Jumping Problems for Fully Nonlinear Elliptic Variational Inequalities

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Received June 15, 2000
Revised manuscript received January 22, 2001

By means of nonsmooth critical point theory we prove existence of at least two solutions for a general class of variational inequalities when between the obstacle and the behavior at $+\infty$ there is a situation of jumping type.

Keywords: Jumping problems, variational inequalities, nonsmooth critical point theory

1991 Mathematics Subject Classification: 35D05, 58J05

1. Introduction

Starting from the pioneering paper of Ambrosetti and Prodi [1], jumping problems for semilinear elliptic equations of the type

$$
\begin{cases}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

have been extensively treated (see e.g. [14, 18, 20, 21]).

Moreover, also the case of semilinear variational inequalities with a situation of jumping type has been discussed in [12, 19]. Very recently, quasilinear inequalities of the form:

$$
\begin{cases}
\int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(x,u)D_i u D_j (v - u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u (v - u) \right\} \, dx + \\
- \int_{\Omega} g(x,u)(v - u) \, dx \geq \langle \omega, v - u \rangle & \forall v \in \tilde{K}_{\vartheta},
\end{cases}
$$

where $K_{\vartheta} = \{ u \in H^1_0(\Omega) : u \geq \vartheta \ \text{a.e. in } \Omega \}$, $\tilde{K}_{\vartheta} = \{ v \in K_{\vartheta} : (v - u) \in L^\infty(\Omega) \}$ and $\vartheta \in H^1_0(\Omega)$, have been considered in [11].

When $\vartheta \equiv -\infty$, namely we have no obstacle and the variational inequality becomes an equation, the problem has been also studied in [5, 6] by A. Canino and has been extended in [13] by the authors to the case of fully nonlinear operators.
The purpose of this paper is to study the more general class of nonlinear variational inequalities of the type:

\[
\begin{aligned}
& \int_{\Omega} \left\{ \nabla_\xi L(x, u, \nabla u) \cdot \nabla (v - u) + D_s L(x, u, \nabla u) (v - u) \right\} \, dx + \\
& - \int_{\Omega} g(x, u) (v - u) \, dx \geq \langle \omega, v - u \rangle \quad \forall v \in \tilde{K}_\vartheta, \\
& u \in K_\vartheta.
\end{aligned}
\] (1)

In the main result we shall prove the existence of at least two solutions of (1). The framework is the same of [13], but technical difficulties arise, mainly in the verification of the Palais–Smale condition. This is due to the fact that such condition is proved in [13] using in a crucial way test functions of exponential type. Such test functions are not admissible for the variational inequality, so that a certain number of modifications is required in particular in the proofs of Theorem 4.4 and Theorem 5.2.

As in the previous papers dealing with quasilinear equations and inequalities (see e.g. [3, 5, 6, 7, 11, 22]) we will use variational methods based on the nonsmooth critical point theory of [9, 10]. Let us mention that similar abstract techniques have been developed independently in [15, 16].

2. The main result

In the following, \( \Omega \) will denote a bounded domain of \( \mathbb{R}^n \), \( 1 < p < n \), \( \vartheta \in W^{1,p}_0(\Omega) \) with \( \vartheta^- \in L^\infty(\Omega) \), \( \omega \in W^{-1,p'}(\Omega) \) and

\[
L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}
\]

is measurable in \( x \) for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and of class \( C^1 \) in \( (s, \xi) \) a. e. in \( \Omega \). We shall assume that \( L(x, s, \cdot) \) is strictly convex and for each \( t \in \mathbb{R} \)

\[
L(x, s, t\xi) = |t|^p L(x, s, \xi)
\] (2)

for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \). Furthermore, we assume that:

(i) there exist \( \nu > 0 \) and \( b_1 \in \mathbb{R} \) such that

\[
\nu |\xi|^p \leq L(x, s, \xi) \leq b_1 |\xi|^p,
\] (3)

for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \);

(ii) there exist \( b_2, b_3 \in \mathbb{R} \) such that

\[
|D_s L(x, s, \xi)| \leq b_2 |\xi|^p,
\] (4)

for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and

\[
|\nabla_\xi L(x, s, \xi)| \leq b_3 |\xi|^{p-1},
\] (5)

for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).
(iii) there exist \( R > 0 \) and a bounded Lipschitzian function \( \psi : [R, +\infty[ \to [0, +\infty[ \) such that
\[
s \geq R \implies D_s L(x, s, \xi) \geq 0,
\]
\[
s \geq R \implies D_s L(x, s, \xi) \leq \psi'(s) \nabla \xi L(x, s, \xi) \cdot \xi,
\]
for a. e. \( x \in \Omega \) and for all \( \xi \in \mathbb{R}^n \). We denote by \( \overline{\psi} \) the limit of \( \psi(s) \) as \( s \to +\infty \).

(iv) \( g(x, s) \) is a Carathéodory function and \( G(x, s) = \int_0^s g(x, \tau) \, d\tau \). We assume that there exist \( a \in L^n_{p-1} (\Omega) \) and \( b \in L_\infty (\Omega) \) such that
\[
|g(x, s)| \leq a(x) + b(x)|s|^{p-1},
\]
for a. e. \( x \in \Omega \) and all \( s \in \mathbb{R} \). Moreover, there exists \( \alpha \in \mathbb{R} \) such that
\[
\lim_{s \to +\infty} \frac{g(x, s)}{s^{p-1}} = \alpha,
\]
for a. e. \( x \in \Omega \).

Set now:
\[
\lim_{s \to +\infty} L(x, s, \xi) = L_\infty (x, \xi)
\]
(this limit exists by (6)). We also assume that \( L_\infty (x, \cdot) \) is strictly convex for a. e. \( x \in \Omega \).

Let us remark that we are not assuming the strict convexity uniformly in \( x \) so that such \( L_\infty \) is pretty general. Moreover, assume that
\[
s_h \to +\infty, \quad \xi_h \to \xi \implies \nabla \xi L(x, s_h, \xi_h) \to \nabla \xi L_\infty (x, \xi),
\]
for a. e. \( x \in \Omega \). Let now
\[
\lambda_1 = \min \left\{ p \int_\Omega L_\infty (x, \nabla u) \, dx : u \in W_0^1 p (\Omega), \int_\Omega |u|^p \, dx = 1 \right\},
\]
be the first (nonlinear) eigenvalue of
\[
\left\{ u \mapsto -\text{div} (\nabla \xi L_\infty (x, \nabla u)) \right\}.
\]

Observe that by [2, Lemma 1.4] the first eigenfunction \( \phi_1 \) belongs to \( L_\infty (\Omega) \) and by [23, Theorem 1.1] is strictly positive.

Our purpose is to study (1) when \( \omega = -t^{p-1} \phi_1^{p-1} \), namely the family of problems
\[
(P_t)
\]
\[
\begin{cases}
\int_\Omega \left\{ \nabla \xi L(x, u, \nabla u) \cdot \nabla (v - u) + D_s L(x, u, \nabla u) (v - u) \right\} \, dx + \\
- \int_\Omega g(x, u)(v - u) \, dx + t^{p-1} \int_\Omega \phi_1^{p-1}(v - u) \, dx \geq 0 & \forall v \in \tilde{K}_\vartheta,
\end{cases}
\]

where
\[
K_\vartheta = \left\{ u \in W_0^1 p (\Omega) : u \geq \vartheta \text{ a. e. in } \Omega \right\}
\]
and \( \tilde{K}_\vartheta = \{ v \in K_\vartheta : (v - u) \in L_\infty (\Omega) \} \).

Under the previous assumptions, the following is our main result:

\[
(\text{iii})
\]

\[
(\text{iv})
\]

\[
(\text{vi})
\]

\[
(\text{vii})
\]
Theorem 2.1. Assume that $\alpha > \lambda_1$. Then there exists $\bar{t} \in \mathbb{R}$ such that for all $t \geq \bar{t}$ the problem $(P_t)$ has at least two solutions.

This result extends [11, Theorem 2.1] dealing with Lagrangians of the type

$$L(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i \xi_j - G(x,s)$$

for a. e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

In this particular case, existence of at least three solutions has been proved in [6] for equations assuming $\alpha > \mu_2$ where $\mu_2$ is the second eigenvalue of the operator

$$\{ u \mapsto - \sum_{i,j=1}^{n} D_j(A_{ij}D_i u) \}.$$

In our general setting, since $L_{\infty}$ is not quadratic with respect to $\xi$, we only have the existence of the first eigenvalue $\lambda_1$ and it is not clear how to define higher order eigenvalues $\lambda_2, \lambda_3, \ldots$. Therefore in our case the comparison of $\alpha$ with higher eigenvalues has no obvious formulation.

3. Recalls from nonsmooth critical point theory

Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a function. We denote by $B_r(u)$ the open ball of center $u$ and radius $r$ and set $\text{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R} : f(u) \leq \lambda \}$. In the following, the space $X \times \mathbb{R}$ will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = (d(u,v))^2 + (\lambda - \mu)^2)^{\frac{1}{2}}$$

and $\text{epi}(f)$ with the induced metric. Finally, we set $D(f) = \{ u \in X : f(u) < +\infty \}$.

Definition 3.1. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the $\sigma$’s in $[0, +\infty]$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : (B_\delta(u, f(u)) \cap \text{epi}(f)) \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}((v, \mu), t), v) \leq t, \quad f(\mathcal{H}((v, \mu), t)) \leq \mu - \sigma t,$$

whenever $(v, \mu) \in B_\delta(u, f(u)) \cap \text{epi}(f)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$.

The above notion has been introduced, in an equivalent way, independently in [10, 16], while a variant has been considered in [15]. The form mentioned here is taken from [4]. For further details see [11, Section 3].

Definition 3.2. An element $u \in X$ is said to be a (lower) critical point of $f$ if $|df|(u) = 0$. A real number $c$ is said to be a (lower) critical value of $f$ if there exists a critical point $u \in X$ of $f$ such that $f(u) = c$. Otherwise $c$ is said to be a regular value of $f$. 

We say that Theorem 4.2. if every (sequence at level Definition 4.1. sequence (\( \parallel \)).

It is sufficient to combine [10, Theorem 3.12] with [11, Proposition 3.4].

Proof. It is sufficient to combine [10, Theorem 3.12] with [11, Proposition 3.4].

4. The bounded Palais–Smale condition

In this section we shall consider the more general variational inequalities (1). To this aim let us now introduce the functional \( f : W^{1,p}_0(\Omega) \to \mathbb{R} \cup \{ +\infty \} \)

\[
f(u) = \begin{cases} 
\int_{\Omega} L(x,u,\nabla u) \, dx - \int_{\Omega} G(x,u) \, dx - \langle \omega, u \rangle & u \in K_0 \\
+\infty & u \notin K_0.
\end{cases}
\]

Definition 4.1. Let \( c \in \mathbb{R} \). A sequence \((u_h)\) in \( K_0 \) is said to be a concrete Palais–Smale sequence at level \( c \), \((CPS)_{c}-sequence, for short) for \( f \), if \( f(u_h) \to c \) and there exists a sequence \((\varphi_h)\) in \( W^{-1,p'}(\Omega) \) such that \( \varphi_h \to 0 \) and

\[
\int_{\Omega} \nabla \xi L(x,u_h,\nabla u_h) \cdot \nabla (v-u_h) \, dx + \int_{\Omega} D_s L(x,u_h,\nabla u_h)(v-u_h) \, dx + \\
- \int_{\Omega} g(x,u_h)(v-u_h) \, dx - \langle \omega, v-u_h \rangle \geq \langle \varphi_h, v-u_h \rangle \quad \forall v \in \bar{K}_0.
\]

We say that \( f \) satisfies the concrete Palais–Smale condition at level \( c \), \((CPS)_{c}-sequence for \( f \) admits a strongly convergent subsequence in \( W^{1,p'}_0(\Omega) \).

Theorem 4.2. Let \( u \in K_0 \) be such that \( |df|(u) < +\infty \). Then there exists \( \varphi \) in \( W^{-1,p'}(\Omega) \) such that \( \| \varphi \|_{-1,p'} \leq |df|(u) \) and

\[
\int_{\Omega} \nabla \xi L(x,u,\nabla u) \cdot \nabla (v-u) \, dx + \int_{\Omega} D_s L(x,u,\nabla u)(v-u) \, dx + \\
- \int_{\Omega} g(x,u)(v-u) \, dx - \langle \omega, v-u \rangle \geq \langle \varphi, v-u \rangle \quad \forall v \in \bar{K}_0.
\]
Proof. Argue as in [11, Theorem 4.6].

Proposition 4.3. Let \( c \in \mathbb{R} \) and assume that \( f \) satisfies the \((CPS)_c\) condition. Then \( f \) satisfies the \((PS)_c\) condition.

Proof. It is an easy consequence of Theorem 4.2.

Let us note that by combining (3) with the convexity of \( L(x, s, \cdot) \), we get
\[
\nabla_{\xi} L(x, s, \xi) \cdot \xi \geq \nu|\xi|^p
\]
for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \). Moreover, there exists \( M > 0 \) such that
\[
|D_s L(x, s, \xi)| \leq M \nabla_{\xi} L(x, s, \xi) \cdot \xi
\]
for a. e. \( x \in \Omega \) and for all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \).

Suppose now that \( \vartheta(x) > -R \) for a. e. \( x \in \Omega \), where \( R > 0 \) is as in \((iii)\) and define
\[
\tilde{L}(x, s, \xi) = \begin{cases} L(x, s, \xi) & s > -R \\ L(x, -R, \xi) & s \leq -R. \end{cases}
\]
Such \( \tilde{L} \) satisfy our assumptions. On the other hand, if \( u \) satisfies
\[
(\tilde{P}_t) \quad \left\{ \begin{array}{l} \int_{\Omega} \left\{ \nabla_{\xi} \tilde{L}(x, u, \nabla u) \cdot \nabla (v - u) + D_s \tilde{L}(x, u, \nabla u)(v - u) \right\} dx + \\ -\int_{\Omega} g(x, u)(v - u) dx + t^{p-1} \int_{\Omega} \phi_1^{p-1}(v - u) dx \geq 0 \quad \forall v \in \tilde{K}_\vartheta, \\ u \in K_\vartheta, \end{array} \right.
\]
then \( u \) satisfies \( (P_t) \). Therefore, up to substituting \( L \) with \( \tilde{L} \), we can assume that \( L \) satisfies (6) for any \( s \in \mathbb{R} \) with \( |s| > R \). (Actually \( \tilde{L} \) is only locally Lipschitz in \( s \) but one might always define \( \tilde{L}(x, s, \xi) = L(x, \sigma(s), \xi) \) for a suitable smooth function \( \sigma \).

Now, we want to provide in Theorem 4.5 a very useful criterion for the verification of \((CPS)_c\) condition. Let us first prove a local compactness property for \((CPS)_c\) sequences.

Theorem 4.4. Let \((u_h)\) be a sequence in \( K_\vartheta \) and \((\varphi_h)\) a sequence in \( W^{-1,p'}(\Omega) \) such that \((u_h)\) is bounded in \( W_0^{1,p}(\Omega) \), \( \varphi_h \rightharpoonup \varphi \) and
\[
\int_{\Omega} \nabla_{\xi} L(x, u_h, \nabla u_h) \cdot \nabla (v - u_h) dx + \int_{\Omega} D_s L(x, u_h, \nabla u_h)(v - u_h) dx \geq \langle \varphi_h, v - u_h \rangle \quad \forall v \in \tilde{K}_\vartheta.
\]
Then it is possible to extract a subsequence \((u_{h_k})\) strongly convergent in \( W_0^{1,p}(\Omega) \).

Proof. Up to a subsequence, \((u_h)\) converges to some \( u \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^p(\Omega) \) and a. e. in \( \Omega \). Moreover, arguing as in step I of [11, Theorem 4.18] it follows that
\[
\nabla u_h(x) \rightharpoonup \nabla u(x) \quad \text{for a. e. } x \in \Omega.
\]
We divide the proof into several steps.

I) Let us prove that

\[
\limsup_h \int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (-u_h^-) \exp \{-M(u_h - R^-)\} \, dx \leq \int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla (-u^-) \exp \{-M(u - R^-)\} \, dx
\]

where \(M > 0\) is defined in (13) and \(R > 0\) has been introduced in hypothesis (6).

Consider the test functions

\[
v = u_h + \zeta \exp \{-M(u_h + R^+)\}
\]

in (14) where \(\zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\) and \(\zeta \geq 0\). Then

\[
\int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla \zeta \exp \{-M(u_h + R^+)\} \, dx +
\]

\[
+ \int_\Omega [D_x L(x, u_h, \nabla u_h) - M\nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h + R^+)] \zeta \exp \{-M(u_h + R^+)\} \, dx
\]

\[
\geq \langle \varphi, \zeta \exp \{-M(u_h + R^+)\} \rangle.
\]

From (6) and (13) we deduce that

\[
[D_x L(x, u_h, \nabla u_h) - M\nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h + R^+)] \zeta \exp \{-M(u_h + R^+)\} \leq 0,
\]

so that by the Fatou’s Lemma we get

\[
\int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla \zeta \exp \{-M(u + R^+)\} \, dx +
\]

\[
+ \int_\Omega [D_x L(x, u, \nabla u) - M\nabla \xi L(x, u, \nabla u) \cdot \nabla (u + R^+)] \zeta \exp \{-M(u + R^+)\} \, dx \geq
\]

\[
\geq \langle \varphi, \zeta \exp \{-M(u + R^+)\} \rangle \quad \forall \zeta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \zeta \geq 0.
\] (16)

Now, let us consider the functions

\[
\eta_k = \eta \exp \{M(u + R^+)\} \vartheta_k(u),
\]

where \(\eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\) with \(\eta \geq 0\) and \(\vartheta_k \in C^\infty(\mathbb{R})\) is such that \(0 \leq \vartheta_k(s) \leq 1\), \(\vartheta_k = 1\) on \([-k, k]\), \(\vartheta_k = 0\) outside \([-2k, 2k]\) and \(|\vartheta_k'| \leq c/k\) for some \(c > 0\).

Putting them in (16), for each \(k \geq 1\) we obtain

\[
\int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla (\eta \vartheta_k(u)) \, dx + \int_\Omega D_x L(x, u, \nabla u) \eta \vartheta_k(u) \, dx \geq
\]

\[
\geq \langle \varphi, \eta \vartheta_k(u) \rangle \quad \forall \eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \eta \geq 0.
\]
Passing to the limit as \( k \to +\infty \) we obtain

\[
\int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla \eta \, dx + \int_\Omega D_s L(x, u, \nabla u) \eta \, dx \geq \langle \varphi, \eta \rangle \quad \forall \eta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \eta \geq 0. \tag{17}
\]

Taking \( \eta = (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \) in (17) we get

\[
\int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx \geq - \int_\Omega [D_s L(x, u, \nabla u) - M \nabla \xi L(x, u, \nabla u) \cdot \nabla (u - R)^-] (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx + \langle \varphi, (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \rangle. \tag{18}
\]

On the other hand, taking

\[
v = u_h - (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \geq u_h - (\vartheta^- - u_h^-) = u_h^+ - \vartheta^-
\]
in (14) we obtain

\[
\int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx + \\
\int_\Omega [D_s L(x, u_h, \nabla u_h) - M \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^-] (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx \leq \langle \varphi_h, (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \rangle. \tag{19}
\]

From (6) and (13) we deduce that

\[
D_s L(x, u_h, \nabla u_h) - M \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^- \geq 0.
\]

From (19), using Fatou’s Lemma and (18) we obtain

\[
\limsup_{h} \int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (\vartheta^- - u_h^-) \exp \{-M(u_h - R)^-\} \, dx \leq \\
\int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla (\vartheta^- - u^-) \exp \{-M(u - R)^-\} \, dx. \tag{20}
\]

Since

\[
\lim_{h} \int_\Omega \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla \vartheta^- \exp \{-M(u_h - R)^-\} \, dx = \\
= \int_\Omega \nabla \xi L(x, u, \nabla u) \cdot \nabla \vartheta^- \exp \{-M(u - R)^-\} \, dx,
\]
then from (20) we deduce (15).
II) Let us now prove that

\[
\limsup_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h^+ \exp \{ -M(u_h - R)^- \} \, dx \leq 1.
\]

We consider the test functions

\[
v = u_h - \left[ (u_h^+ - \vartheta^+) \land k \right] \exp \{ -M(u_h - R)^- \} \geq \vartheta + (\vartheta^- - u_h^-)
\]

in (14). By Fatou’s Lemma, we get

\[
\int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{ -M(u_h - R)^- \} \, dx + \\
\int_{\Omega} \left[ D_s L(x, u_h, \nabla u_h) - M \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^- \right] (u_h^+ - \vartheta^+) \exp \{ -M(u_h - R)^- \} \, dx
\]

from which we deduce that

\[
\left[ D_s L(x, u_h, \nabla u_h) - M \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h - R)^- \right] (u_h^+ - \vartheta^+) \exp \{ -M(u_h - R)^- \}
\]

belongs to \( L^1(\Omega) \). Using Fatou’s Lemma in (22) we obtain

\[
\limsup_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{ -M(u_h - R)^- \} \, dx \leq \\
\leq - \int_{\Omega} \int [ D_s L(x, u, \nabla u) - M \nabla \xi L(x, u, \nabla u) \cdot \nabla (u - R)^- ] \\
( u^+ - \vartheta^+ ) \exp \{ -M(u - R)^- \} \, dx + \langle \varphi, ( u^+ - \vartheta^+ ) \exp \{ -M(u - R)^- \} \rangle,
\]

from which we also deduce that

\[
\left[ D_s L(x, u, \nabla u) - M \nabla \xi L(x, u, \nabla u) \cdot \nabla (u - R)^- \right] (u^+ - \vartheta^+) \exp \{ -M(u - R)^- \}
\]

belongs to \( L^1(\Omega) \). Now, taking \( \eta_k = [(u^+ - \vartheta^+) \land k] \exp \{ -M(u - R)^- \} \) in (17), we have

\[
\int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla \left[ (u^+ - \vartheta^+) \land k \right] \exp \{ -M(u - R)^- \} \, dx + \\
\int_{\Omega} \left[ D_s L(x, u, \nabla u) - M \nabla \xi L(x, u, \nabla u) \cdot \nabla (u - R)^- \right] \left[ (u^+ - \vartheta^+) \land k \right] \exp \{ -M(u - R)^- \} \, dx
\]

\[
\geq \langle \varphi, \left[ (u^+ - \vartheta^+) \land k \right] \exp \{ -M(u - R)^- \} \rangle.
\]

Using (24) and passing to the limit as \( k \to +\infty \) in (25), it results

\[
\int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla (u^+ - \vartheta^+) \exp \{ -M(u - R)^- \} \, dx + \\
\int_{\Omega} \left[ D_s L(x, u, \nabla u) - M \nabla \xi L(x, u, \nabla u) \cdot \nabla (u - R)^- \right] (u^+ - \vartheta^+) \exp \{ -M(u - R)^- \} \, dx
\]

\[
\geq \langle \varphi, (u^+ - \vartheta^+) \exp \{ -M(u - R)^- \} \rangle.
\]
Combining (26) with (23) we obtain

\[
\limsup_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (u_h^+ - \vartheta^+) \exp \{-M(u_h - R)\} \, dx \leq \\
\leq \int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla (u^+ - \vartheta^+) \exp \{-M(u - R)\} \, dx
\]

(27)

Since

\[
\lim_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla \vartheta^+ \exp \{-M(u_h - R)\} \, dx = \\
= \int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla \vartheta^+ \exp \{-M(u - R)\} \, dx
\]

from (27) we deduce (21).

III) Let us finally prove that \( u_h \to u \) strongly in \( W^{1,p}_0(\Omega) \). We claim that

\[
\limsup_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)\} \, dx \leq \\
\leq \int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla u \exp \{-M(u - R)\} \, dx
\]

In fact using (15) and (21) we get

\[
\limsup_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)\} \, dx \leq \\
\leq \limsup_h \int_{\Omega \cap \{u_h > 0\}} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h^+ \exp \{-M(u_h - R)\} \, dx + \\
+ \limsup_h \int_{\Omega \cap \{u_h \leq 0\}} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla (-u_h^-) \exp \{-M(u_h - R)\} \, dx \leq \\
\leq \int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla u \exp \{-M(u - R)\} \, dx
\]

(28)

From (28) using the Fatou Lemma we get

\[
\lim_h \int_{\Omega} \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)\} \, dx = \\
= \int_{\Omega} \nabla \xi L(x, u, \nabla u) \cdot \nabla u \exp \{-M(u - R)\} \, dx.
\]

Therefore, since by (12) we have

\[
\nu \exp \{-M(R + \|\vartheta^-\|_\infty\)} \|\nabla u_h\|^p \leq \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \exp \{-M(u_h - R)\},
\]

it follows that

\[
\lim_h \int_{\Omega} |\nabla u_h|^p \, dx = \int_{\Omega} |\nabla u|^p \, dx,
\]

namely the strong convergence of \((u_h)\) to \( u \) in \( W^{1,p}_0(\Omega) \).
In view of Theorem 4.2, any critical point of $x$

Lemma 5.1.

Of course, the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. Since the map $\{u \mapsto g(x, u)\}$ is completely continuous from $W_0^{1,p}(\Omega)$ to $L^{np'}(\Omega)$, the proof goes like [11, Theorem 4.37].

5. The Palais–Smale condition

Let us now set

$$g_0(x, s) = g(x, s) - \alpha(s^+)^{p-1}, \quad G_0(x, s) = \int_0^s g_0(x, t)\,dt.$$ 

Of course, $g_0$ is a Carathéodory function satisfying

$$\lim_{s \to +\infty} \frac{g_0(x, s)}{s^{p-1}} = 0, \quad |g_0(x, s)| \leq a(x) + b(x)|s|^{p-1},$$

for a. e. $x \in \Omega$ and all $s \in \mathbb{R}$ where $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b \in L^\frac{n}{p}(\Omega)$. Then (P$_1$) is equivalent to finding $u \in K_\vartheta$ such that

$$\int_{\Omega} \nabla \xi \cdot (L(x, u, \nabla u)(v-u) + \int_{\Omega} D_s L(x, u, \nabla u)(v-u)\,dx +$$

$$- \alpha \int_{\Omega} (u^+)^{p-1}(v-u)\,dx - \int_{\Omega} g_0(x, u)(v-u)\,dx + \int_{\Omega} \phi_1^{p-1}(v-u)\,dx \geq 0 \quad \forall v \in K_\vartheta.\]  

Let us define the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f(u) = \begin{cases} 
\int_{\Omega} L(x, u, \nabla u)\,dx - \frac{a}{p} \int_{\Omega} (u^+)^p\,dx - \int_{\Omega} G_0(x, u)\,dx + t^{p-1} \int_{\Omega} \phi_1^{p-1} u\,dx & \text{if } u \in K_\vartheta \\
+\infty & \text{if } u \notin K_\vartheta.
\end{cases}$$

In view of Theorem 4.2, any critical point of $f$ is a weak solutions of (P$_1$). Let us introduce a new functional $f_t : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting for each $t > 0$

$$f_t(u) = \begin{cases} 
\int_{\Omega} L(x, tu, \nabla u)\,dx - \frac{a}{p} \int_{\Omega} (u^+)^p\,dx - \frac{1}{p} \int_{\Omega} G_0(x, tu)\,dx + \int_{\Omega} \phi_1^{p-1} u\,dx & \text{if } u \in K_t \\
+\infty & \text{if } u \notin K_t.
\end{cases}$$

where we have set

$$K_t = \{ u \in W_0^{1,p}(\Omega) : tu \geq \vartheta \text{ a. e. in } \Omega \}.$$ 

From Theorem 4.2 it follows that if $u$ is a critical point of $f_t$ then $tu$ satisfies (P$_1$).

Lemma 5.1. Let $(u_n)$ a sequence in $W_0^{1,p}(\Omega)$ and $\theta_n \in [0, +\infty]$ with $\theta_n \to +\infty$. Assume that the sequence $\left(\frac{u_n}{\theta_n}\right)$ is bounded in $W_0^{1,p}(\Omega)$. Then

$$\frac{g_0(x, u_n)}{\theta_n^{p-1}} \to 0 \quad \text{in} \quad L^{\frac{np'}{n(p-1)+p}}(\Omega), \quad \frac{G_0(x, u_n)}{\theta_n^p} \to 0 \quad \text{in} \quad L^1(\Omega).$$
Consider the test functions

We firstly prove that we shall divide the proof into several steps.

\[ s \geq -N \implies D_s L(x, s, \xi) \leq \psi'(s)\nabla_x L(x, s, \xi) \cdot \xi. \]  

(29)

**Theorem 5.2.** Let \( \alpha > \lambda_1, c \in \mathbb{R} \) and let \((u_h)\) in \( K_\vartheta \) be a \((CPS)_c\) sequence for \( f \). Then \((u_h)\) is bounded in \( W_0^{1,p}(\Omega) \).

**Proof.** By Definition 4.1, there exists a sequence \((\varphi_h)\) in \( W^{-1,p'}(\Omega) \) with \( \varphi_h \to 0 \) and

\[
\int_{\Omega} \nabla_x L(x, u_h, \nabla u_h) \cdot \nabla (v - u_h) \, dx + \int_{\Omega} D_s L(x, u_h, \nabla u_h) (v - u_h) \, dx +
- \alpha \int_{\Omega} (u_h^+)_{p-1} (v - u_h) \, dx - \int_{\Omega} g_0(x, u_h) (v - u_h) \, dx + t^{p-1} \int_{\Omega} \varphi_h^{p-1} (v - u_h) \, dx \geq
\geq \langle \varphi_h, v - u_h \rangle \quad \forall v \in K_\vartheta : (v - u_h) \in L^\infty(\Omega).
\]

(30)

We set now \( g_h = \|u_h\|_{1,p} \), and suppose by contradiction that \( g_h \to +\infty \). If we set \( z_h = g_h^{-1} u_h \), up to a subsequence, \( z_h \) converges to some \( z \) weakly in \( W_0^{1,p}(\Omega) \), strongly in \( L^p(\Omega) \) and a.e. in \( \Omega \). Note that \( z \geq 0 \) a.e. in \( \Omega \).

We shall divide the proof into several steps.

I) We firstly prove that

\[
\int_{\Omega} \nabla_x L_\infty(x, \nabla z) \cdot \nabla z \, dx \geq \alpha \int_{\Omega} z^p \, dx.
\]

(31)

Consider the test functions \( v = u_h + (z \wedge k) \exp \{-\psi(u_h)\} \), where \( \psi \) is the function defined in (7). Putting such \( v \) in (30) and dividing by \( g_h^{p-1} \), we obtain

\[
\int_{\Omega} \nabla_x L(x, u_h, \nabla z_h) \cdot \nabla (z \wedge k) \exp \{-\psi(u_h)\} \, dx +
+ \frac{1}{g_h^{p-1}} \int_{\Omega} [D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla_x L(x, u_h, \nabla u_h) \cdot \nabla u_h] (z \wedge k) \exp \{-\psi(u_h)\} \, dx \geq
\geq \alpha \int_{\Omega} (z_h^+)_{p-1} (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \int_{\Omega} g_0(x, u_h) (z \wedge k) \exp \{-\psi(u_h)\} \, dx +
- t^{p-1} \int_{\Omega} \varphi_h^{p-1} (z \wedge k) \exp \{-\psi(u_h)\} \, dx + \frac{1}{g_h^{p-1}} (\varphi_h, (z \wedge k) \exp \{-\psi(u_h)\}).
\]

Observe now that the first term

\[
\int_{\Omega} \nabla_x L(x, u_h, \nabla z_h) \cdot \nabla (z \wedge k) \exp \{-\psi(u_h)\} \, dx
\]

passes to the limit, yielding

\[
\int_{\Omega} \nabla_x L_\infty(x, \nabla z) \cdot \nabla (z \wedge k) \exp \{-\overline{\psi}\} \, dx.
\]
Indeed, by taking into account assumptions (10) and (5), we may apply [8, Theorem 5] and deduce that, up to a subsequence,

\[ a \text{ e. in } \Omega \setminus \{ z = 0 \}: \nabla z_h(x) \to \nabla z(x). \]

Since of course, being \( u_h(x) \to +\infty \) a. e. in \( \Omega \setminus \{ z = 0 \} \), again recalling (10), we have

\[ a \text{ e. in } \Omega \setminus \{ z = 0 \}: \nabla \xi L(x, u_h(x), \nabla z_h(x)) \to \nabla \xi L_\infty(x, \nabla z(x)). \]

Since by (5) the sequence \( (\nabla \xi L(x, u_h(x), \nabla z_h(x))) \) is bounded in \( L^p(\Omega) \), the assertion follows. Note also that the term

\[ \frac{1}{\varrho_h} \langle \varphi_h, (z \wedge k) \exp \{-\psi(u_h)\} \rangle, \]

goes to 0 even if \( 1 < p < 2 \). Indeed, in this case, one could use the Cerami–Palais–Smale condition, which yields \( \varrho_h \varphi_h \to 0 \) in \( W_0^{-1,p'}(\Omega) \).

Now, by (29) we have

\[ D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \leq 0, \]

then, passing to the limit as \( h \to +\infty \), we get

\[ \int_{\Omega} \nabla \xi L_\infty(x, \nabla z) \cdot \nabla (z \wedge k) \exp \{-\tilde{\psi}\} \, dx \geq \alpha \int_{\Omega} z^{p-1}(z \wedge k) \exp \{-\tilde{\psi}\} \, dx. \]

Passing to the limit as \( k \to +\infty \), we obtain (31).

II) Let us prove that \( z_h \to z \) strongly in \( W_0^{1,p}(\Omega) \), so that of course \( \|z\|_{1,p} = 1 \). Consider the function \( \zeta : [-R, +\infty[ \to \mathbb{R} \) defined by

\[ \zeta(s) = \begin{cases} MR & \text{if } s \geq R, \\ Ms & \text{if } |s| < R. \end{cases} \]

(32)

where \( M \in \mathbb{R} \) is such that for a. e. \( x \in \Omega \), each \( s \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \)

\[ |D_s L(x, s, \xi)| \leq M \nabla \xi L(x, s, \xi) \cdot \xi. \]

If we choose the test functions

\[ v = u_h - \frac{u_h - \vartheta}{\exp(MR) \exp(\zeta(u_h))} \]

in (30), we have

\[ \int_{\Omega} \nabla L(x, u_h, \nabla u_h) \cdot \nabla (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \int_{\Omega} [D_s L(x, u_h, \nabla u_h) + \zeta'(u_h) \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h] \exp\{\zeta(u_h)\} \, dx \leq \]
\begin{align*}
\leq & \alpha \int_{\Omega} (u_h^*)^{p-1} (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \int_{\Omega} g_0(x, u_h) (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \\
& \quad - t^{p-1} \int_{\Omega} \phi_{\vartheta}^{p-1} (u_h - \vartheta) \exp\{\zeta(u_h)\} \, dx + \langle \varphi_h , (u_h - \vartheta) \exp\{\zeta(u_h)\} \rangle.
\end{align*}

Note that it results

\[ [D_s L(x, u_h, \nabla u_h) + \zeta'(u_h) \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h] (u_h - \vartheta) \geq 0. \]

Therefore, after division by \( \vartheta_h \) we get

\[ \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla \left( z_h - \frac{\vartheta}{\vartheta_h} \right) \exp\{\zeta(u_h)\} \, dx \leq \]

\[ \leq \alpha \int_{\Omega} (z_h^*)^{p-1} \left( z_h - \frac{\vartheta}{\vartheta_h} \right) \exp\{\zeta(u_h)\} \, dx + \frac{1}{\vartheta_h^{p-1}} \int_{\Omega} g_0(x, u_h) \left( z_h - \frac{\vartheta}{\vartheta_h} \right) \exp\{\zeta(u_h)\} \, dx + \\
& \quad - \frac{t^{p-1}}{\vartheta_h^{p-1}} \int_{\Omega} \phi_{\vartheta_h}^{p-1} \left( z_h - \frac{\vartheta}{\vartheta_h} \right) \exp\{\zeta(u_h)\} \, dx + \frac{1}{\vartheta_h^{p-1}} \langle \varphi_h , \left( z_h - \frac{\vartheta}{\vartheta_h} \right) \exp\{\zeta(u_h)\} \rangle,
\]

which yields

\[ \limsup_{h} \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \alpha \exp\{MR\} \int_{\Omega} z^p \, dx. \quad (33) \]

By combining (33) with (31) we get

\[ \limsup_{h} \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \exp\{MR\} \int_{\Omega} \nabla \xi L_{\infty}(x, \nabla z) \cdot \nabla z \, dx. \]

In particular, by Fatou’s Lemma, it results

\[ \exp\{MR\} \int_{\Omega} \nabla \xi L_{\infty}(x, \nabla z) \cdot \nabla z \, dx \leq \]

\[ \leq \liminf_{h} \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \]

\[ \leq \limsup_{h} \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \leq \]

\[ \leq \exp\{MR\} \int_{\Omega} \nabla \xi L_{\infty}(x, \nabla z) \cdot \nabla z \, dx, \]

namely, we get

\[ \int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\} \, dx \rightarrow \int_{\Omega} \exp\{MR\} \nabla \xi L_{\infty}(x, \nabla z) \cdot \nabla z \, dx. \]

Therefore, since

\[ \nu \exp\{-MR\} |\nabla z_h|^p \leq \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla z_h \exp\{\zeta(u_h)\}, \]

thanks to the generalized Lebesgue’s theorem, we conclude that

\[ \lim_{h} \int_{\Omega} |\nabla z_h|^p \, dx = \int_{\Omega} |\nabla z|^p \, dx, \]
and $z_h$ converges to $z$ in $W_0^{1,p}(\Omega)$.

III) Let us consider the test functions $v = u_h + \varphi \exp \{-\psi(u_h)\}$ with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geq 0$. Taking such $v$ in (30) and dividing by $\varrho_h^{p-1}$ we obtain

$$\int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp \{-\psi(u_h)\} \, dx + \frac{1}{\varrho_h^{p-1}} \int_{\Omega} [D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h] \varphi \exp \{-\psi(u_h)\} \, dx \geq$$

$$\geq \alpha \int_{\Omega} (z_h)^{p-1} \varphi \exp \{-\psi(u_h)\} \, dx + \int_{\Omega} \frac{g_0(x, u_h)}{\varrho_h^{p-1}} \varphi \exp \{-\psi(u_h)\} \, dx + \frac{1}{\varrho_h^{p-1}} (\varphi_h, \varphi \exp \{-\psi(u_h)\}).$$

Note that, since by step II we have $z_h \to z$ in $W_0^{1,p}(\Omega)$, the term

$$\int_{\Omega} \nabla \xi L(x, u_h, \nabla z_h) \cdot \nabla \varphi \exp \{-\psi(u_h)\} \, dx$$

passes to the limit, yielding

$$\int_{\Omega} \nabla \xi L(x, \nabla z) \cdot \nabla \varphi \exp \{-\psi\} \, dx.$$

By means of (29), we have

$$D_s L(x, u_h, \nabla u_h) - \psi'(u_h) \nabla \xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \leq 0,$$

then passing to the limit as $h \to +\infty$, we obtain

$$\int_{\Omega} \nabla \xi L(x, \nabla z) \cdot \nabla \varphi \exp \{-\psi\} \, dx - \alpha \int_{\Omega} z^{p-1} \varphi \exp \{-\psi\} \, dx \geq 0,$$

for each $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ with $\varphi \geq 0$ which yields

$$\int_{\Omega} \nabla \xi L(x, \nabla z) \cdot \nabla \varphi \, dx \geq \alpha \int_{\Omega} z^{p-1} \varphi \, dx \tag{34}$$

for each $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$.

In a similar fashion, considering in (30) the admissible test functions

$$v = u_h - \left( \varphi \wedge \frac{z_h - \partial / \varrho_h}{\exp(\psi)} \right) \exp(\psi(u_h))$$

with $\varphi \in W_0^{1,p} \cap L^\infty(\Omega)$ and $\varphi \geq 0$ and dividing by $\varrho_h^{p-1}$, recalling that $z_h \to z$ strongly, we get

$$\int_{\Omega} \nabla \xi L(x, \nabla z) \cdot \nabla \left[ \varphi \wedge \frac{z}{\exp(\psi)} \right] \, dx \leq \alpha \int_{\Omega} z^{p-1} \left[ \varphi \wedge \frac{z}{\exp(\psi)} \right] \, dx,$$
for each \( \varphi \in W_0^{1,p} \cap L^\infty(\Omega) \) with \( \varphi \geq 0 \). Actually this holds for any \( \varphi \in W_0^{1,p}(\Omega) \) with \( \varphi \geq 0 \). By substituting \( \varphi \) with \( t\varphi \) with \( t > 0 \) we obtain
\[
\int_\Omega \nabla \xi L_\infty(x, \nabla z) \cdot \nabla \left[ \varphi \wedge \frac{z}{t \exp \psi} \right] dx \leq \alpha \int_\Omega z^{p-1} \left[ \varphi \wedge \frac{z}{t \exp \psi} \right] dx.
\]
Letting \( t \to +\infty \), and taking into account (34), it results
\[
\int_\Omega \nabla \xi L_\infty(x, \nabla z) \cdot \nabla \varphi dx = \alpha \int_\Omega z^{p-1} \varphi dx \tag{35}
\]
for each \( \varphi \in W_0^{1,p}(\Omega) \) with \( \varphi \geq 0 \). Clearly (35) holds for any \( \varphi \in W_0^{1,p}(\Omega) \), so that \( z \) is a positive eigenfunction related to \( \alpha \). This is a contradiction by [17, Remark 1, pp. 161].

**Theorem 5.3.** Let \( c \in \mathbb{R}, \alpha > \lambda_1 \) and \( t > 0 \). Then \( f_t \) satisfies the \((PS)_c\) condition.

**Proof.** Since \( f_t(u) = \frac{f(u)}{t^p} \), it is sufficient to combine Theorem 5.2, Theorem 4.5 and Proposition 4.3.

6. **Min–Max estimates**

Let us first introduce the “asymptotic functional” \( f_\infty : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\} \) by setting
\[
f_\infty(u) = \begin{cases} \int_\Omega L_\infty(x, \nabla u) dx - \frac{a}{p} \int_\Omega u^p dx + \int_\Omega \phi_1^{p-1} u dx & \text{if } u \in K_\infty \\ +\infty & \text{if } u \notin K_\infty \end{cases}
\]
where
\[K_\infty = \{ u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ a. e. in } \Omega \}.
\]

**Proposition 6.1.** There exist \( r > 0, \sigma > 0 \) such that
(a) for every \( u \in W_0^{1,p}(\Omega) \) with \( 0 < \|u\|_{1,p} \leq r \) then \( f_\infty(u) > 0 \);
(b) for every \( u \in W_0^{1,p}(\Omega) \) with \( \|u\|_{1,p} = r \) then \( f_\infty(u) \geq \sigma > 0 \).

**Proof.** Let us consider the weakly closed set
\[K^* = \left\{ u \in K_\infty : \int_\Omega L_\infty(x, \nabla u) dx - \frac{a}{p} \int_\Omega u^p dx \leq \frac{1}{2} \int_\Omega L_\infty(x, \nabla u) dx \right\}.
\]
In \( K_\infty \setminus K^* \) the statements are evident. On the other hand, it is easy to see that
\[
\inf \left\{ \int_\Omega v \phi_1^{p-1} dx : v \in K^*, \|v\|_{1,p} = 1 \right\} = \varepsilon > 0
\]
arguing by contradiction. Therefore for each \( u \in K^* \) we have
\[
f_\infty(u) = \int_\Omega L_\infty(x, \nabla u) dx - \frac{a}{p} \int_\Omega u^p dx + \int_\Omega \phi_1^{p-1} u dx \geq c\|u\|_{1,p}^p + \varepsilon\|u\|_{1,p}
\]
where \( c \in \mathbb{R} \) is a suitable constant. Thus the statements follow. \( \square \)
Proposition 6.2. Let $r > 0$ be as in the Proposition 6.1. Then there exist $\bar{t} > 0$, $\sigma' > 0$ such that for every $t \geq \bar{t}$ and for every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = r$, then $f_t(u) \geq \sigma'$.

Proof. By contradiction, we can find two sequences $(t_h) \subseteq \mathbb{R}$ and $(u_h) \subseteq W_0^{1,p}(\Omega)$ such that $t_h \geq h$ for each $h \in \mathbb{N}$, $\|u_h\|_{1,p} = r$ and $f_{t_h}(u_h) < \frac{1}{h}$. Up to a subsequence, $(u_h)$ weakly converges in $W_0^{1,p}(\Omega)$ to some $u \in K_\infty$. Using (b) of [13, Theorem 5], it follows that $f_\infty(u) \leq \liminf_h f_{t_h}(u_h) \leq 0$.

By (a) of Proposition 6.1, we have $u = 0$. On the other hand, since

$$\limsup_h f_{t_h}(u_h) \leq 0 = f_\infty(u),$$

using (c) of [13, Theorem 5] we deduce that $(u_h)$ strongly converges to $u$ in $W_0^{1,p}(\Omega)$, namely $\|u\|_{1,p} = r$. This is impossible. □

Proposition 6.3. Let $\sigma', \bar{t}$ as in Proposition 6.2. Then there exists $\tilde{t} \geq \bar{t}$ such that for every $t \geq \tilde{t}$ there exist $v_t, w_t \in W_0^{1,p}(\Omega)$ such that $\|v_t\|_{1,p} < r$, $\|w_t\|_{1,p} > r$, $f_t(v_t) \leq \frac{\sigma'}{2}$ and $f_t(w_t) \leq \frac{\sigma'}{2}$. Moreover we have

$$\sup \left\{ f_t((1-s)v_t + sw_t) : 0 \leq s \leq 1 \right\} < +\infty.$$ 

Proof. We argue by contradiction. We set $\tilde{t} = \bar{t} + h$ and suppose that there exists $(t_h)$ such that $t_h \geq h + \bar{t}$ and such that for every $v_{t_h}, w_{t_h}$ in $W_0^{1,p}(\Omega)$ with $\|v_{t_h}\|_{1,p} < r$, $\|w_{t_h}\|_{1,p} > r$ it results $f_{t_h}(v_{t_h}) > \frac{\sigma'}{2}$ and $f_{t_h}(w_{t_h}) > \frac{\sigma'}{2}$. It is easy to prove that there exists a sequence $(u_h)$ in $K_{t_h}$ which strongly converges to 0 in $W_0^{1,p}(\Omega)$ and therefore $\|u_h\|_{1,p} < r$ and $f_{t_h}(u_{t_h}) \leq \frac{\sigma'}{2}$ eventually as $h \to +\infty$. This contradicts our assumptions. In a similar way one can prove the statement for $w_t$, while the last statement is straightforward. □

7. Proof of the main result

Proof of Theorem 2.1. By combining Theorem 5.3, Propositions 6.2 and 6.3 we can apply Theorem 3.4 and deduce the assertion. □

Acknowledgements. The authors wish to thank the referee for some very useful remarks on our paper. Moreover, the authors warmly thank M. Degiovanni for providing some helpful discussions.

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