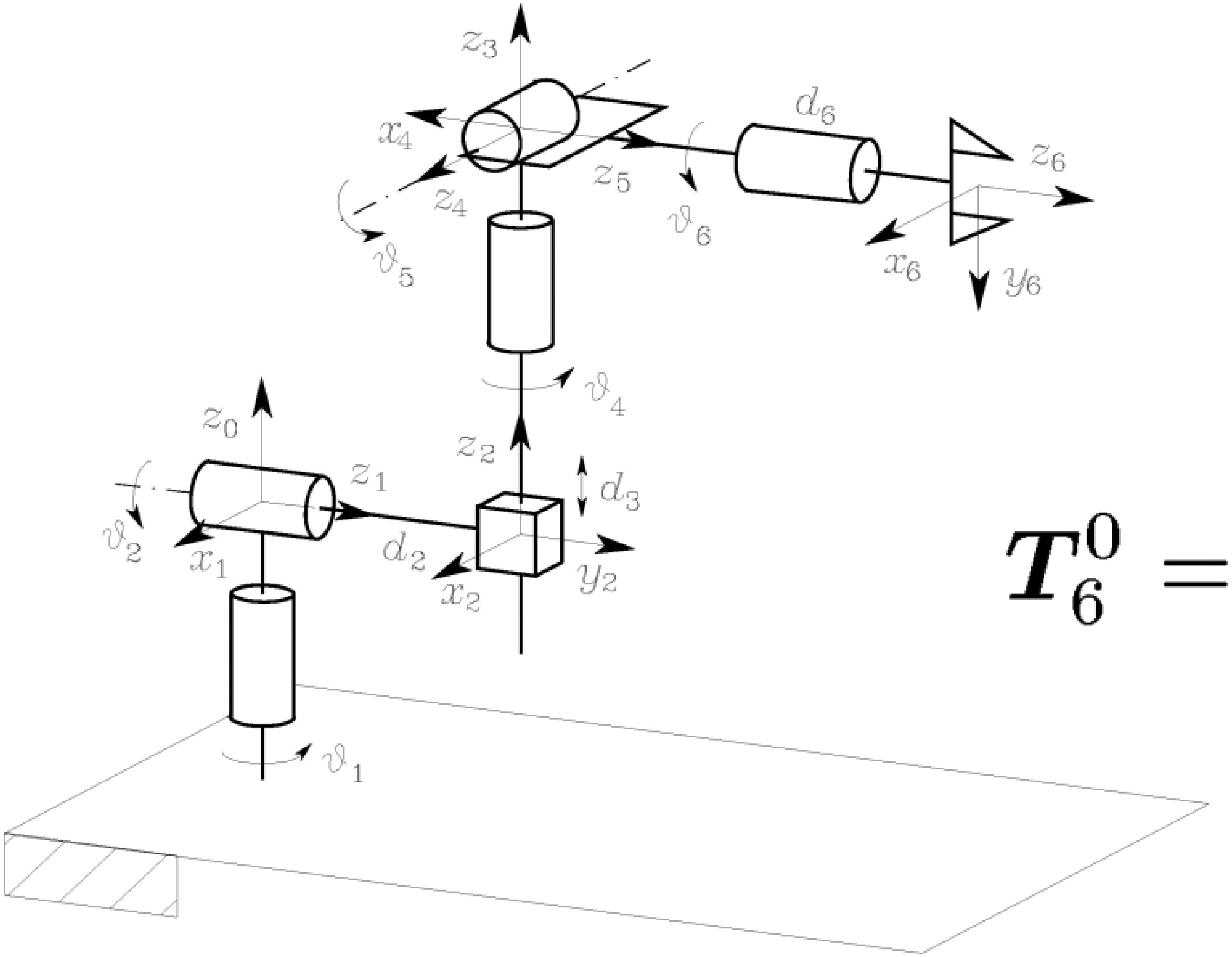
$$\begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3.$$

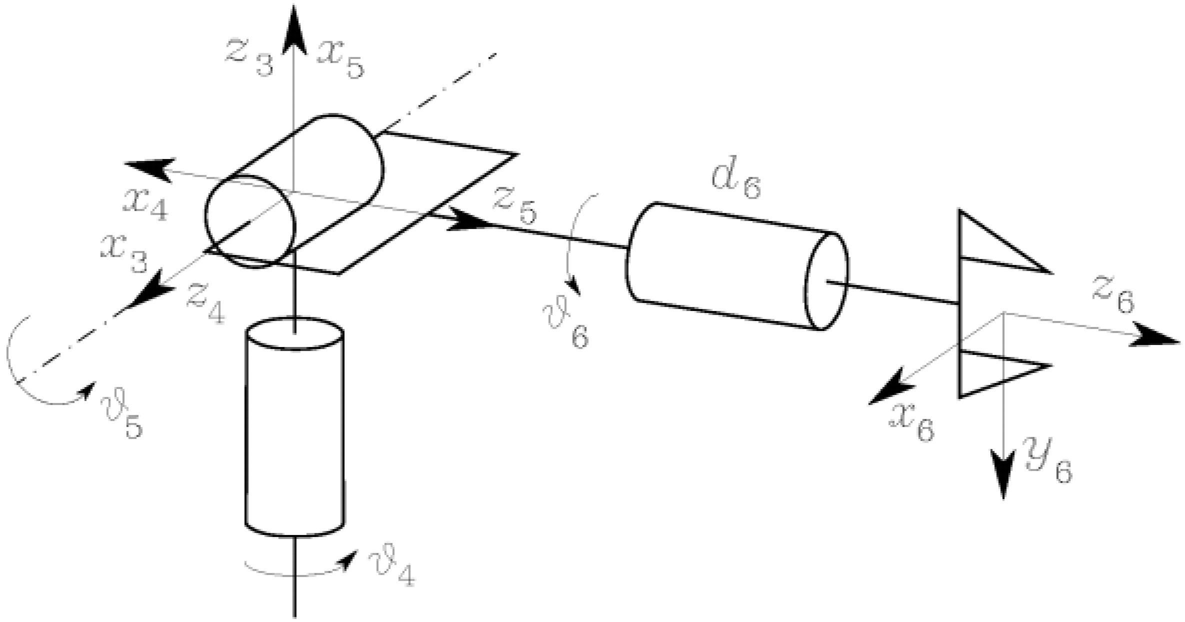
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise



$$T_6^0 = T_3^0 T_6^3$$

Exercise



Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

Exercise

$$A_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Computation of the direct kinematics function as in (2.45) yields

$$T_6^3(q) = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise

$$\boldsymbol{a}^0 = \begin{bmatrix} c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\ s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix}$$

$$\boldsymbol{T}_6^0 = \boldsymbol{T}_3^0\boldsymbol{T}_6^3 = \begin{bmatrix} \boldsymbol{n}^0 & \boldsymbol{s}^0 & \boldsymbol{a}^0 & \boldsymbol{p}^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Carrying out the products yields

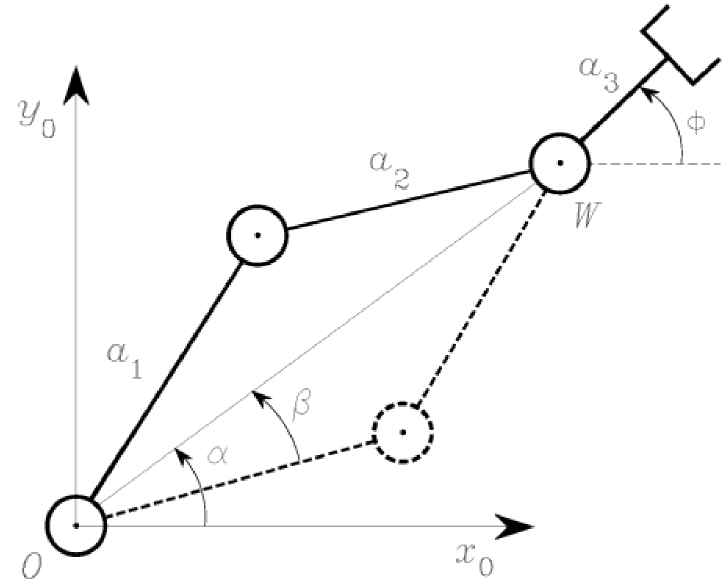
$$\boldsymbol{p}^0 = \begin{bmatrix} c_1s_2d_3 - s_1d_2 + \big(c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5\big)d_6 \\ s_1s_2d_3 + c_1d_2 + \big(s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5\big)d_6 \\ c_2d_3 + (-s_2c_4s_5 + c_2c_5)d_6 \end{bmatrix} \tag{2.64}$$

for the end-effector position, and

$$\boldsymbol{n}^0 = \begin{bmatrix} c_1\big(c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6\big) - s_1(s_4c_5c_6 + c_4s_6) \\ s_1\big(c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6\big) + c_1(s_4c_5c_6 + c_4s_6) \\ -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \end{bmatrix}$$

$$\boldsymbol{s}^0 = \begin{bmatrix} c_1\big(-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6\big) - s_1(-s_4c_5s_6 + c_4c_6) \\ s_1\big(-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6\big) + c_1(-s_4c_5s_6 + c_4c_6) \\ s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \end{bmatrix} \tag{2.65}$$

Inverse kinematics



$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$
$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12}$$

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2a_1 a_2 c_2$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2} \qquad s_2 = \pm\sqrt{1 - c_2^2}, \qquad \vartheta_2 = \mathrm{Atan2}(s_2, c_2).$$

$$s_1 = \frac{(a_1 + a_2 c_2)p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2} \qquad \vartheta_1 = \mathrm{Atan2}(s_1, c_1)$$

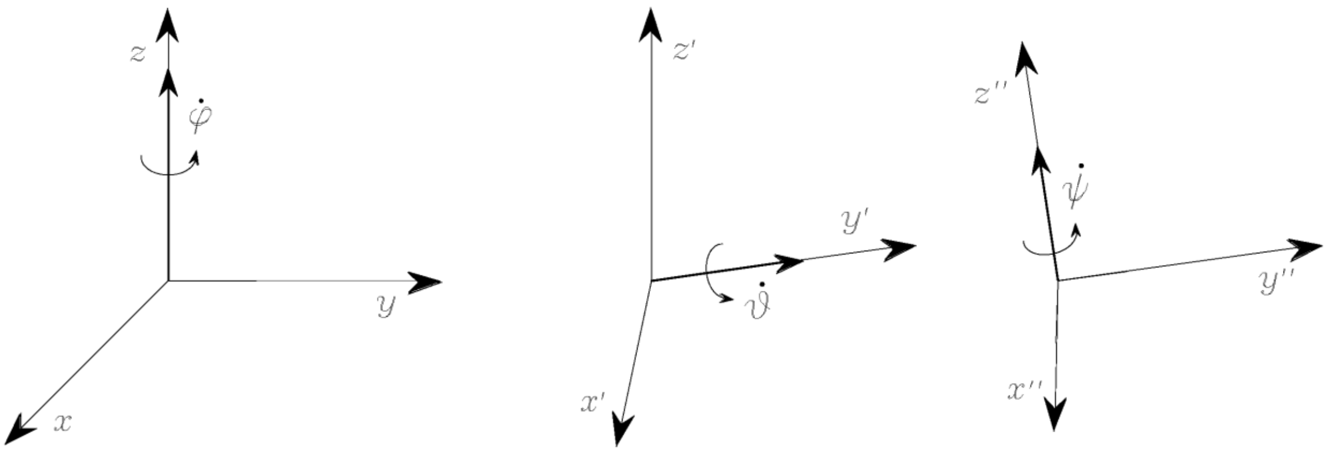
$$c_1 = \frac{(a_1 + a_2 c_2)p_{Wx} + a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2} \qquad \vartheta_3 = \phi - \vartheta_1 - \vartheta_2$$

Euler Angles & Angular Velocity

- ω è un vettore, ovvero un elemento di uno spazio vettoriale: si può ottenere come somma dei contributi ω_1 , … , ω_n (in qualsiasi ordine)

- viceversa, Φ (e $\dot{\Phi}$) non è un elemento di uno spazio vettoriale: la rappresentazione di rotazioni successive, in generale, non si ottiene sommando gli angoli corrispondenti

Angular Velocity from ZYZ



- as a result of $\dot{\varphi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\varphi}\,[0 \quad 0 \quad 1]^T$
- as a result of $\dot{\vartheta}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\vartheta}\,[-s_\varphi \quad c_\varphi \quad 0]^T$
- as a result of $\dot{\psi}$: $[\omega_x \quad \omega_y \quad \omega_z]^T = \dot{\psi}\,[c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta]^T$,

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}.$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}. \tag{3.60}$$

The determinant of matrix $\boldsymbol{T}$ is $-s_\vartheta$, which implies that the relationship cannot be inverted for $\vartheta = 0, \pi$. This means that, even though all rotational velocities of the end-effector frame can be expressed by means of a suitable angular velocity vector $\boldsymbol{\omega}$, there exist angular velocities which cannot be expressed by means of $\dot{\boldsymbol{\phi}}$ when the orientation of the end-effector frame causes $s_\vartheta = 0$[4]. In fact, in this situation, the angular velocities that can be described by $\dot{\boldsymbol{\phi}}$ shall have linearly dependent components in the directions orthogonal to axis z ($\omega_x^2 + \omega_y^2 = \dot{\vartheta}^2$). An orientation for which the determinant of the transformation matrix vanishes is termed *representation singularity* of $\boldsymbol{\phi}$.

From a physical viewpoint, the meaning of $\boldsymbol{\omega}$ is more intuitive than that of $\dot{\boldsymbol{\phi}}$. The three components of $\boldsymbol{\omega}$ represent the components of angular velocity with respect to the base frame. Instead, the three elements of $\dot{\boldsymbol{\phi}}$ represent nonorthogonal components of angular velocity defined with respect to the axes of a frame that varies as the end-effector orientation varies. On the other hand, while the integral of $\dot{\boldsymbol{\phi}}$ over time gives $\boldsymbol{\phi}$, the integral of $\boldsymbol{\omega}$ does not admit a clear physical interpretation, as can be seen in the following example.

Example 3.3

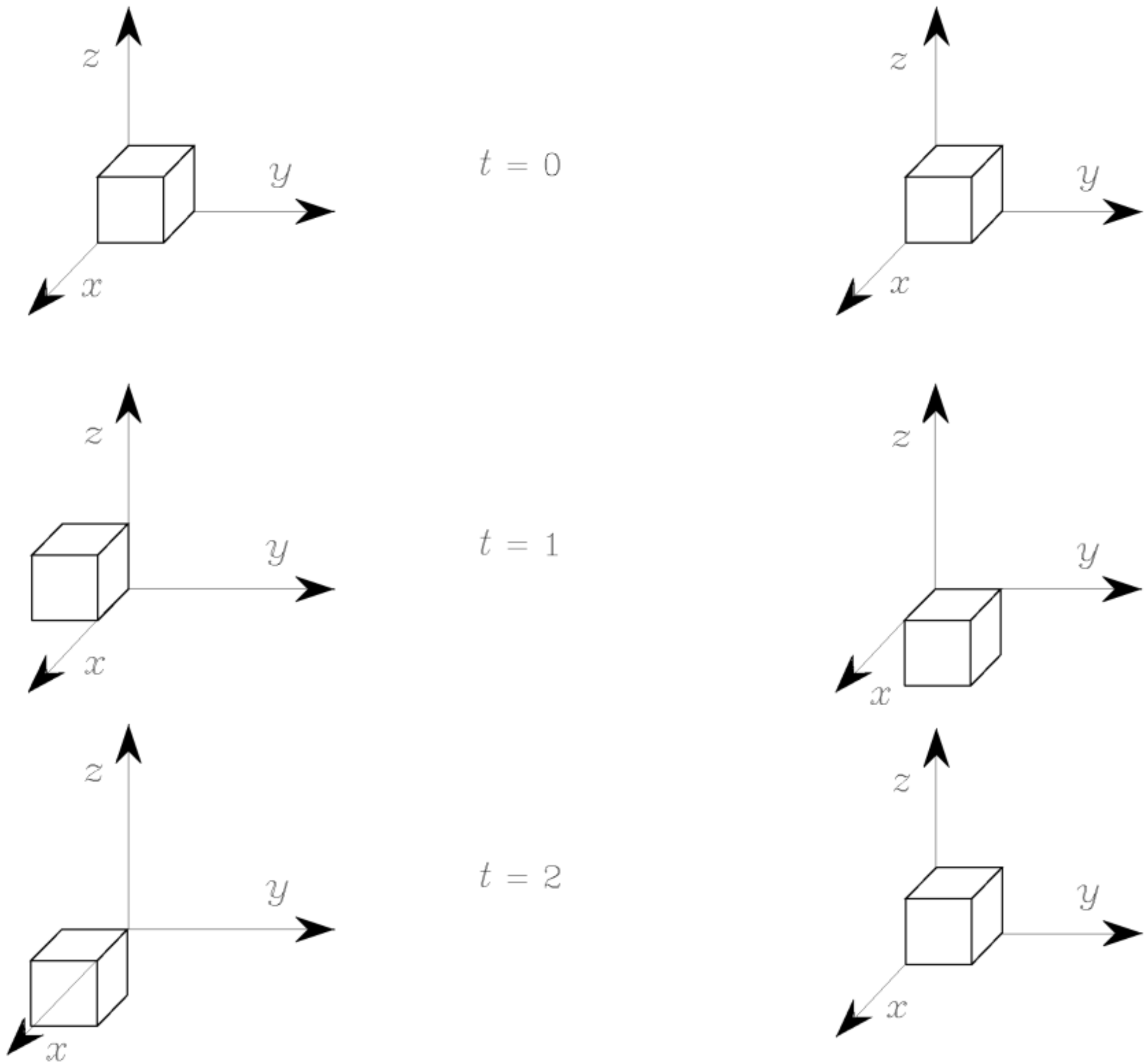
Consider an object whose orientation with respect to a reference frame is known at time $t = 0$. Assign the following time profiles to $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 1 < t \le 2,$$
$$\boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 1 < t \le 2.$$

The integral of $\boldsymbol{\omega}$ gives the same result in the two cases

$$\int_0^2 \boldsymbol{\omega} dt = [\,\pi/2 \quad \pi/2 \quad 0\,]^T$$

but the final object orientation corresponding to the second time law is clearly different from the one obtained with the first time law (Figure 3.10).


z
y
x
$t = 0$
$t = 1$
$t = 2$

Analytic and Geometric Jacobian

Once the transformation $\boldsymbol{T}$ between $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\phi}}$ is given, the analytical Jacobian can be related to the geometric Jacobian as

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}) \end{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{T}_A(\boldsymbol{\phi})\dot{\boldsymbol{x}} \tag{3.61}$$

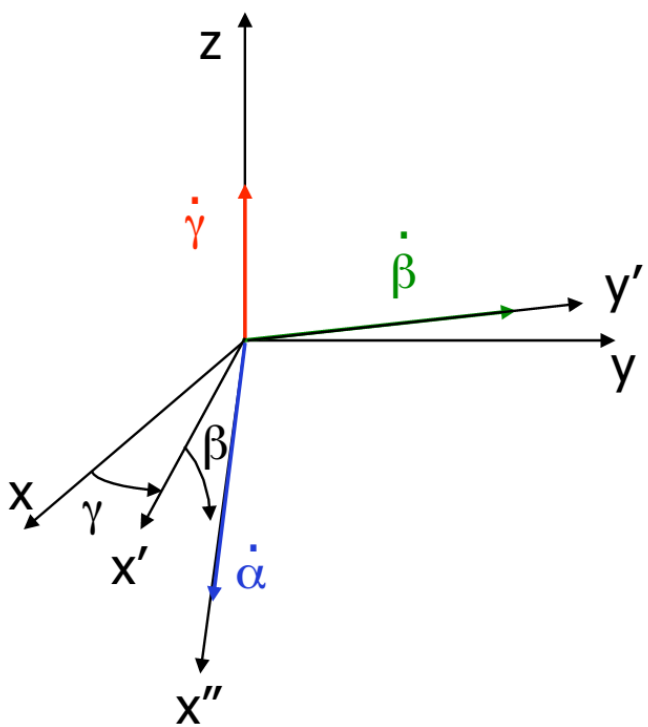
which, in view of (3.3) and (3.58), yields

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A. \tag{3.62}$$

For certain manipulator geometries, it is possible to establish a substantial equivalence between $\boldsymbol{J}$ and $\boldsymbol{J}_A$. In fact, when the degrees of mobility cause rotations of the end effector all about the same fixed axis in space, the two Jacobians are essentially the same. This is the case of the above three-link planar arm. Its geometric Jacobian (3.31)

Angular Velocity from RPY

$$R_{RPY}(\alpha_x, \beta_y, \gamma_z) = R_{ZY'X''}(\gamma_z, \beta_y, \alpha_x)$$



$$\omega = \underbrace{\begin{array}{c} \begin{bmatrix} c\beta\, c\gamma & -s\gamma & 0 \\ c\beta\, s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} x'' & y' & z \end{array} \end{array}}_{}\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

$T_{RPY}(\beta,\gamma)$

1a col in $R_{ZY'}(\gamma_z,\beta_y)$ 2a col in $R_Z(\gamma_z)$

$$\det T_{RPY}(\beta,\gamma) = c\beta = 0$$

per $\beta = \pm\pi/2$
(singolarità della rappresentazione RPY)

N.B. la trattazione è analoga per gli altri 11 set di rappresentazioni minimali

Courtesy of Prof. Alessandro De Luca

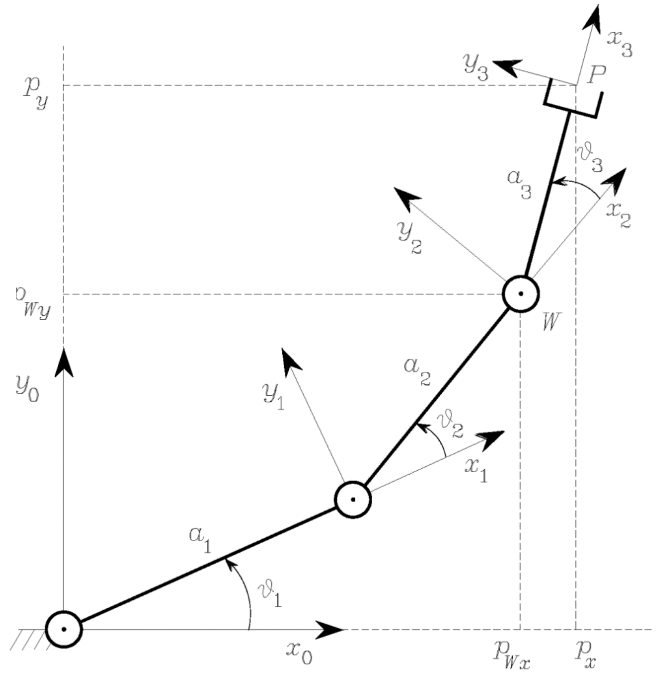
Jacobian of Typical Manipulator Structures

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix},$$

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Jacobian of Typical Manipulator Structures:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

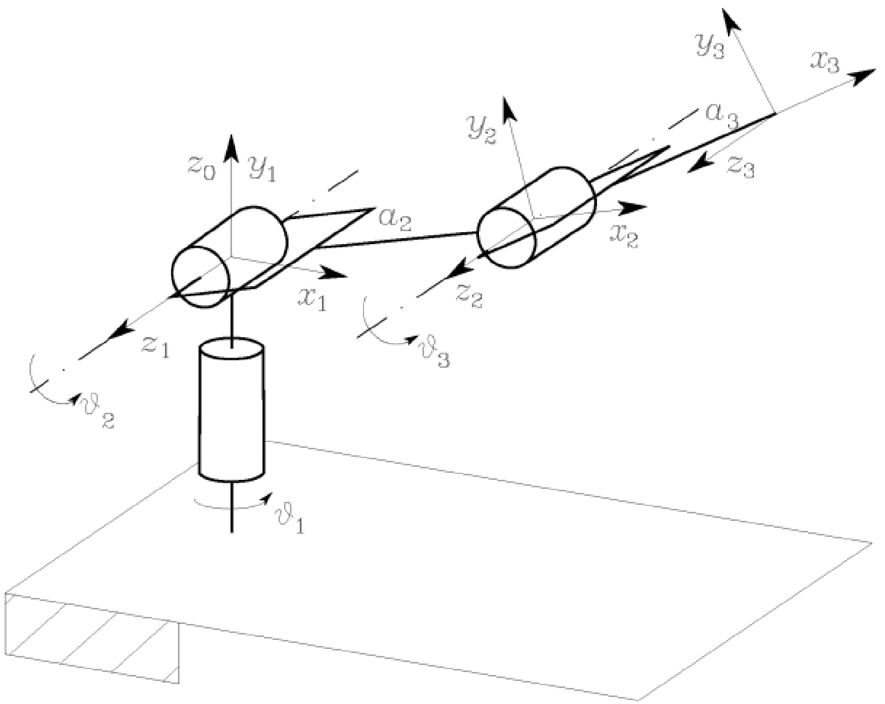
Jacobian of Typical Manipulator Structures

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{p}_2 = \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ a_2 s_2 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} c_1(a_2 c_2 + a_3 c_{23}) \\ s_1(a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{bmatrix},$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$



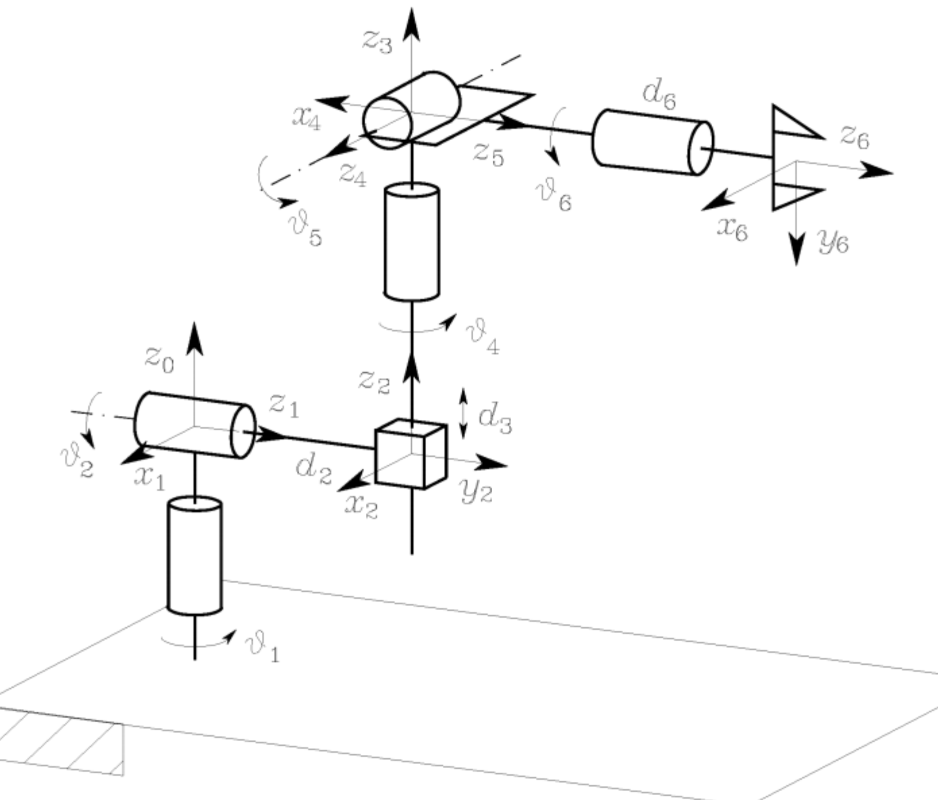
Jacobian of Typical Manipulator Structures

$$\boldsymbol{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

Only three of the six rows of the Jacobian (3.33) are linearly independent. Having three degrees of mobility only, it is worth considering the upper (3×3) block of the Jacobian

$$\boldsymbol{J}_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix} \tag{3.34}$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 & \boldsymbol{z}_3 \times (\boldsymbol{p} - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p} - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p} - \boldsymbol{p}_5) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$



$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_3 = \boldsymbol{p}_4 = \boldsymbol{p}_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

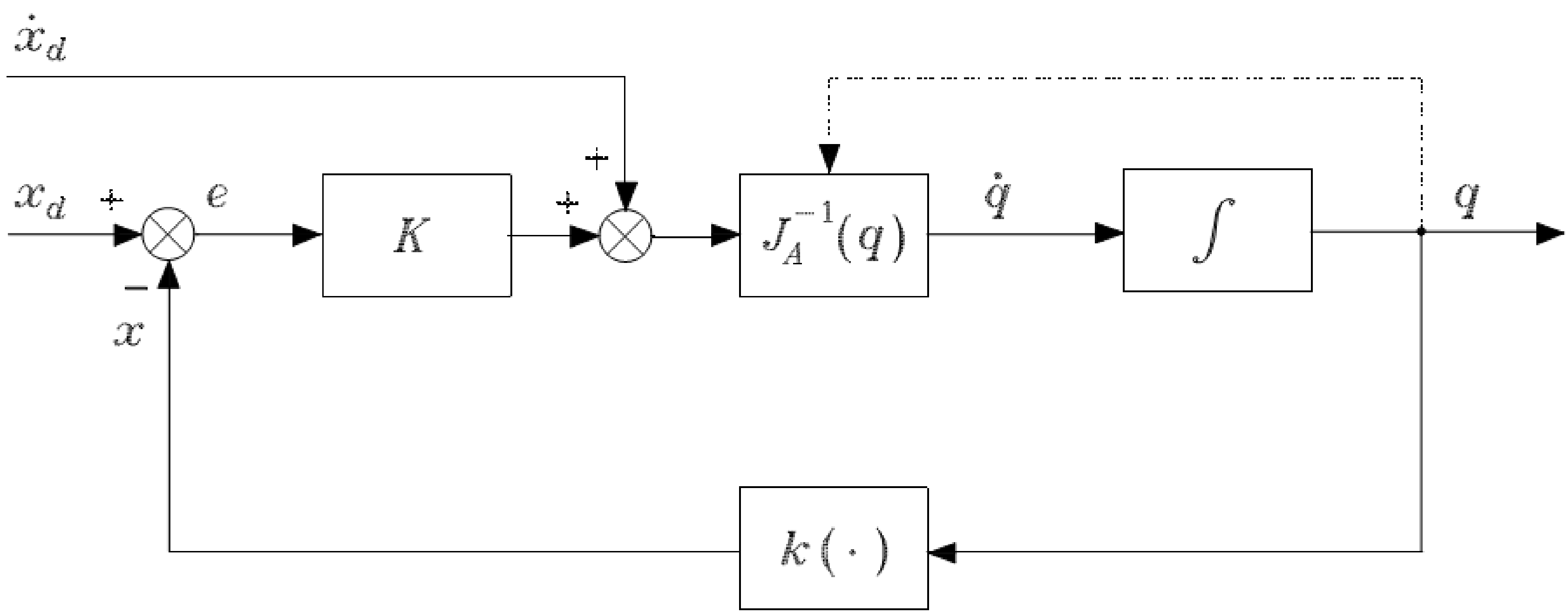
$$\boldsymbol{p} = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + \big(c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5\big) d_6 \\ s_1 s_2 d_3 + c_1 d_2 + \big(s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5\big) d_6 \\ c_2 d_3 + (-s_2 c_4 s_5 + c_2 c_5) d_6 \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \boldsymbol{z}_2 = \boldsymbol{z}_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$\boldsymbol{z}_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad \boldsymbol{z}_5 = \begin{bmatrix} c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5 \\ s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

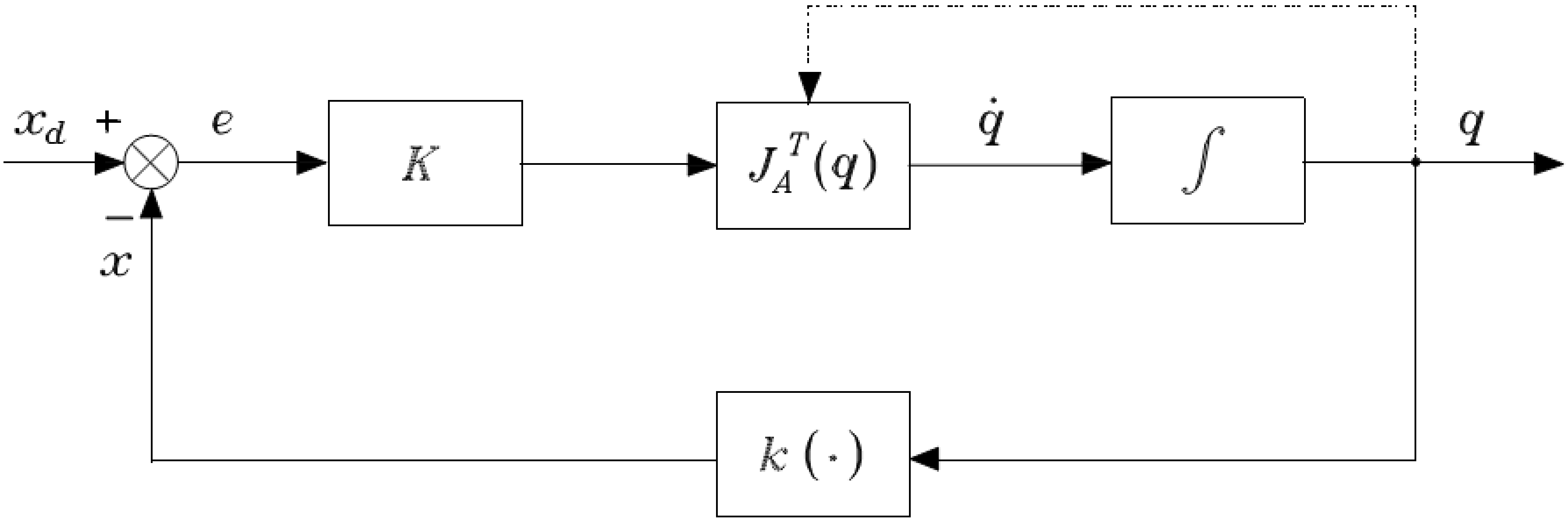
Inverse kinematics by Jacobian

Inverse Jacobian



Inverse kinematics by Jacobian

Transpose Jacobian



Inverse kinematics by Jacobian

- Proof by Lyapuanov function $V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{K}\boldsymbol{e}$

$$V(\boldsymbol{e}) > 0 \quad \forall \boldsymbol{e} \neq \boldsymbol{0} \qquad V(\boldsymbol{0}) = 0$$

$$\begin{aligned}\dot{V}(\boldsymbol{e}) &= \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}} \\ &= \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

$$\boxed{\dot{\boldsymbol{q}} = \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}}$$

$$\dot{V}(\boldsymbol{e}) = \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}$$