Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2012/13 - N.7

Pb 1. Prove that the set K of functions in $C^1([0,1])$ such that

$$\int_0^1 (|f(\sigma)|^2 + |f'(\sigma)|^2) d\sigma \leq C$$

for some positive constant C is relatively compact in $C([0,1])$.

Pb 2. Prove that the set K of functions in $C^1([0,1])$ such that $|f'(x)| \leq C$ for all $x \in [0,1]$ and some $C > 0$ and such that any $f \in K$ admits a root in $[0,1]$ is relatively compact in $C([0,1])$.

Pb 3. Let M be a bounded set in $C([0,1])$. Prove that

$$K = \left\{ y(t) = \int_0^t x(\sigma) d\sigma : x \in M \right\}$$

is relatively compact in $C([0,1])$.

Pb 4. Let (f_n) be a sequence of functions in $C^2([0,1])$ such that $f_n(0) = f_n'(0) = 0$ and $|f_n''(x)| \leq 1$ for every $x \in [0,1]$ and $n \in \mathbb{N}$. Prove that there exists a subsequence of (f_n) which converges in $C([0,1])$.

Pb 5. Let K be a compact metric space and consider a bounded sequence $(f_n) \subset C(K)$. Let $\psi : K \to \ell^\infty$ be the function defined by setting $\psi(x) = (f_n(x))_{n \in \mathbb{N}}$. Prove that $(f_n) \subset C(K)$ is relatively compact if and only if the function g is continuous.

Pb 6. Find the function $\varphi : [0,1] \to \mathbb{R}$ such that the set

$$K = \{ f \in C([0,1]) : |f(x)| \leq \varphi(x), \text{ for all } x \in [0,1] \}$$

is relatively compact in $C([0,1])$.

Pb 7. Let X be a Banach space and $(x_n) \subset X$ be a Cauchy sequence. Prove that the set $K = \{x_n : n \in \mathbb{N}\}$ is relatively compact in X.

Pb 8. Let K be a compact metric space. Assume that M is a relatively compact set of $C(K)$. Prove that M is equicontinuous.

Pb 9. Consider the sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x < n, \\ \arctan(x - n) & \text{if } x \geq n. \end{cases}$$

Prove that (f_n) is bounded and equicontinuous. Is (f_n) relatively compact?

Pb 10. Consider the sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Prove that (f_n) cannot be equicontinuous.

Pb 11. Let X be a complete metric space and $Y \subset X$. The Y is relatively compact in X (i.e. $\bar{Y}$ is compact) if and only if every sequence $(x_n) \subset Y$ admits a subsequence converging in X.

Pb 12. Let X and Y be Banach spaces and $L_n \in \mathcal{L}(X,Y)$ such that for every $(x_n) \subset X$ with $\|x_n\|_X \to 0$ it holds $\|L_n x_n\|_Y \to 0$ as $n \to \infty$. Prove that $\sup_{n\in\mathbb{N}} \|L_n\|_{\mathcal{L}(X,Y)} < \infty$.

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