Approximation Preserving Reductions

Summary

- Memo
- AP-reducibility
  - L-reduction technique
- Complete problems
- Examples: MAXIMUM CLIQUE, MAXIMUM INDEPENDENT SET, MAXIMUM 2-SAT, MAXIMUM NAE 3-SAT, MAXIMUM SAT(B)

Memo: Approximation classes inclusions

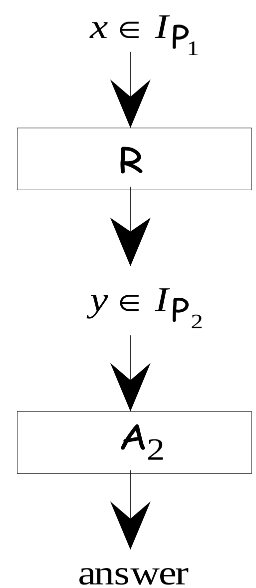
- It holds that
  PTAS$\subseteq$APX$\subseteq$log-APX$\subseteq$poly-APX$\subseteq$exp-APX$\subseteq$NPO
  where
  - log-APX={polynomial-time O(log n)-approximate problem}
  - poly-APX={polynomial-time $O(n^k)$-approximate problem, for any k>0}
  - exp-APX={polynomial-time $O(2^{n^k})$-approximate problem, for any k>0}

Memo: Approximation classes inclusions

- Polynomial bound on m() implies that any NPO problem is $h2^{n^k}$-approximable for some h and k... so every problem is in exp-APX?
- NO! There are problems for which it is even hard (NP-hard) to decide if any feasible solution exists.
  - Example: MIN {0,1}-linear programming
    - Given an integer matrix A and an integer vector b, deciding whether a binary vector x exists such that $Ax \geq b$ is NP-hard
- If P≠NP, then PTAS⊂APX⊂log-APX⊂poly-APX⊂exp-APX⊂NPO

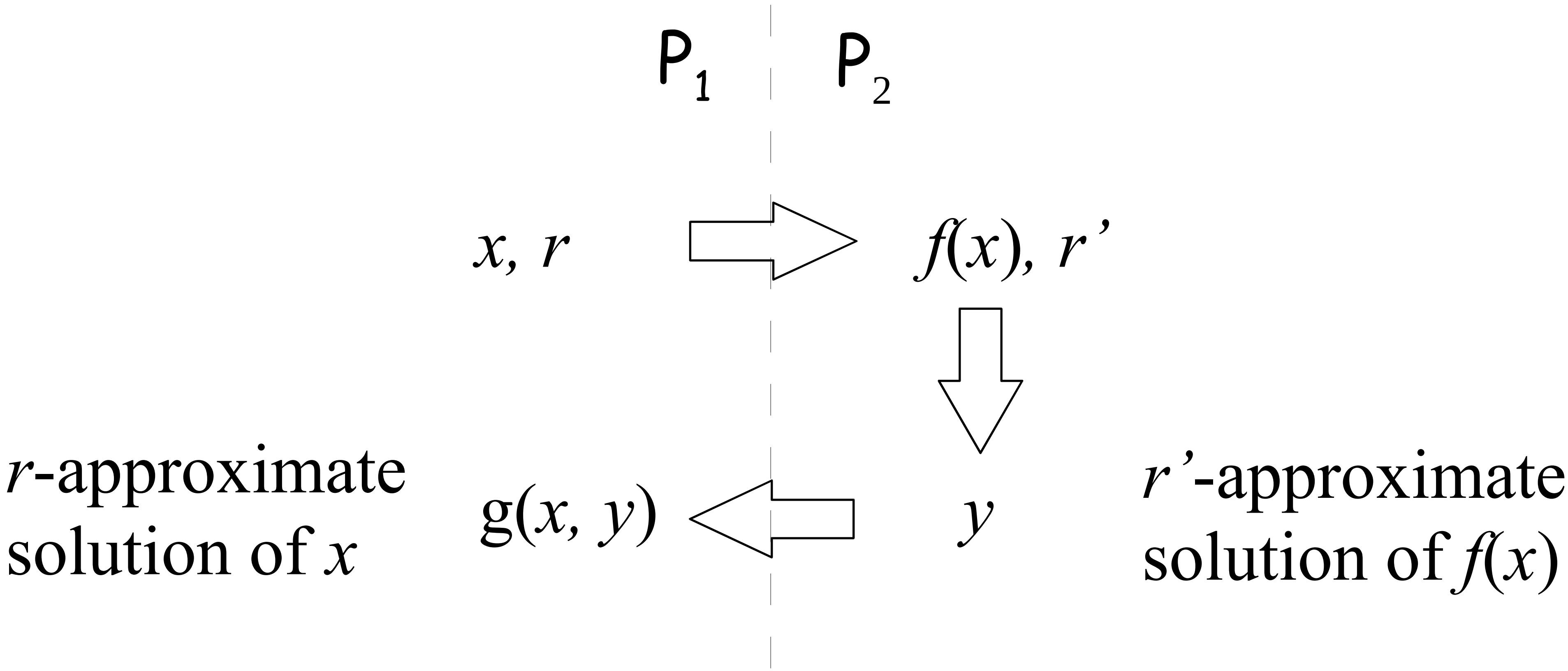
Memo: Karp reducibility

- A decision problem P_1 is *Karp reducible* to a decision problem P_2 (in short, $\mathsf{P}_1 \leq \mathsf{P}_2$) if there exists a polynomial-time computable function R such that, for any x, x is a YES-instance of P_1 if and only if $\mathsf{R}(x)$ is a YES-instance of P_2

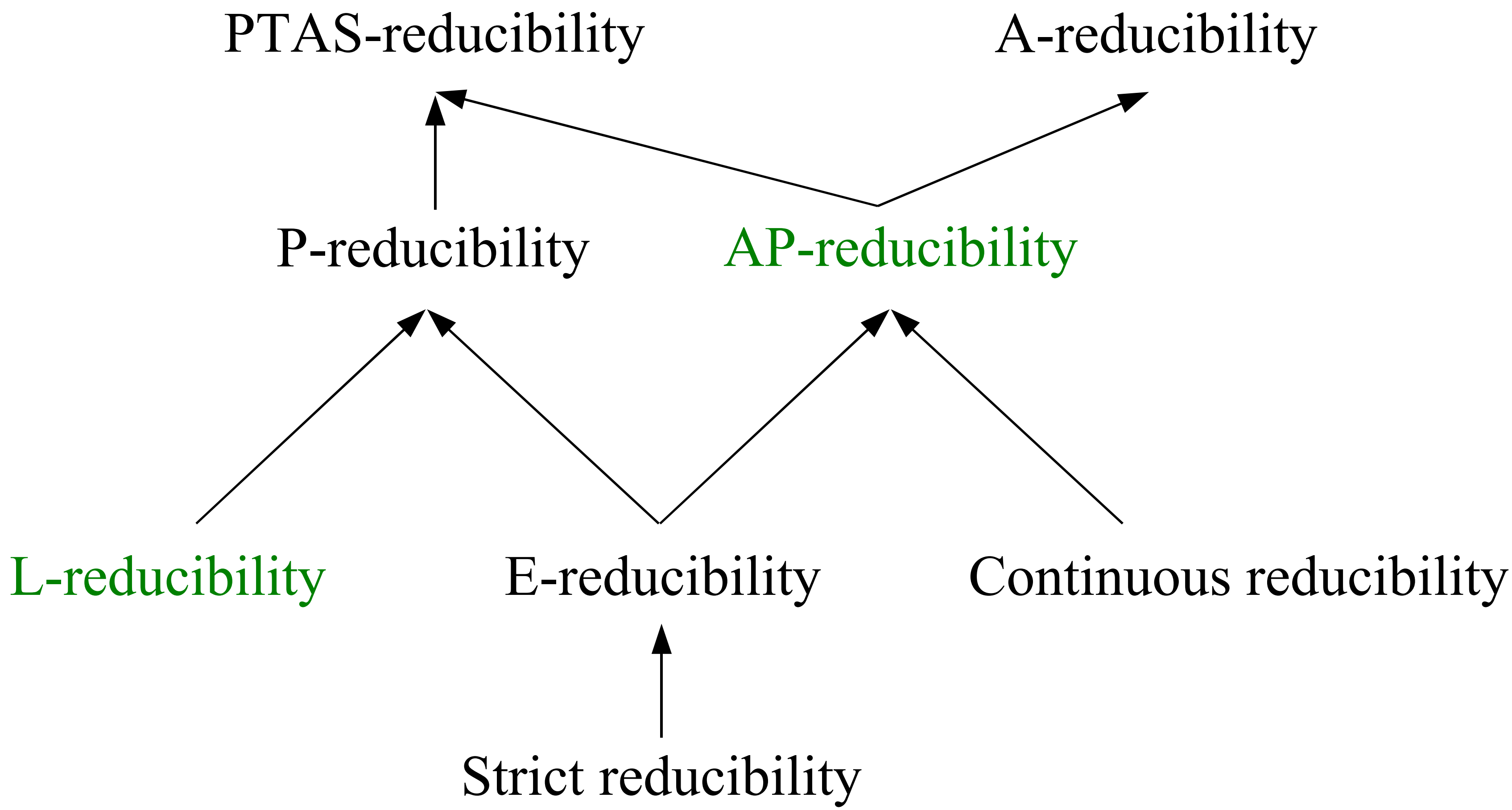


- If $\mathsf{P}_1 \leq \mathsf{P}_2$ and P_2 is in P, then P_1 is in P

Reducibility and NPO problems



A taxonomy


PTAS-reducibility
A-reducibility
P-reducibility
AP-reducibility
L-reducibility
E-reducibility
Continuous reducibility
Strict reducibility

AP-reducibility ($\leq_{AP}$)

- P_1 is AP-reducible to P_2 ($P_1 \leq_{AP} P_2$) if two functions f and g and a constant $c \geq 1$ exist such that:
  - For any instance x of P_1 and for any $r>1$, $f(x, r)$ is an instance of P_2
  - For any instance x of P_1, for any $r>1$, and for any solution y of $f(x,r)$, $g(x,y,r)$ is a solution of x
  - For any fixed $r>1$, f and g are computable in polynomial time
  - For any instance x of P_1, for any $r>1$, and for any solution y of $f(x,r)$, if $R_{P_2}(f(x,r), y) \leq r$, then $R_{P_1}(x, g(x,y,r)) \leq 1+c(r-1)$

Basic properties

- **Theorem:** If $P_1 \leq_{AP} P_2$ and $P_2 \in$ APX, then $P_1 \in$ APX
  - If A is an r-approximation algorithm for P_2 then

    $$A_{P_1}(x) = g(x, A(f(x,r)), r)$$

    is a $(1+c(r-1))$-approximation algorithm for P_1

Basic properties

- **Theorem:** If $P_1 \leq_{AP} P_2$ and $P_2 \in$ PTAS, then $P_1 \in$ PTAS
  - If A is a polynomial-time approximation scheme for P_2 then

    $$A_{P_1}(x, r) = g(x, A(f(x, r'), r'), r')$$

    is a polynomial-time approximation scheme for P_1, where $r' = 1 + (r-1)/c$

L-reducibility

- P_1 is L-reducible to P_2 ($P_1 \leq_L P_2$) if two functions f and g and two constants a and b exist such that:
  - For any instance x of P_1, $f(x)$ is an instance of P_2
  - For any instance x of P_1, and for any solution y of $f(x)$, $g(x, y)$ is a solution of x
  - f and g are computable in polynomial time
  - For any instance x of P_1, $m^*(f(x)) \leq a\, m^*(x)$
  - For any instance x of P_1 and for any solution y of $f(x)$, $|m^*(x) - m(x, g(x,y))| \leq b\, |m^*(f(x)) - m(f(x), y)|$

Basic property of L-reductions

- **Theorem:** If $P_1 \leq_L P_2$ and $P_2 \in$ PTAS, then $P_1 \in$ PTAS
  - Relative error in P_1 is bounded by $a\ b$ times the relative error in P_2

- However, in general, it is not true that if $P_1 \leq_L P_2$ and $P_2 \in$ APX, then $P_1 \in$ APX ⇐ **NO!**
  - The problem is that the relation between r and r' may be non-invertible

Basic property of L-reductions

- **Lemma:** Let P_1 and P_2 be two NPO problems such that is $P_1 \leq_L P_2$.

$$\text{If } P_1 \in \text{APX}, \text{ then } P_1 \leq_{AP} P_2$$

Inapproximability of independent set

- **Theorem:** MAX CLIQUE $\leq_{AP}$ MAX INDEPENDENT SET
  - $G=(V, E)$, $G^c=(V,V^2-E)$ is the complement graph
  - $f(G) = G^c$
  - $g(G,U)=U$
  - $c=1$
    - Each clique in G is an independent set in G^c

- **Corollary:** MAX INDEPENDENT SET $\notin$ APX

Complete problems

- AP-reduction is transitive
- AP-reduction induces a partial order among problems in the same approximation classes
- Given a class C of NPO problems, a problem P is C-hard (with respect to the AP-reduction) if, for any $P' \in C$, $P' \leq_{AP} P$. If $P \in C$, then P is C-complete (with respect to the AP-reduction).
  - For any class $C \not\subset$ APX ($\not\subset$ PTAS), if P is C-complete, then $P \notin$ APX ($\notin$ PTAS), unless P=NP

MAXIMUM WEIGHTED SAT

- INSTANCE: CNF Boolean formula φ with variables $x_1, x_2, \ldots x_n$, with non negative weights $w_1, w_2, \ldots w_n$

- SOLUTION: A satisfied truth-assignment f to φ

- MEASURE: $\max(1, \Sigma\, w_i f(x_i))$, where $f(x_i)$=true is calculated as $f(x_i)=1$ and $f(x_i)$=false as $f(x_i)=0$

NPO-complete problems

- Finding a feasible solution is as hard as SAT
  - MAX WEIGHTED SAT $\notin$ exp-APX
- MAX WEIGHTED SAT is NPO-complete
  - MAX WEIGHTED SAT $\leq_{AP}$ MAX WEIGHTED 3-SAT
  - MAX WEIGHTED 3-SAT $\leq_{AP}$ MIN WEIGHTED 3-SAT
  - MIN WEIGHTED 3-SAT $\leq_{AP}$ MIN {0,1}-LINEAR PROGRAMMING
  - MIN {0,1}-LINEAR PROGRAMMING is NPO-complete

APX-complete problems

- The PCP theorem permits to show that MAX 3-SAT is complete for the class of maximization problems in APX
- For any minimization problem P in APX, a maximization problem P' in APX exists such that $P \leq_{\mathrm{AP}} P'$
- MAX 3-SAT is APX-complete

Inapproximability of 2-satisfiability

- **Theorem:** MAX 3-SAT $\leq_L$ MAX 2-SAT
  - f transforms each clause (x **or** y **or** z) into the following set of 10 clauses where i is a new variable:
    - (x), (y), (z), (i), (**not** x **or not** y), (**not** x **or not** z), (**not** y **or not** z), (x **or not** i), (y **or not** i), (z **or not** i)
  - $g(C,t)$=restriction of t to original variables
  - a=13, b=1
    - $m^*(f(x))=6|C|+m^*(x) \leq 12m^*(x)+m^*(x)=13m^*(x)$
    - $m^*(f(x))-m(f(x),t) \leq m^*(x)-m(x,g(C,t))$
- **Corollary:** MAX 2-SAT is APX-complete

MAXIMUM NOT-ALL-EQUAL SAT

- INSTANCE: CNF Boolean formula, that is, set C of clauses over set of variables V

- SOLUTION: A truth-assignment f to V

- MEASURE: Number of clauses that contain both a false and a true literal

Inapproximability of NAE 3-SAT

- **Theorem:** MAX 2-SAT $\leq_L$ MAX NAE 3-SAT
  - f transforms each clause x **or** y into new clause x **or** y **or** z where z is a new global variable
  - $g(C,t)$=restriction of t to original variables
  - a=1, b=1
    - z may be assumed false
    - each new clause is not-all-equal satisfied iff the original clause is satisfied

- **Corollary:** MAX NAE 3-SAT is APX-complete

Other inapproximability results

- **Theorem:** MIN VERTEX COVER is APX-complete
  - Reduction from MAX 3-SAT(3)

- **Theorem:** MAX CUT is APX-complete
  - Reduction from MAX NAE 3-SAT

- **Theorem:** MIN GRAPH COLORING $\notin$ APX
  - Reduction from variation of independent set

The NPO world if P $\neq$ NP

NPO	MINIMUM TSP MAXIMUM INDEPENDENT SET MAXIMUM CLIQUE MINIMUM GRAPH COLORING
APX	MINIMUM BIN PACKING MAXIMUM SATISFIABILITY MINIMUM VERTEX COVER MAXIMUM CUT
PTAS	MINIMUM PARTITION
PO	MINIMUM PATH